http://elec3004.com

Aliasing & Antialiasing

ELEC 3004: **Systems**: Signals & Controls
Dr. Surya Singh

Lecture 6

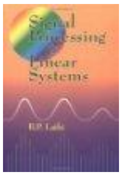
elec3004@itee.uq.edu.au
http://robotics.itee.uq.edu.au/~elec3004/

March 15, 2019

Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	Systems: Linear Differential Systems
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	**Aliasing & Antialiasing**
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	Digital Filter (FIR)
7	10-Apr	Digital Windows
	12-Apr	FFT
8	17-Apr	Active Filters & Estimation & Holiday
	19-Apr	**Holiday**
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation/PID
10	8-May	PID & State-Space
	10-May	State-Space Control
11	15-May	Digital Control Design
	17-May	Stability
12	22-May	State Space Control System Design
	24-May	Shaping the Dynamic Response
13	29-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 5:
  Sampling
  - § 5.1 **The Sampling Theorem**
  - § 5.2 Numerical Computation of Fourier Transform: The Discrete Fourier Transform (DFT)

Also:
- § 4.6 Signal Energy

Recap: Sampling Theorem

- The Nyquist criterion states:

*To prevent aliasing, a **bandlimited** signal of bandwidth w_B rad/s must be sampled at a rate greater than $2w_B$ rad/s*

– $w_s > 2w_B$

Note: this is a > sign not a ≥

Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

Sampling & The Fourier Transform

- Samples are like snacks – You don't just take one ☺
- Thus, by definition, sampling is a sequence
- And if we assume a fixed sample Period "*T*"
  Then it is a periodic sequence
- Any periodic sequence has a frequency
- A nice tool for mathematically modelling frequencies is the Fourier Transform

Recall: Fourier Transform & Fourier Transform Tables

Table 4.1
A Short Table of Fourier Transforms

	$f(t)$	$F(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$a > 0$
2	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$	$a > 0$
3	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$	$a > 0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$a > 0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$a > 0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	

Ref: Lathi, p. 252

Fourier Transform Pairs [2]

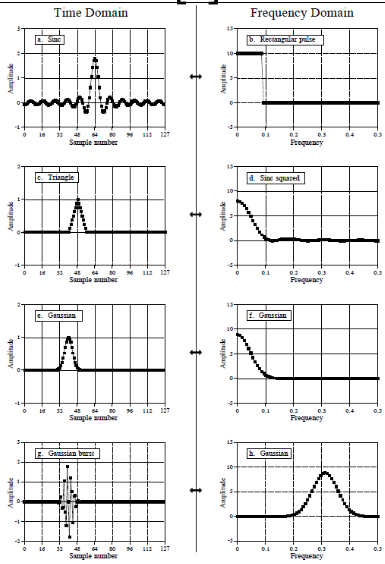
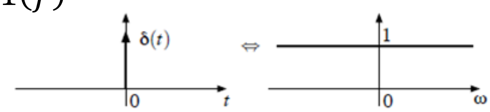


FIGURE 11-5
Common transform pairs.

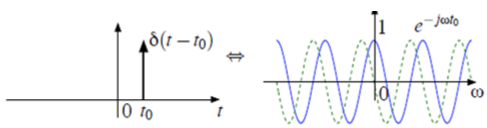
Ref: Analog Devices, DSP Book, Ch. 11, p. 217

Fourier Transform Pairs [3]

- $\delta(t) \Leftrightarrow 1(f)$

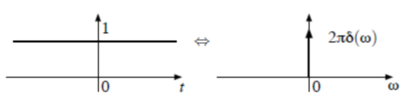


- Shifted δ



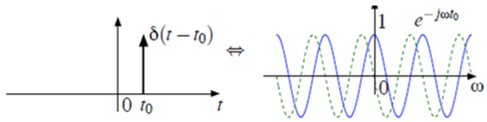
- $1(t) \;\; \Leftrightarrow \delta(f)$

$1 \Leftrightarrow \delta(f)$

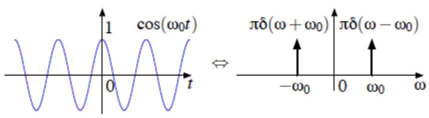


Fourier Transform Pairs [4]

- Shifted δ



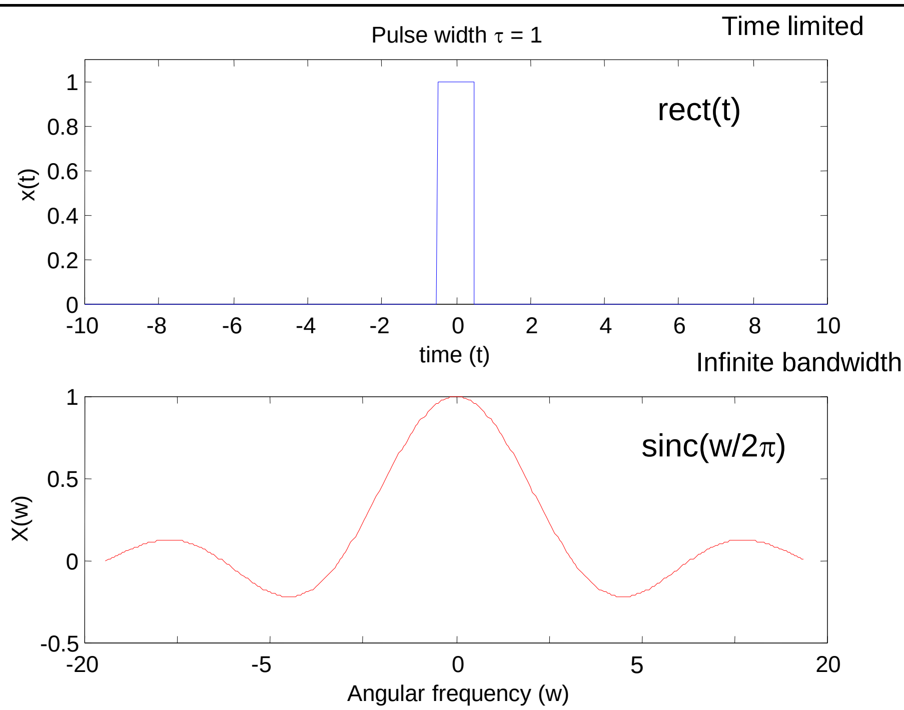
- $\cos(2\pi f_0 t) \Leftrightarrow \frac{1}{2}[\,\delta(f-f_0)+\delta(f+f_0)]$

