Discrete Time Analysis & $\mathcal{Z}$-Transforms

ELEC 3004: **Systems**: Signals & Controls
Dr. Surya Singh

Lecture 7 (Revised)

elec3004@itee.uq.edu.au
http://robotics.itee.uq.edu.au/~elec3004/

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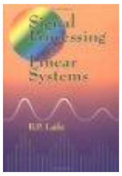
http://elec3004.com

Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	Systems: Linear Differential Systems
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	Aliasing & Antialiasing
4	**20-Mar**	**Discrete Time Analysis & Z-Transforms**
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	Digital Filter (FIR)
7	10-Apr	Digital Windows
	12-Apr	FFT
8	17-Apr	Active Filters & Estimation & Holiday
	19-Apr	Holiday
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation/PID
10	8-May	PID & State-Space
	10-May	State-Space Control
11	15-May	Digital Control Design
	17-May	Stability
12	22-May	State Space Control System Design
	24-May	Shaping the Dynamic Response
13	29-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

Today

- Chapter 8 (**Discrete-Time Signals and Systems**)
  - § 8.1 Introduction
  - § 8.2 **Some Useful Discrete-Time Signal Models**
  - § 8.3 **Sampling Continuous-Time Sinusoids & Aliasing**
  - § 8.4 Useful Signal Operations
  - § 8.5 Examples of Discrete-Time Systems

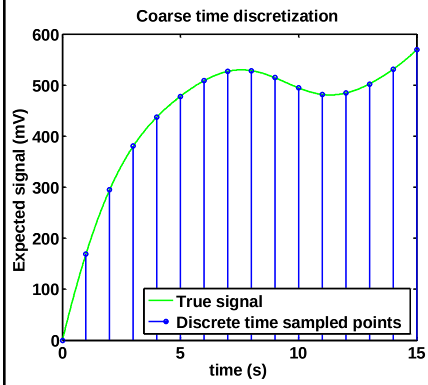
- Chapter 11 (**Discrete-Time System Analysis Using the *z*-Transform**)
  - § 11.1 The $\mathscr{Z}$-Transform
  - § 11.2 Some Properties of the *Z*-Transform

Next Time

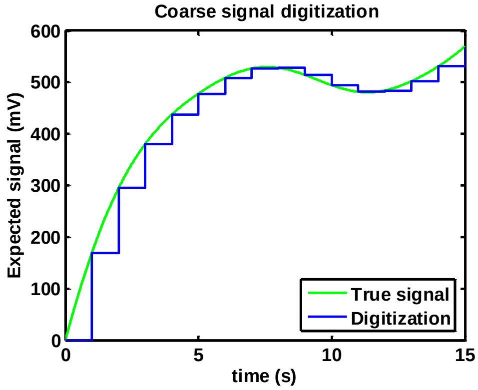
Data Acquisition (A/D Conversion)

Representation of Signal

- Time Discretization



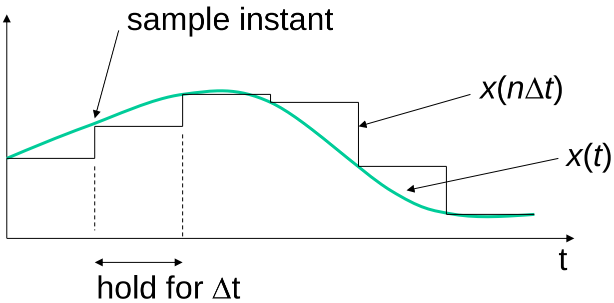
- Digitization



Practical Sampling

- Sample and Hold (S/H)
  - takes a sample every Δt seconds
  - holds that value constant until next sample

- Produces 'staircase' waveform, x(nΔt)



Practical Reconstruction

Two stage process:

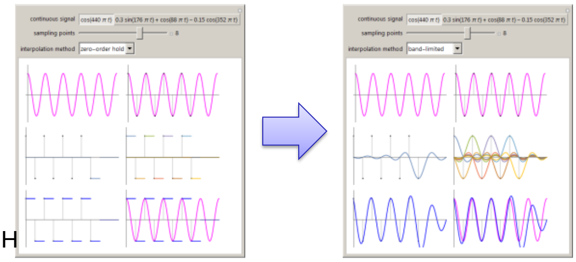
1. Digital to analogue converter (D/A)

- zero order hold filter
- produces ‘staircase’ analogue output

2. Reconstruction filter

- non-ideal filter: $\omega_c = \frac{\omega_s}{2}$
- further reduces replica spectrums
- usually 4th – 6th order (Butterworth)
  - 4th-6th Order: for acceptable phase response

Ex$_1$:ZOH



Ex$_2$: Sinc

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Quantisation

- Analogue to digital converter (A/D)
  - Calculates nearest binary number to $x(n\Delta t)$
    - $x_q[n] = q(x(n\Delta t))$, where q() is non-linear rounding fctn
  - output modeled as $x_q[n] = x(n\Delta t) + e[n]$