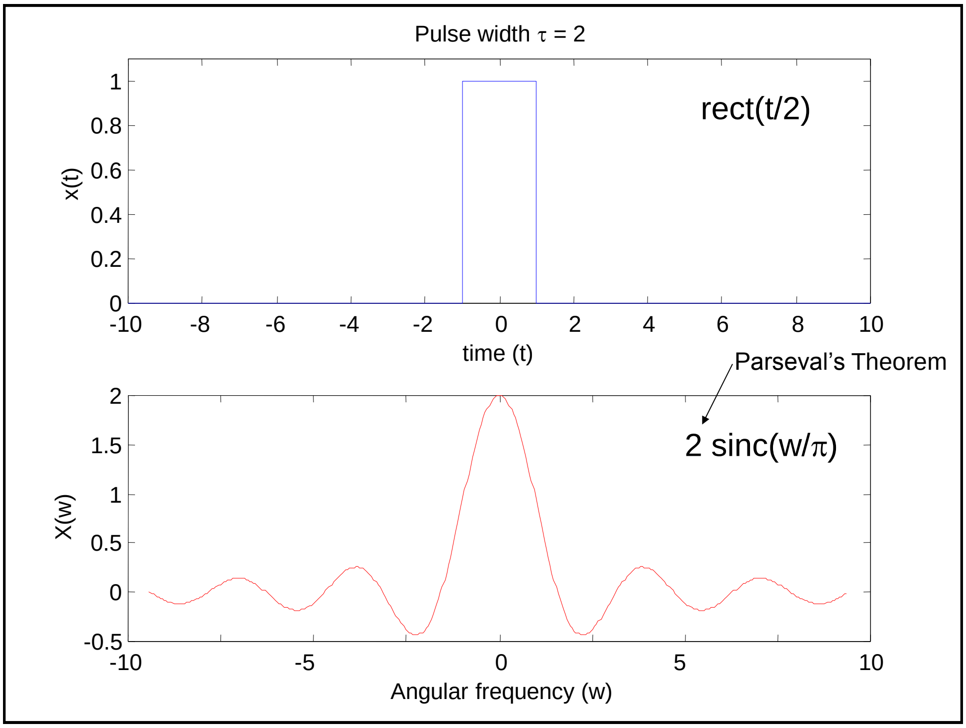


- Duality:

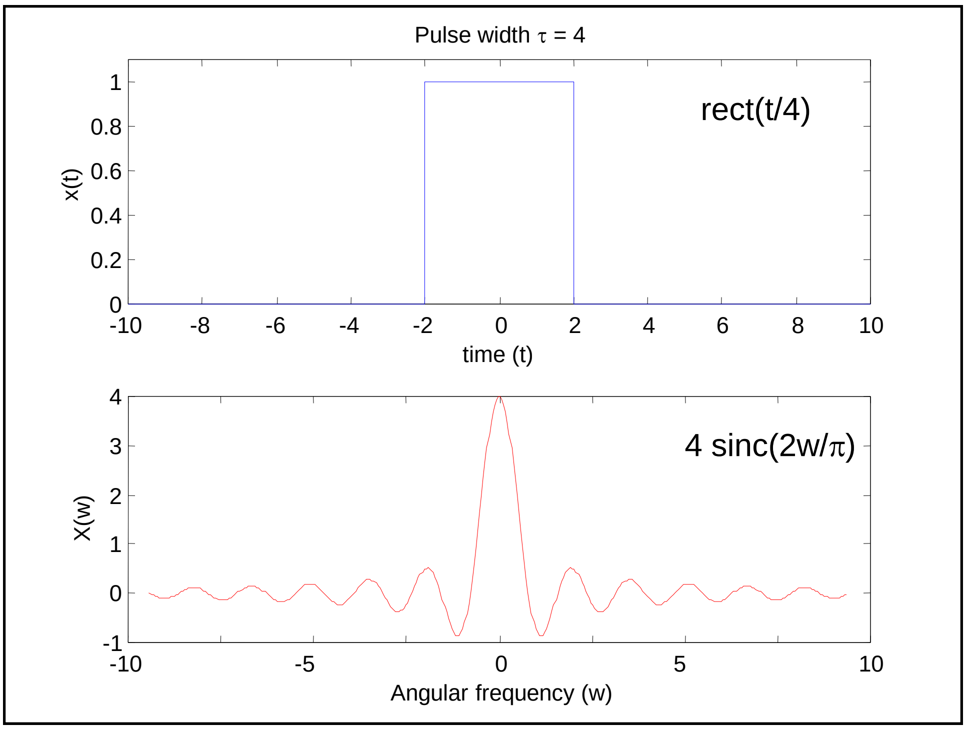
Duality Theorem: If $x(t) \Leftrightarrow X(f)$, then $X(t) \Leftrightarrow x(-f)$.

In other words, $\mathcal{F}[\mathcal{F}[x(t)]] = x(-t)$.

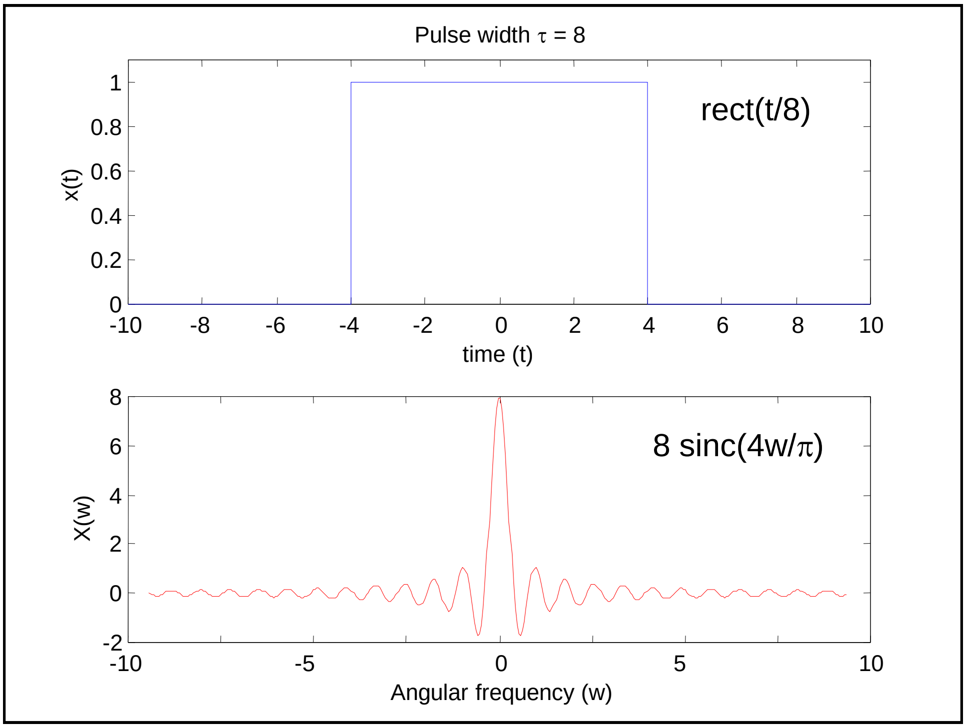




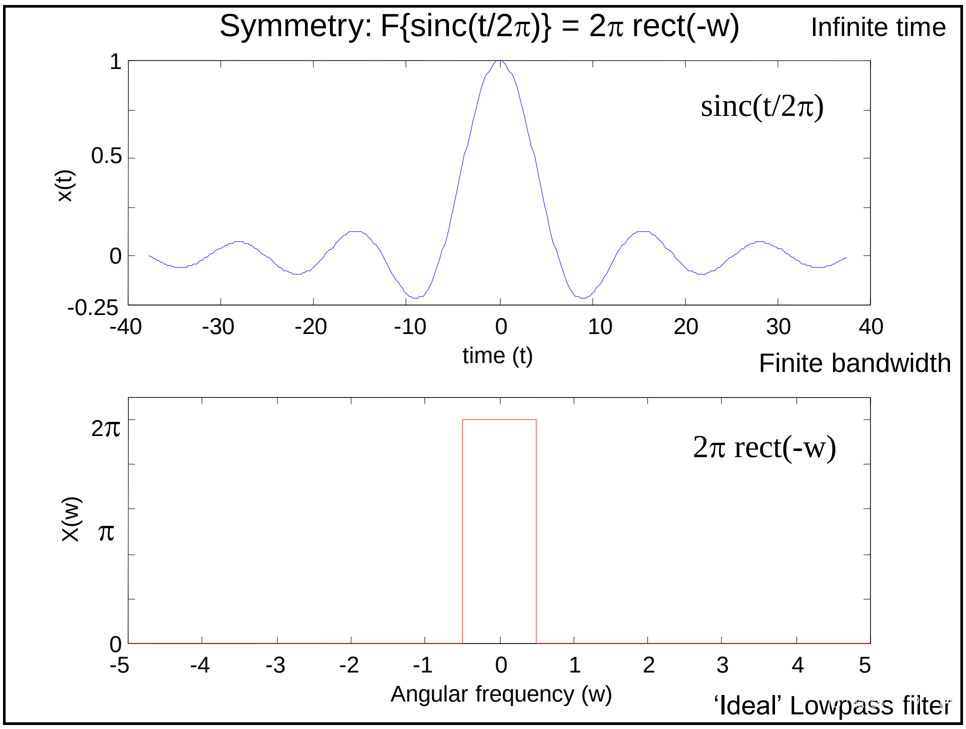
Pulse width τ = 2
rect(t/2)
x(t)
time (t)
Parseval's Theorem
2 sinc(w/π)
X(w)
Angular frequency (w)



Pulse width τ = 4
rect(t/4)
x(t)
time (t)
4 sinc(2w/π)
X(w)
Angular frequency (w)



Pulse width τ = 8
rect(t/8)
x(t)
time (t)
8 sinc(4w/π)
X(w)
Angular frequency (w)



Symmetry: F{sinc(t/2π)} = 2π rect(-w)
Infinite time
sinc(t/2π)
x(t)
time (t)
Finite bandwidth
2π rect(-w)
X(w)
Angular frequency (w)
'Ideal' Lowpass filter

Reconstruction

Time Domain Analysis of Reconstruction

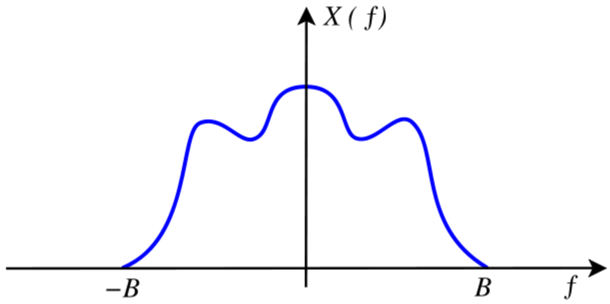
- Frequency domain: multiply by ideal LPF
  - ideal LPF: 'rect' function (gain Δt, cut off w_c)
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with 'sinc' function
  - as $F^{-1}\{\Delta t\ \text{rect}(w/w_c)\} = \Delta t\, w_c\, \text{sinc}(w_c t/\pi)$
  - i.e., weighted sinc on every sample
- Normally, $w_c = w_s/2$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \operatorname{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$

Reconstruction

- Whittaker–Shannon interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$



Why *sinc*?