- Approximation process
  - therefore, loss of information (unrecoverable)
  - known as ‘quantisation noise’ ($e[n]$)
  - error reduced as number of bits in A/D increased
    - i.e., Δx, quantisation step size reduces

$$|e[n]| \le \frac{\Delta x}{2}$$

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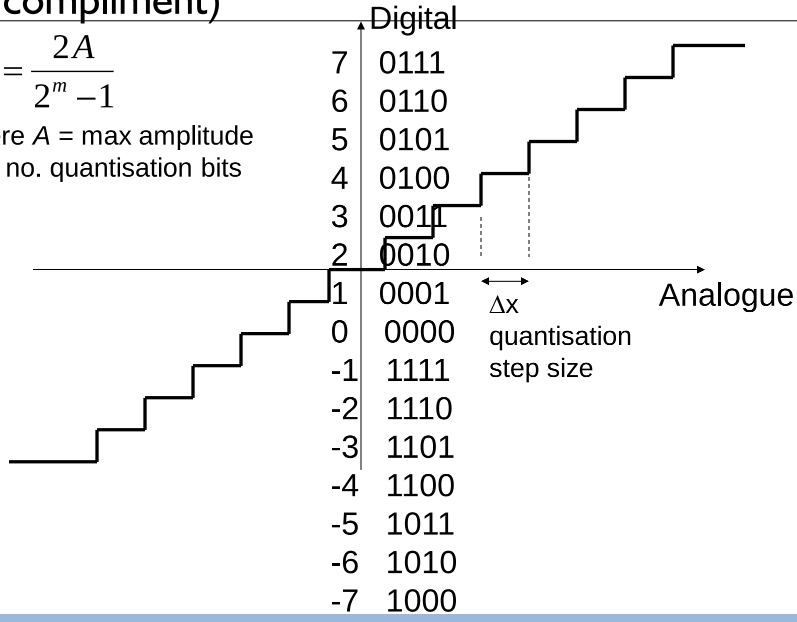
Input-output for 4-bit quantiser (two's compliment)

$$\Delta x = \frac{2A}{2^m - 1}$$

where A = max amplitude
m = no. quantisation bits



D/A Converter

- Analogue output y(t) is
  - convolution of output samples y(nΔt) with $h_{ZOH}(t)$

$$y(t) = \sum_n y(n\Delta t) h_{ZOH}(t - n\Delta t)$$

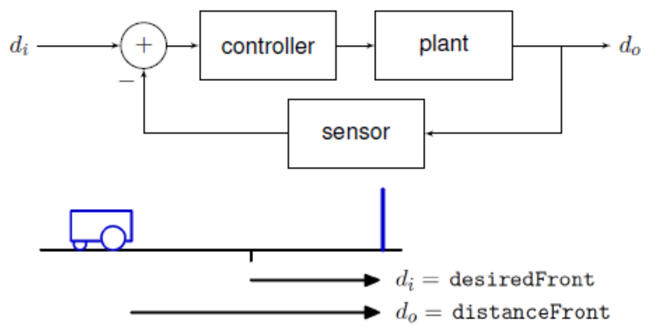
$$h_{ZOH}(t) = \begin{cases} 1, & 0 \le t < \Delta t \\ 0, & \text{otherwise} \end{cases}$$

$$H_{ZOH}(w) = \Delta t \exp\left(\frac{-jw\Delta t}{2}\right) \frac{\sin(w\Delta t / 2)}{w\Delta t / 2}$$

D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional **reconstruction filter required**

Discrete-Time Signal Analysis

Simple Controller Goes Digital



plant: $y[n] = y[n-1] - Tu[n-1]$

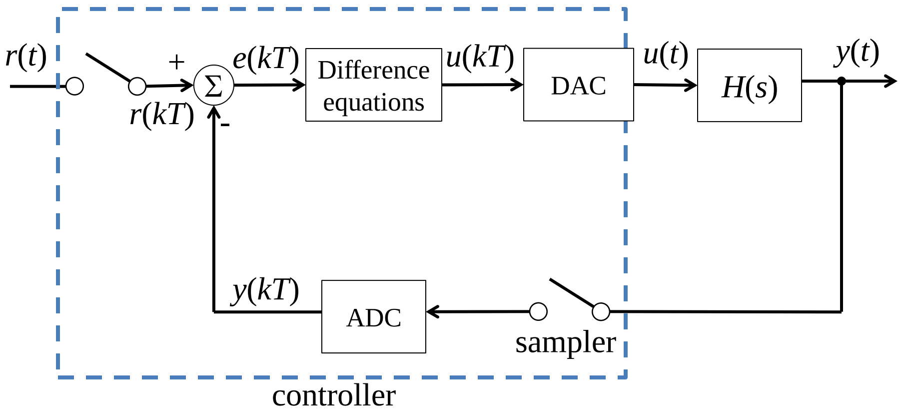
sensor: $y[n] = u[n-1]$

controller: $y[n] = Ku[n]$

Complex system behaviors, depending on K

Digitisation

- Continuous signals sampled with period T
- kth control value computed at $t_k = kT$