Time Domain Analysis of Reconstruction

- Frequency domain: multiply by ideal LPF
  - ideal LPF: 'rect' function (gain Δt, cut off w_c)
  - removes replica spectrums, leaves original

- Time domain: this is equivalent to
  - convolution with 'sinc' function
  - as $F^{-1}\{\Delta t \ \mathrm{rect}(w/w_c)\} = \Delta t \, w_c \, \mathrm{sinc}(w_c t/\pi)$
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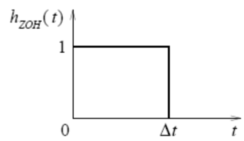
- Normally, $w_c = w_s/2$

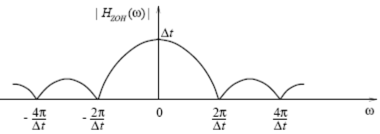
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \ \mathrm{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$

Zero Order Hold (ZOH)

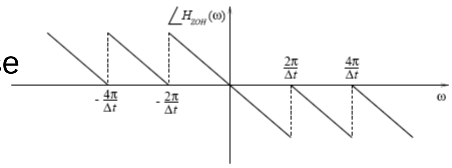
ZOH impulse response



ZOH amplitude response

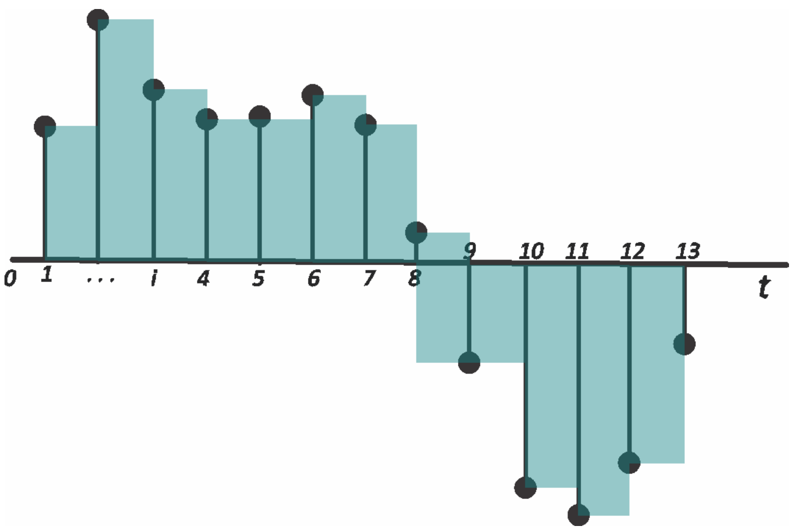


ZOH phase response



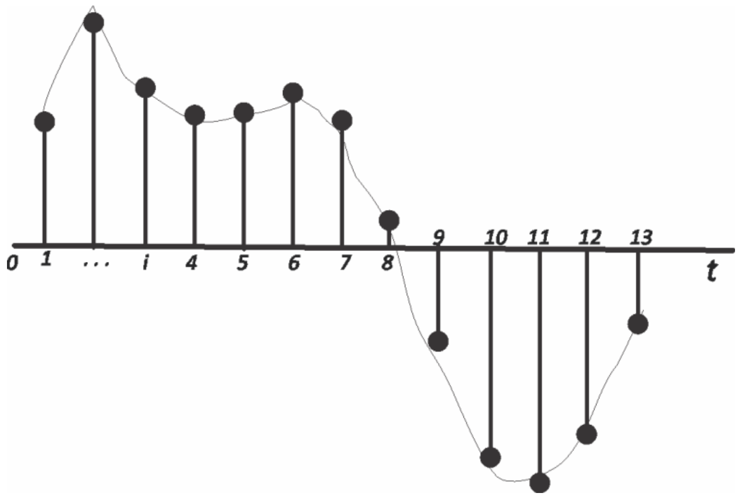
Reconstruction

- Zero-Order Hold [ZOH]

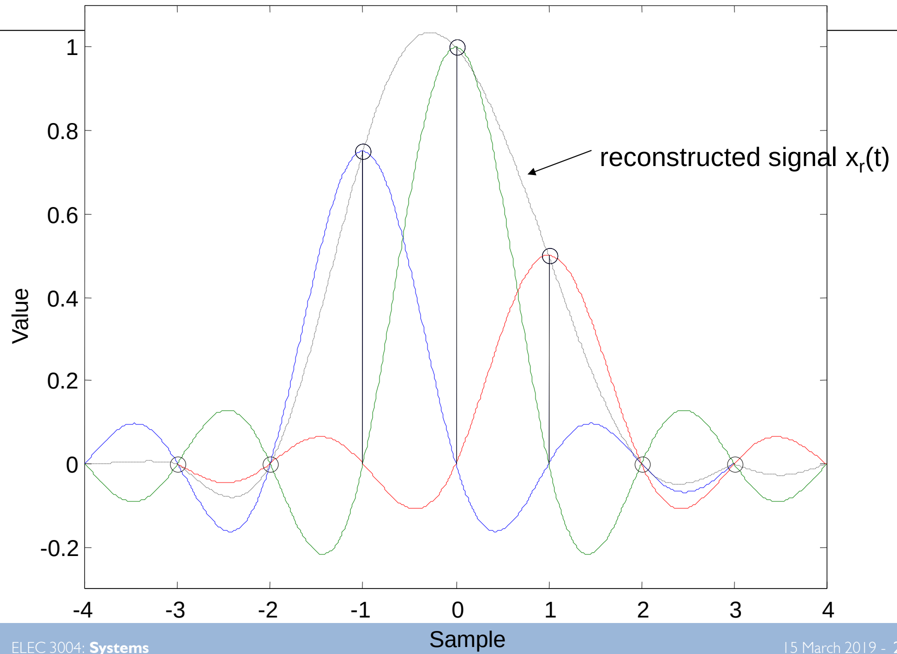


Reconstruction

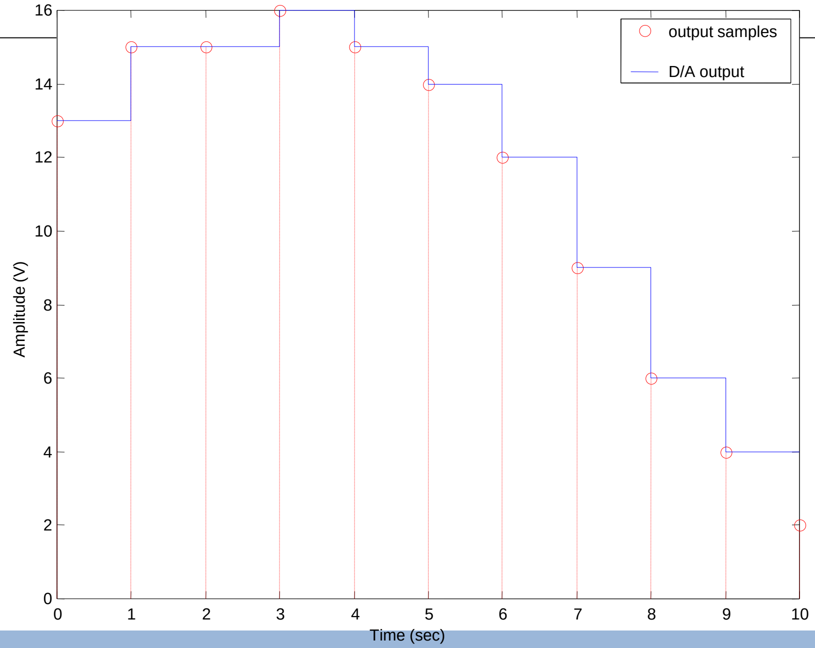
- Whittaker–Shannon interpolation formula



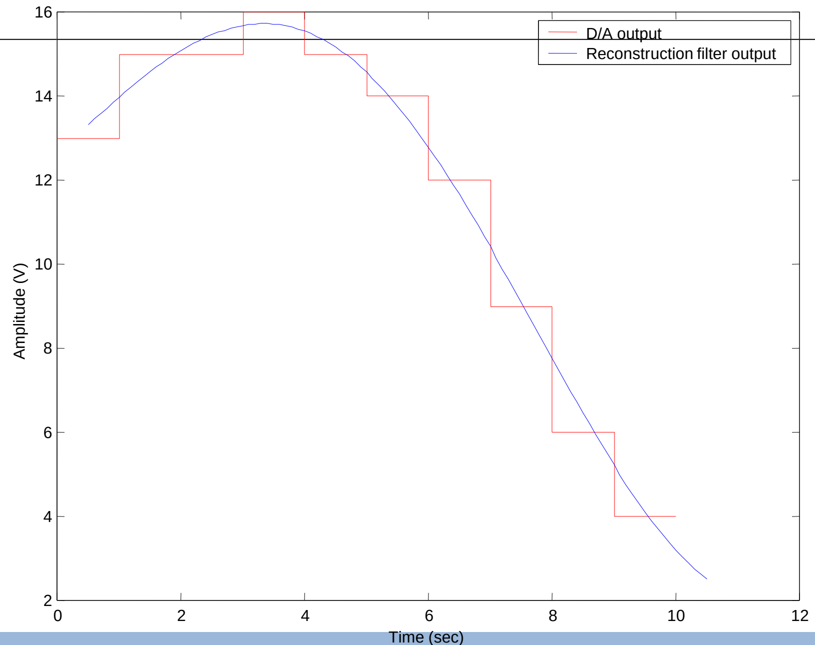
Ideal *sinc* Interpolation of sample values [0 0 0.75 1 0.5 0 0]



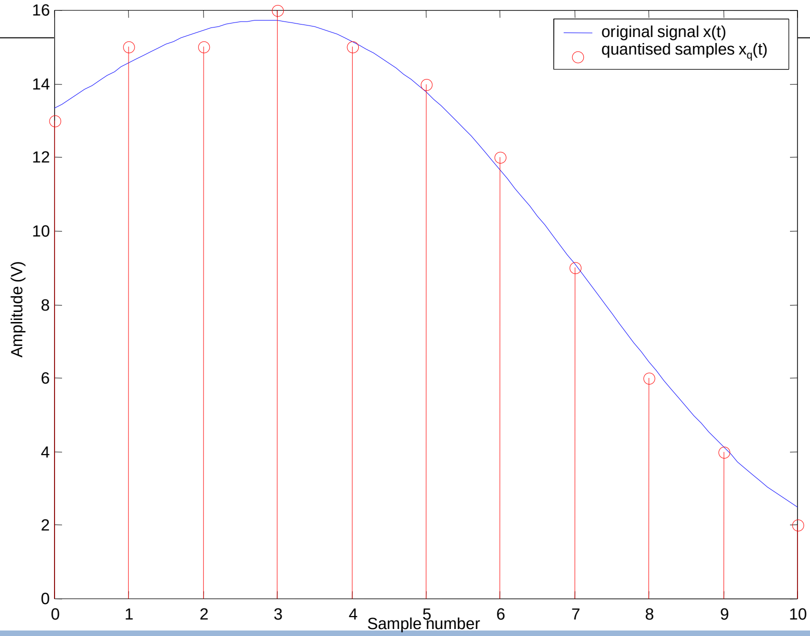
'staircase' output from D/A converter (ZOH)