Digitisation

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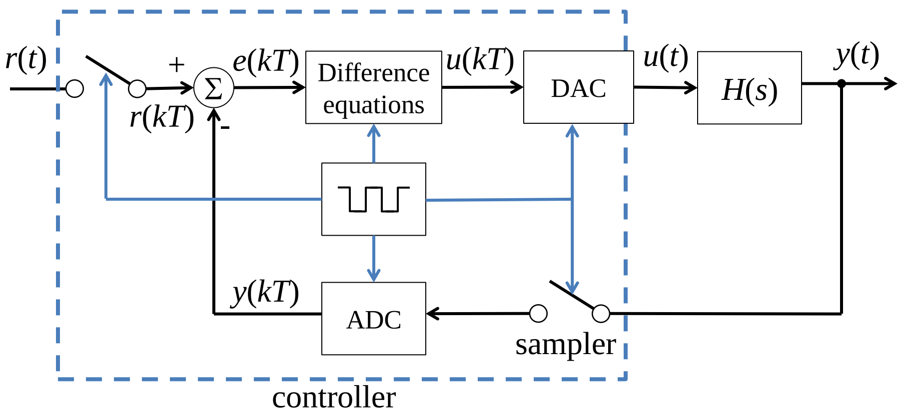
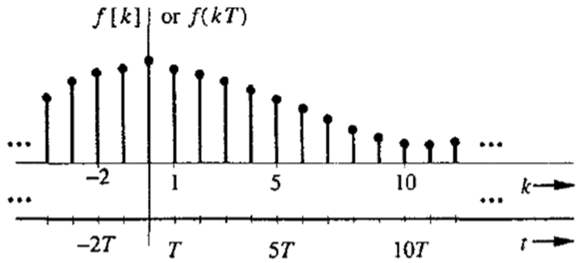
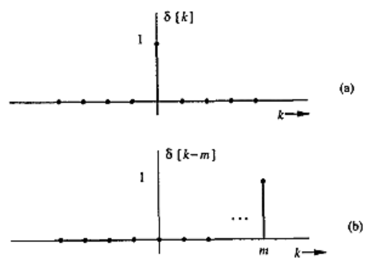


Discrete-Time Signal: *f[k]*



- Impulse Train Model:
  - May be denoted by *f(kT)*, where time *t* values are specified at *t* = *kT*
  - ***OR*** *f[k]* and viewed as a function of *k* (*k* ∈ integer)
- Continuous-time exponential Model:
  - $f(t) = e^{-t}$, sampled at *T* = 0.1 ➔ $f(kT) = e^{-kT} = e^{-0.1k}$

Discrete-Time Impulse Function $\delta[k]$



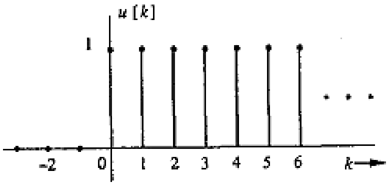
The discrete-time counterpart of the continuous-time impulse function $\delta(t)$ is $\delta[k]$, defined by

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \qquad (8.1)$$

This function, also called the unit impulse sequence, is shown in Fig. 8.3a. The time-shifted impulse sequence $\delta[k-m]$ is depicted in Fig. 8.3b. Unlike its continuous-time counterpart $\delta(t)$, this is a very simple function without any mystery.

Later, we shall express an arbitrary input $f[k]$ in terms of impulse components. The (zero-state) system response to input $f[k]$ can then be obtained as the sum of system responses to impulse components of $f[k]$.

Discrete-Time Unit Step Function $u[k]$



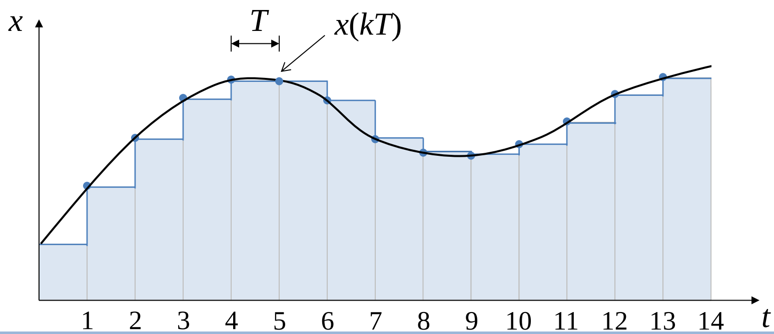
The discrete-time counterpart of the unit step function $u(t)$ is $u[k]$ (Fig. 8.4), defined by

$$u[k] = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases} \tag{8.2}$$

If we want a signal to start at $k = 0$ (so that it has a zero value for all $k < 0$), we need only multiply the signal with $u[k]$.

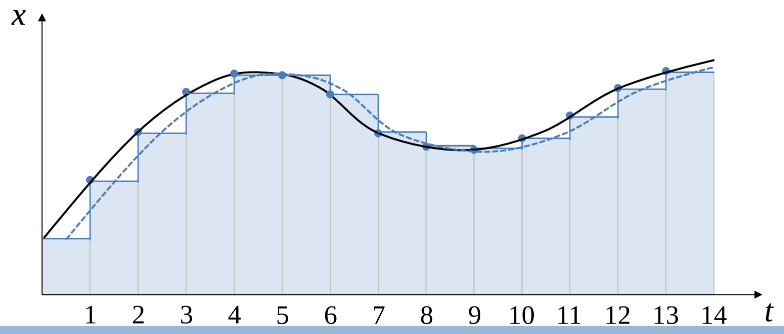
Return to the discrete domain

- Recall that continuous signals can be represented by a series of samples with period T



Zero Order Hold

- An output value of a synthesised signal is held constant until the next value is ready
  - This introduces an effective delay of $T/2$



Effect of ZOH Sampling

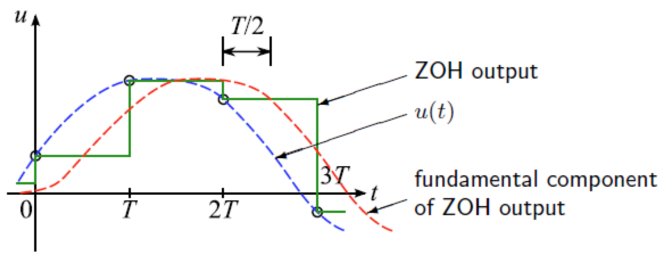
$\boxed{\text{Lower sample rate} \quad \Longrightarrow \quad \text{more oscillatory response}}$

— Why?