Smooth output from reconstruction filter



Example: error due to signal quantisation



Matlab Example

```
%% Sample PSD

%% Set Values
f=1;
phi=0;
fs=1e2;
t0=0;
tf=1;

%% Generate Signal
t=linspace(t0,tf,(fs*(tf-t0)));
x1=cos(2*pi*f*t + phi);
figure(10); plot(t, x1);

%% PSD
[p_x1, f_x1] = pwelch(x1,[],[],[],fs);
figure(20); plot(f_x1, pow2db(p_x1));
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');

%% PSD (Centered)
[p_x1, f_x1] = pwelch(x1,[],[],[],fs, 'centered','power');
figure(30); plot(f_x1, pow2db(p_x1));
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');
```

Hello World

Hello World

A demonstration

No antialiasing

BREAK

Sampling & Aliasing

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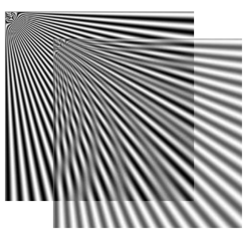
Overview (i.e. today we are going to learn …)

- Aliasing
- Spectral Folding
- Anti-Aliasing
  - Low-pass filtering of signals so as to keep things band limited



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Alliasing

- Aliasing - through sampling, two entirely different analog sinusoids take on the same "discrete time" identity

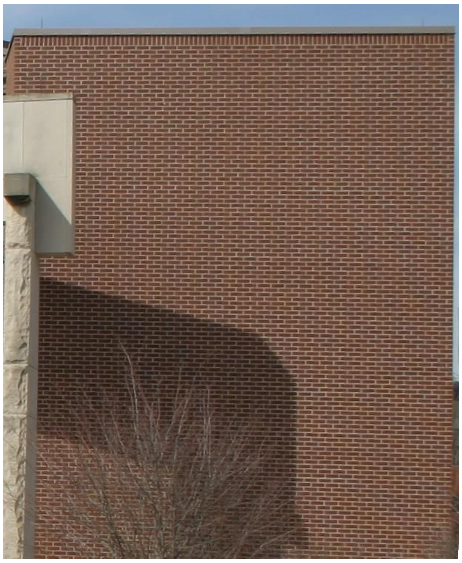
For $f[k] = \cos(\Omega k,)\ \ \Omega = \omega T$:

The period has to be less than F_h (highest frequency): $T \leq \frac{1}{2\mathcal{F}_h}$

Thus: $0 \leq \mathcal{F} \leq \frac{\mathcal{F}_s}{2}$

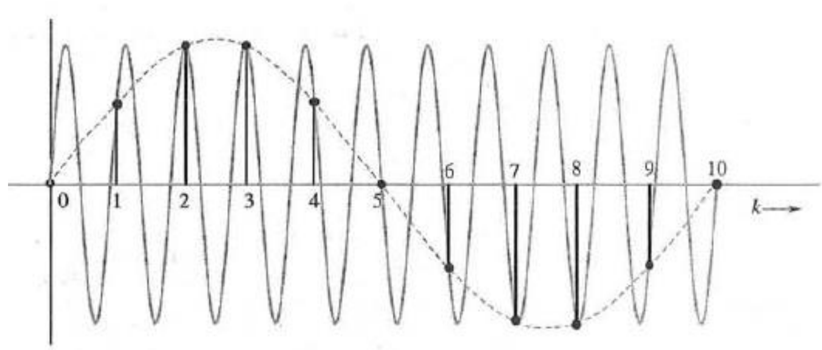
ω_f: aliased frequency: $\omega T = \omega_f T + 2\pi m$

Ex: Moire Effects

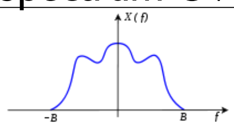


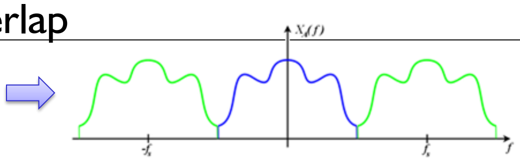
Source: Wikimedia https://en.wikipedia.org/wiki/Aliasing#/media/File:Moire_pattern_of_bricks.jpg (and aliased)

Aliasing: Another view of this



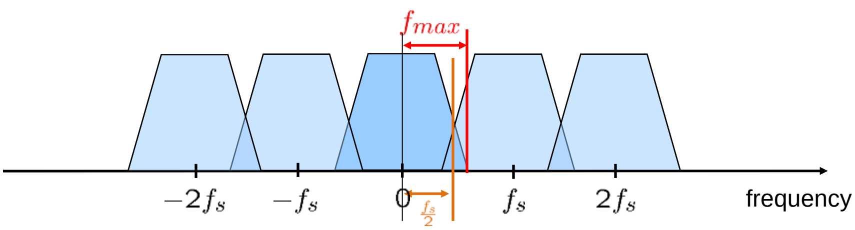
Spectrum Overlap



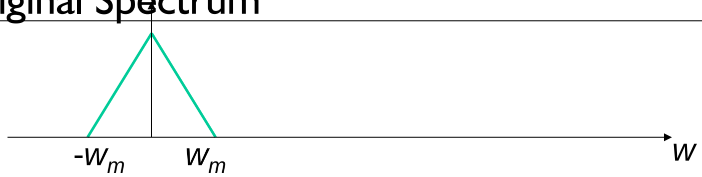


➔ if $f_s < 2B$

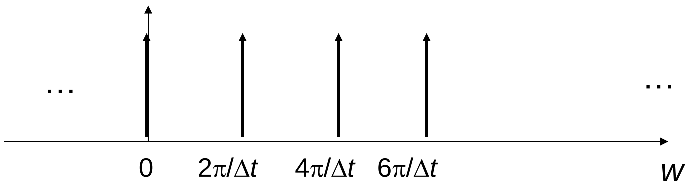
➔ *then* "**Folding**" or "**aliasing**":