Sampling and reconstruction introduces:

delay in time domain
& phase lag in freq. domain $\leftarrow$ can destabilize the closed loop system

On average $u(kT)$ is delayed by $T/2$ relative to $u(t)$ due to the ZOH:



Effect of ZOH Sampling

The ZOH delay of $T/2$ (sec) causes

$$\text{phase lag} = \omega T/2 \text{ (rad)} \qquad \text{at } \omega \text{ rad s}^{-1}$$
$$\text{phase lag} = \pi/2 = 90^\circ \qquad \text{at } \omega = \pi/T \text{ [= Nyquist rate]}$$
$$\text{phase lag} = \pi/30 = 6^\circ \qquad \text{at } \omega = \pi/(15T)$$

⋆ 90° phase lag could be catastrophic

⋆ If $\omega_{\text{samp}} > 30 \times \omega_{\max}$,

then system bandwidth: $\omega_{\max} < \pi/(15T)$,

so the maximum phase lag is less than 6°

usually safe to ignore

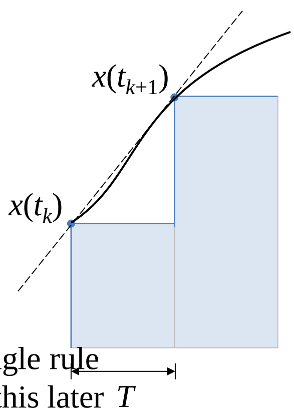
⋆ Any time needed to compute u_k causes additional delay (!)

Euler's method*

- Dynamic systems can be approximated† by recognising that:

$$\dot{x} \cong \frac{x(k+1) - x(k)}{T}$$

- As $T \to 0$, approximation error approaches 0



*Also known as the forward rectangle rule
†Just an approximation – more on this later

Difference Equation: Euler's approximation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} \qquad \Longrightarrow \qquad \frac{\mathrm{d}x}{\mathrm{d}t} \approx \frac{x_{k+1} - x_k}{T}$$

For small enough T, this can be used to approximate a continuous controller by a discrete controller:

1. Laplace transform $\longrightarrow$ differential equation

   e.g.
   $$D(s) = \frac{U(s)}{E(s)} = \frac{K(s+a)}{(s+b)} \quad \Longrightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}t} + b\,u = K\Big(\frac{de}{dt} + a\,e\Big)$$

2. Differential equation $\longrightarrow$ difference equation

   e.g.
   $$\frac{u_{k+1} - u_k}{T} + b\,u_k = K\Big(\frac{e_{k+1} - e_k}{T} + a\,e_k\Big)$$
   $$\Longrightarrow u_{k+1} = (1 - bT)u_k + K e_{k+1} + K(aT - 1)e_k$$
   $$= -a_1 u_k + b_0 e_{k+1} + b_1 e_k$$

So Why Is this a Concern? **Difference equations**

Difference equations arise in problems where the independent variable, usually time, is assumed to have a discrete set of possible values. The nonlinear difference equation

$$\begin{aligned} y(k+n) = f[&y(k+n-1),\ y(k+n-2),\ \dots,\ y(k+1),\ y(k),\ u(k+n),\\ &u(k+n-1),\ \dots,\ u(k+1),\ u(k)] \end{aligned} \tag{2.1}$$

with forcing function $u(k)$ is said to be of order n because the difference between the highest and lowest time arguments of $y(.)$ and $u(.)$ is n. The equations we deal with in this text are almost exclusively linear and are of the form

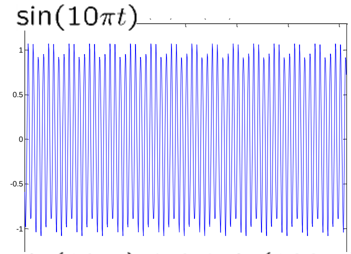
$$\begin{aligned} &y(k+n) + a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k)\\ &= b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k) \end{aligned} \tag{2.2}$$

We further assume that the coefficients a_i, b_i, $i = 0, 1, 2, \dots$, are constant. The difference equation is then referred to as linear time invariant, or LTI. If the forcing function $u(k)$ is equal to zero, the equation is said to be *homogeneous*.

Difference equations can be solved using classical methods analogous to those available for differential equations. Alternatively, z-transforms provide a convenient approach for solving LTI equations, as discussed in the next section.

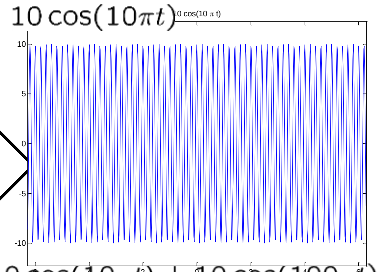
Delay is a Derivative **& Derivatives magnify noise!**

- $\sin(10\pi t)$

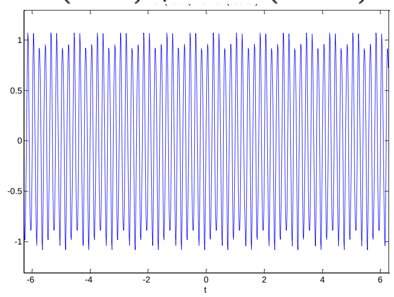


$\frac{d}{dt}$

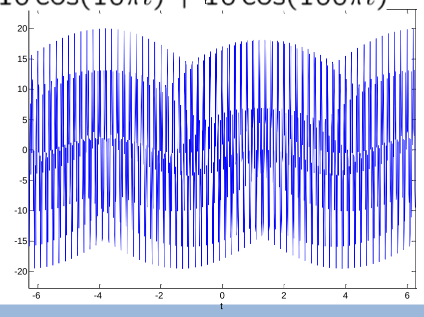
- $10\cos(10\pi t)$



- $\sin(10\pi t) + 0.1\sin(100\pi t)$



- $10\cos(10\pi t) + 10\cos(100\pi t)$



Discrete-Time
System Analysis

Linear Differential System Order

$$Q(D)y(t) = P(D)f(t)$$

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$
$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0$$

y(t)=P(D)/Q(D) f(t)
P(D): M
Q(D): N
(yes, N is de**N**ominator)

- In practice: $m \leq n$

∵ **if** m > *n:*
then the system is an
(m - n)th -order differentiator of high-frequency signals!

- Derivatives magnify noise!

Linear Differential Systems

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) =$$
$$b_m\frac{d^m f}{dt^m} + b_{m-1}\frac{d^{m-1}f}{dt^{m-1}} + \cdots + b_1\frac{df}{dt} + b_0 f(t) \qquad (2.1a)$$

where all the coefficients a_i and b_i are constants. Using operational notation D to represent d/dt, we can express this equation as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)\,y(t)$$
$$= (b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0)\,f(t) \qquad (2.1b)$$

or

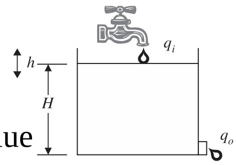
$$Q(D)y(t) = P(D)f(t) \qquad (2.1c)$$

where the polynomials $Q(D)$ and $P(D)$ are

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 \qquad (2.2a)$$
$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0 \qquad (2.2b)$$

Why e^{-kT} ?

- **Solution to First-Order ODE!**
- Ex: “Tank” Fill
- Where:
  - H=steady-state fluid height in the tank
  - h=height perturbation from the nominal value
  - Q=steady-state flow rate through the tank
  - q_i=inflow perturbation from the nominal value
  - q_0=outflow perturbation from the nominal value

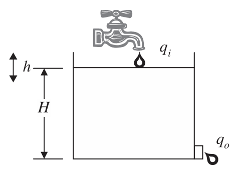


- Goal: Maintain H by adjusting Q.

Why e^{-kT} ? [2]

- $h = Rq_0$
- $\frac{dC(h+H)}{dt} = (q_i+Q) - (q_0 + Q)$
- $\frac{dh}{dt} + \frac{h}{\tau} = \frac{q_i}{C}$
- $\tau = RC$
- Solution:

$$h(t) = e^{\frac{t-t_0}{\tau}} h(t_0) + \frac{1}{C}\int_{t_0}^{t} e^{\frac{t-\lambda}{\tau}} q_i(\lambda) d\lambda$$



- For a fixed period of time (T) and steps k=0,1,2,…:

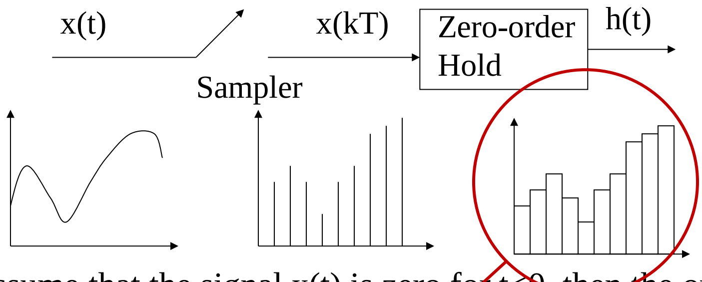
$$h(k+1) = e^{\frac{-T}{\tau}} h(k) + R\left[1 - e^{-\frac{T}{\tau}}\right] q_i(k)$$

BREAK

z Transforms
(Digital Systems Made eZ)

Review and Extended Explanation

Back to the Zero-order **Hold** (ZOH)



- Assume that the signal x(t) is zero for t<0, then the output h(t) is related to x(t) as follows:

$$h(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] + \cdots$$
$$= \sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]$$

➔The “hold” adds a delay. The delay leads to difference equations

Coping with Complexity

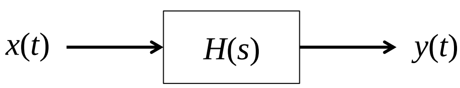
Transfer functions help control complexity

– Recall the Laplace transform:

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}dt = F(s)$$

where

$$\mathcal{L}\{\dot{f}(t)\} = sF(s)$$



- Is there a something similar for sampled systems?

The z-Transform

- It is defined by:

$$z = re^{j\omega}$$

- Or in the Laplace domain:

$$z = e^{sT}$$

- Thus: $Y(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ $\qquad y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$

- **That is → it is a discrete version of the Laplace:**

$$f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{z}{z - e^{-aT}}$$

The z-transform

- The discrete equivalent is the *z*-Transform†:

$$\mathcal{Z}\{f(k)\} = \sum_{k=0}^{\infty} f(k)z^{-k} = F(z)$$

and

$$\mathcal{Z}\{f(k-1)\} = z^{-1}F(z)$$

$$x(k) \longrightarrow \boxed{F(z)} \longrightarrow y(k)$$

Convenient!