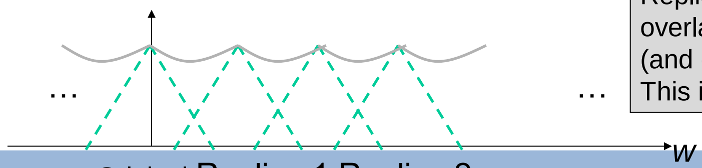
Original Spectrum



Fourier transform of impulse train (sampling signal)



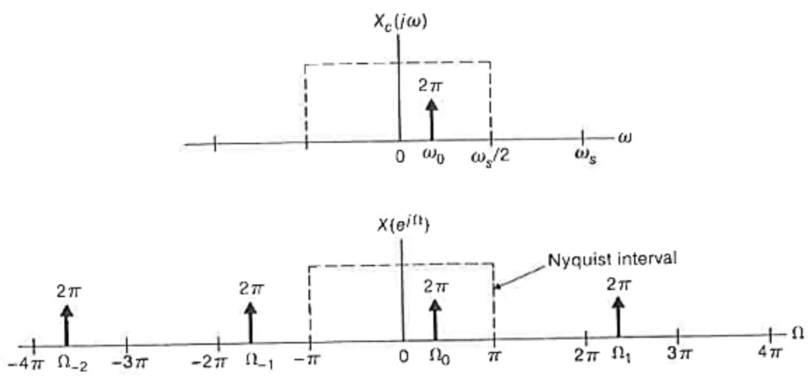
Amplitude spectrum of sampled signal



Replica spectrums overlap with original (and each other)
This is **Aliasing**

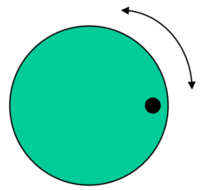
Original Replica 1 Replica 2 …

Original Spectrum



Another way to see Aliasing Too!

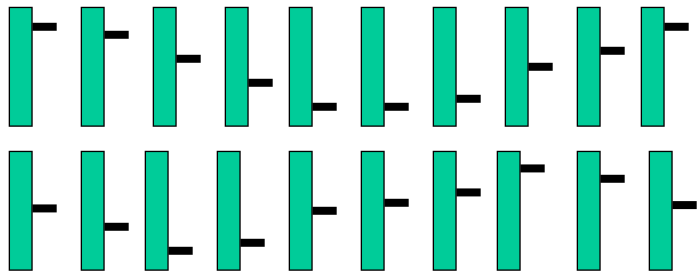
Rotating wheel and peg



Need both top and front view to determine rotation

Top View

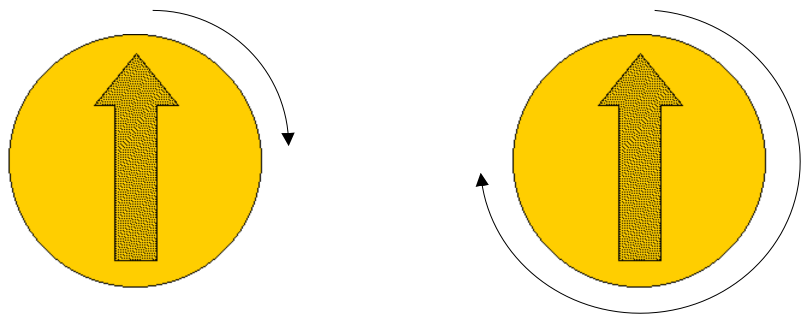
Front View



Temporal Aliasing

90^o clockwise rotation/frame
clockwise rotation perceived

270^o clockwise rotation/frame
(90^o) anticlockwise rotation perceived i.e., aliasing

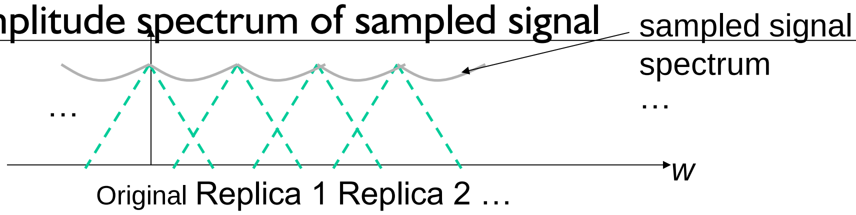


Require LPF to ‘blur’ motion

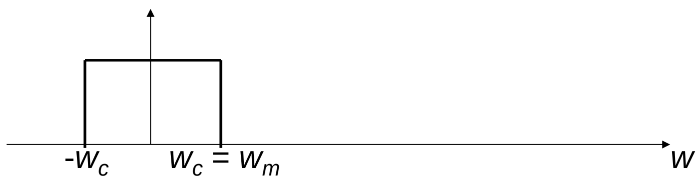
Practical Anti-aliasing Filter

- Non-ideal filter
  - $w_c = \frac{w_s}{2}$
- Filter usually 4th – 6th order (e.g., Butterworth)
  - so frequencies > w_c may still be present
  - not higher order as phase response gets worse
- Luckily, most real signals
  - are lowpass in nature
    - signal power reduces with increasing frequency
  - e.g., speech naturally bandlimited (say < 8KHz)
  - Natural signals have a $\sim\frac{1}{f}$ spectrum
  - so, in practice aliasing is not (usually) a problem