†This is not an approximation, but approximations are easier to derive

The z-Transform [2]

- Thus:

$$Y(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k} \qquad\qquad y[n] \xleftrightarrow{\mathcal{Z}} Y(z)$$

- z-Transform is analogous to other transforms:

$$\mathcal{Z}\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$$

and

$$\boxed{\mathcal{Z}\{f(k-1)\} = z^{-1} F(z)}$$

∴ Giving:

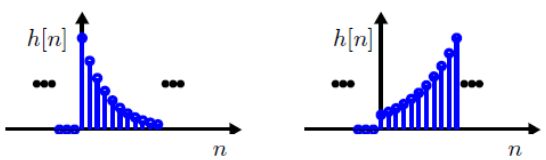
$$x(k) \longrightarrow \boxed{F(z)} \longrightarrow y(k)$$

z-Transforms for Difference Equations

- First-order linear constant coefficient difference equation:

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$



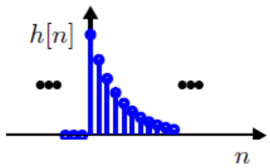
$$h[n] = \begin{cases} ba^n & n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

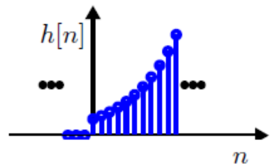
$$H(z) = \sum_{k=0}^{\infty} ba^k z^{-k} = b \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{b}{1 - az^{-1}}, \quad \text{when } |z| > |a|.$$

z-Transforms for Difference Equations

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$





$$\boxed{\begin{array}{c} y[n] - ay[n-1] = bu[n] \\ \updownarrow \\ Y(z) - az^{-1}Y(z) = bU(z) \end{array}}$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1 - az^{-1}},$$ when does it converge?

The z-transform

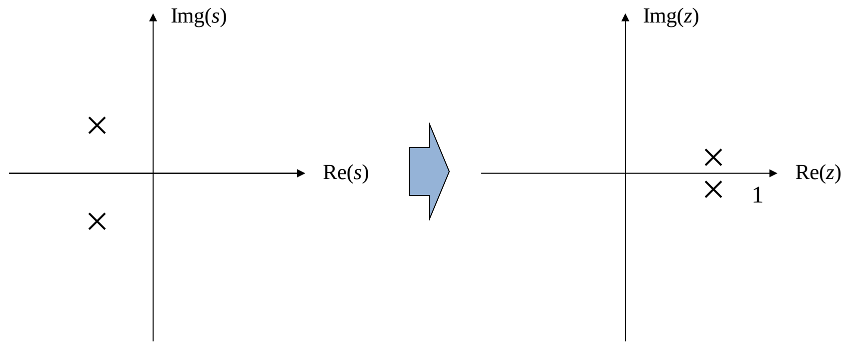
- In practice, you'll use look-up tables or computer tools (ie. Matlab) to find the *z*-transform of your functions

$F(s)$	$F(kt)$	$F(z)$
$\frac{1}{s}$	1	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\frac{1}{s^2+a^2}$	$\sin(akT)$	$\frac{z \sin aT}{z^2 - (2\cos aT)z + 1}$

The z-Plane

z-domain poles and zeros can be plotted just like s-domain poles and zeros (of the $\mathcal{L}$):

- S-plane:
  - λ – Plane
- $\boldsymbol{z = e^{sT}}$ Plane
  - γ – Plane

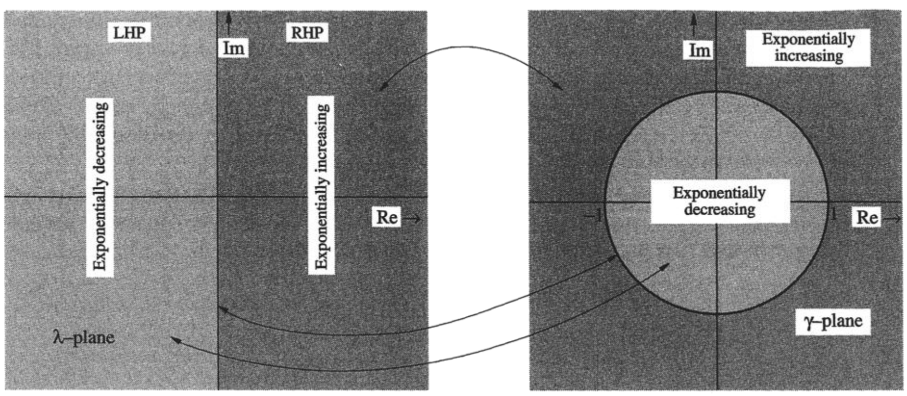


Discrete-Time Exponential γ^k

Recall we defined $z = e^{sT}$

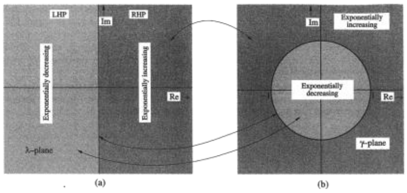
Thus, this is the γ plane ➔

$$e^{\lambda k} = \gamma^k$$



(a) (b)

Discrete-Time Exponential γ^k

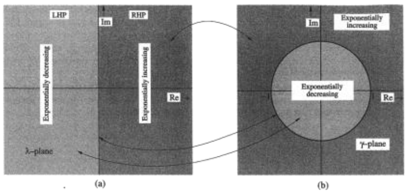


- $e^{\lambda k} = \gamma^k$
- $\gamma = e^{\lambda}$ or $\lambda = \ln \gamma$
- In discrete-time systems, unlike the continuous-time case, the form γ^k proves more convenient than the form $e^{\lambda k}$

Why?

- Consider $e^{j\Omega k}$ ($\lambda = j\Omega \therefore$ constant amplitude oscillatory)
- $e^{j\Omega k}$ ➔ γ^k, for $\gamma \equiv e^{j\Omega}$
- $\left|e^{j\Omega}\right| = 1$, hence $|\gamma| = 1$

Discrete-Time Exponential γ^k



- Consider $e^{\lambda k}$
  When λ: LHP
- Then
- $\gamma = e^{\lambda}$
- $\gamma = e^{\lambda} = e^{a+jb} = e^{a}e^{jb}$
- $|\gamma| = \left|e^{a}e^{jb}\right| = |e^{a}| \because \left|e^{jb}\right| = 1$

Properties of the the *z*-transform

- Some useful properties
  - **Delay by $\boldsymbol{n}$ samples**: $\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$
  - **Linear**: $\mathcal{Z}\{af(k) + bg(k)\} = \mathrm{a}F(z) + bG(z)$
  - **Convolution**: $\mathcal{Z}\{f(k) * g(k)\} = F(z)G(z)$

So, all those block diagram manipulation tools you know and love will work just the same!

More Z-Transform Properties

- Time Reversal

$$x[n] \leftrightarrow X(z) \qquad \mathrm{ROC} = R$$

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \qquad R' = \frac{1}{R}$$

- Multiplication by z^n

$$x[n] \leftrightarrow X(z) \qquad \mathrm{ROC} = R$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad R' = |z_0| R$$

- Multiplication by n (or Differentiation in z):

$$x[n] \leftrightarrow X(z) \qquad \mathrm{ROC} = R$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \qquad R' = R$$

- Convolution

$$x_1[n] \leftrightarrow X_1(z) \qquad \mathrm{ROC} = R_1$$
$$x_2[n] \leftrightarrow X_2(z) \qquad \mathrm{ROC} = R_2$$

$$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \qquad R' \supset R_1 \cap R_2$$

Z-Transform Properties: Time Shifting

$$y[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}Y(z)$$

$$\begin{aligned} y_2[n] &= y[n-n_0] \\ Y_2(e^{jw}) &= \sum_{k=-\infty}^{\infty} y[k-n_0]z^{-k} \\ &= \sum_{l=-\infty}^{\infty} y[l]z^{-(l+n_0)} \\ &= z^{-n_0}Y(z) \end{aligned}$$

- Two Special Cases:
- z^{-1}: the *unit-delay operator:*

$$x[n-1] \leftrightarrow z^{-1}X(z) \qquad R' = R \cap \{0 < |z|\}$$

- z: *unit-advance operator:*

$$x[n+1] \leftrightarrow zX(z) \qquad R' = R \cap \{|z| < \infty\}$$

z-Transform Example

- Obtain the z-Transform of the sequence:

$$x[k] = \{3, 0, 1, 4,1,5, \dots\}$$

- Solution:

$$X(z) = 3 + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5}$$

An example!

- Back to our difference equation:

$$y(k) = x(k) + Ax(k-1) - By(k-1)$$

becomes

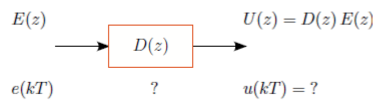
$$Y(z) = X(z) + Az^{-1}X(z) - Bz^{-1}Y(z)$$
$$(z+B)Y(z) \ = (z+A)X(z)$$

which yields the transfer function:

$$\frac{Y(z)}{X(z)} = \frac{z+A}{z+B}$$

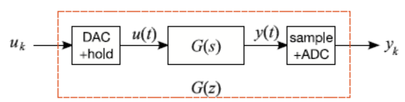
Note: It is also not uncommon to see systems expressed as polynomials in z^{-n}

$s \leftrightarrow z$: Pulse Transfer Function Models



- Pulse in Discrete is equivalent to Dirac-δ

$$e_k = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$



➔

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}_{t=kT}\right\} = \mathbf{(1 - z^{-1})}\boldsymbol{\mathcal{Z}}\left\{\frac{\mathbf{G(s)}}{\mathbf{s}}\right\}$$

Source: Oxford 2A2 Discrete Systems, Tutorial Notes p. 26

Relationship with s-plane poles and z-plane transforms

If $F(s)$ has a pole at $s = a$
then $F(z)$ has a pole at $z = e^{aT}$

$\uparrow$

consistent with $z = e^{sT}$

What about transfer functions?

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

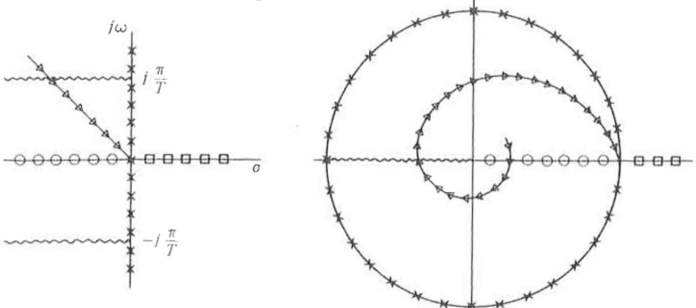
$\downarrow$

If $G(s)$ has poles $s = a_i$
then $G(z)$ has poles $z = e^{a_i T}$

but the zeros are unrelated

$\mathcal{F}(s)$	$f(kT)$	$F(z)$
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-1}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$
$\frac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$

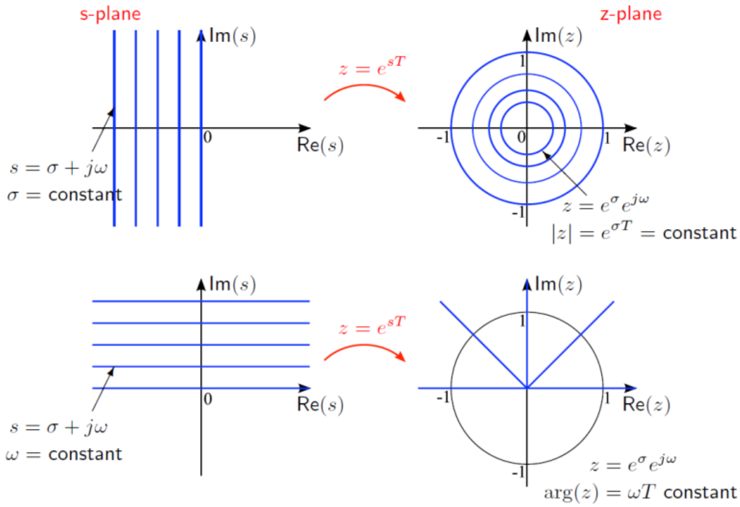
Hint: Use γ to Transform s ↔ z: z=e^{sT}



s-plane	Symbol	z-plane
$s = j\omega$ Real frequency axis	× × ×	$\lvert z\rvert = 1$ Unit circle
$s = \sigma \geq 0$	□□□	$z = r \geq 1$
$s = \sigma \leq 0$	○○○	$z = r,\ 0 \leq r \leq 1$
$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -a + jb$ Constant damping ratio if ζ is fixed and ω_n varies	△△△	$z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T) = e^{-aT}$, $\theta = \omega_n T\sqrt{1-\zeta^2} = bT$ Logarithmic spiral
$s = \pm j(\pi/T) + \sigma, \sigma \leq 0$	∿∿∿	$z = -r$

(a) s-plane (b) z-plane

S-Plane to z-Plane [1/2]



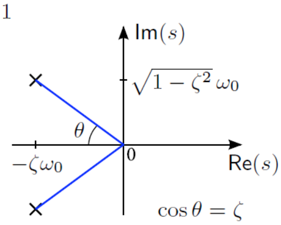
S-Plane to z-Plane [2/2]

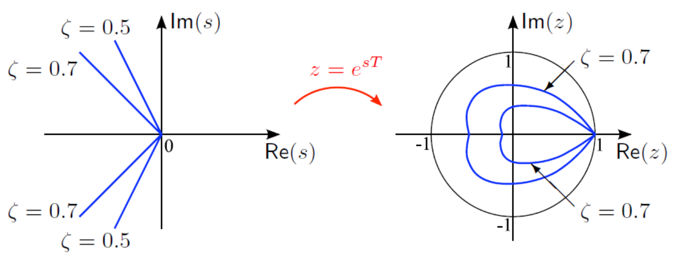
Pole locations for constant damping ratio $\zeta < 1$

$$s^2 + \zeta\omega_0 s + \omega_0^2 = 0$$

$$\Downarrow$$

$$s = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\,\omega_0$$





$s = -\zeta\omega_0 + j\sqrt{1-\zeta^2}\,\omega_0$: $\zeta =$ constant $\qquad z = e^{-\zeta\omega_0 T} e^{-j\sqrt{1-\zeta^2}\omega_0 T}$

Transfer function of Zero-order-hold (ZOH)

- Recall the Laplace Transforms ($\mathcal{L}$) of:

$$\mathcal{L}[\delta(t)] = 1 \qquad \mathcal{L}[f(t-kT)] = F(s)e^{-kTs}$$

$$\mathcal{L}\left[\delta(t-kT)\right] = e^{-kTs} \qquad \mathcal{L}[1(t-kT)] = \frac{e^{-kTs}}{s}$$

- Thus the $\mathcal{L}$ of h(t) becomes:

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Transfer function of Zero-order-hold (ZOH)

… Continuing the $\mathcal{L}$ of h(t) …

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\to X(s) = \mathcal{L}\left[\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right] = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\therefore H(s) = \mathcal{L}[h(t)] = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}X(s)$$

➔ Thus, giving the transfer function as:

$$\boxed{G_{\mathrm{ZOH}}(s) = \frac{H(s)}{X(s)} = \frac{1-e^{-Ts}}{s}} \xrightarrow{\mathcal{Z}} \boxed{G_{ZOH}(z) = \frac{\left(1-e^{-aT}\right)}{z-e^{-aT}}}$$

Next Time…

- **Digital Systems**

- Review:
  - Chapter 8 of Lathi

- A signal has many signals ☺
  [Unless it's bandlimited. Then there is the one ω]