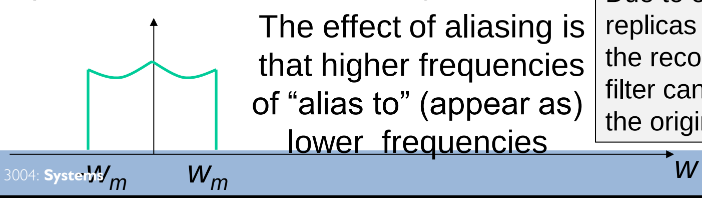
Amplitude spectrum of sampled signal



Reconstruction filter (ideal lowpass filter)



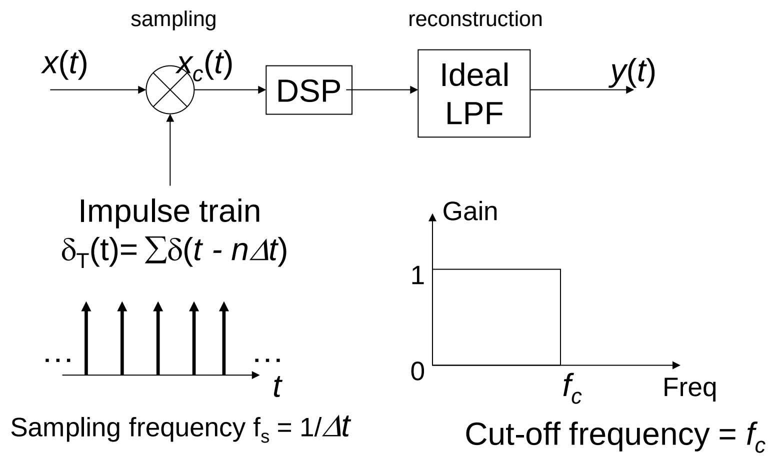
Spectrum of reconstructed signal



The effect of aliasing is that higher frequencies of "alias to" (appear as) lower frequencies

Due to overlapping replicas (aliasing) the reconstruction filter cannot recover the original spectrum

Mathematics of Sampling and Reconstruction



Frequency Domain Analysis of Sampling

- Consider the case where the DSP performs no filtering operations
  - i.e., only passes $x_c(t)$ to the reconstruction filter
- To understand we need to look at the frequency domain
- Sampling: we know
  - multiplication in time ≡ convolution in frequency
  - F{x(t)} = X(w)
  - F{δT(t)} = $\sum \boldsymbol{\delta}(w - 2\pi n/\Delta t)$,
  - i.e., an impulse train in the frequency domain

Frequency Space

Signal	Time domain	Transform
Impulse	$\delta[n]$	1
	$\delta[n-n_0]$	$e^{-j\Omega n_0}$
Unit step	$u[n]$	$\frac{1}{1-e^{-j\Omega}} + \pi\,\delta(\Omega), \quad \lvert\Omega\rvert \le \pi$
	$-u[-n-1]$	$\frac{1}{1-e^{-j\Omega}} - \pi\,\delta(\Omega), \quad \lvert\Omega\rvert \le \pi$
Exponential	$a^n u[n]$	$\frac{1}{1-ae^{-j\Omega}}, \quad \lvert a\rvert < 1$
	$-a^n u[-n-1]$	$\frac{1}{1-ae^{-j\Omega}}, \quad \lvert a\rvert > 1$
Weighted exponential	$(n+1)a^n u[n]$	$\frac{1}{(1-ae^{-j\Omega})^2}, \quad \lvert a\rvert < 1$
DC signal	1, for all n	$2\pi\,\delta(\Omega), \quad \lvert\Omega\rvert \le \pi$
Complex sinusoid	$e^{j\Omega_0 n}$	$2\pi\,\delta(\Omega-\Omega_0), \quad \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
Sine wave	$\sin \Omega_0 n$	$-j\pi[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)], \quad \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
Cosine wave	$\cos \Omega_0 n$	$\pi[\delta(\Omega-\Omega_0) + \delta(\Omega+\Omega_0)], \quad \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$

Frequency Domain Analysis of Sampling

- In the frequency domain we have

$$X_c(w) = \frac{1}{2\pi}\left(X(w) * \frac{2\pi}{\Delta t} \sum_n \delta\left(w - \frac{2\pi n}{\Delta t} \right) \right)$$

$$= \frac{1}{\Delta t} \sum_n X\left(w - \frac{2\pi n}{\Delta t} \right)$$

Remember convolution with an impulse? Same idea for an impulse train

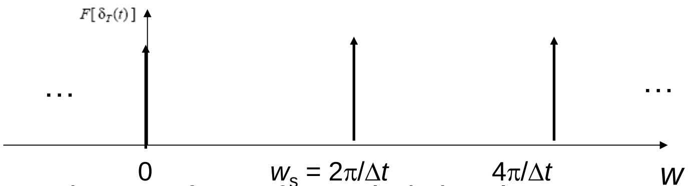
- Let's look at an example
  - where X(w) is triangular function
  - with maximum frequency w_m rad/s
  - being sampled by an impulse train, of frequency w_s rad/s

Sampling Frequency

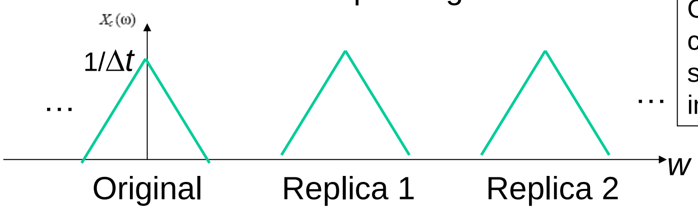
- In this example it was possible to recover the original signal from the discrete-time samples
- But is this always the case?
- Consider an example where the sampling frequency w_s is reduced
  - i.e., Δt is increased

Fourier transform of original signal X(ω) (signal spectrum)

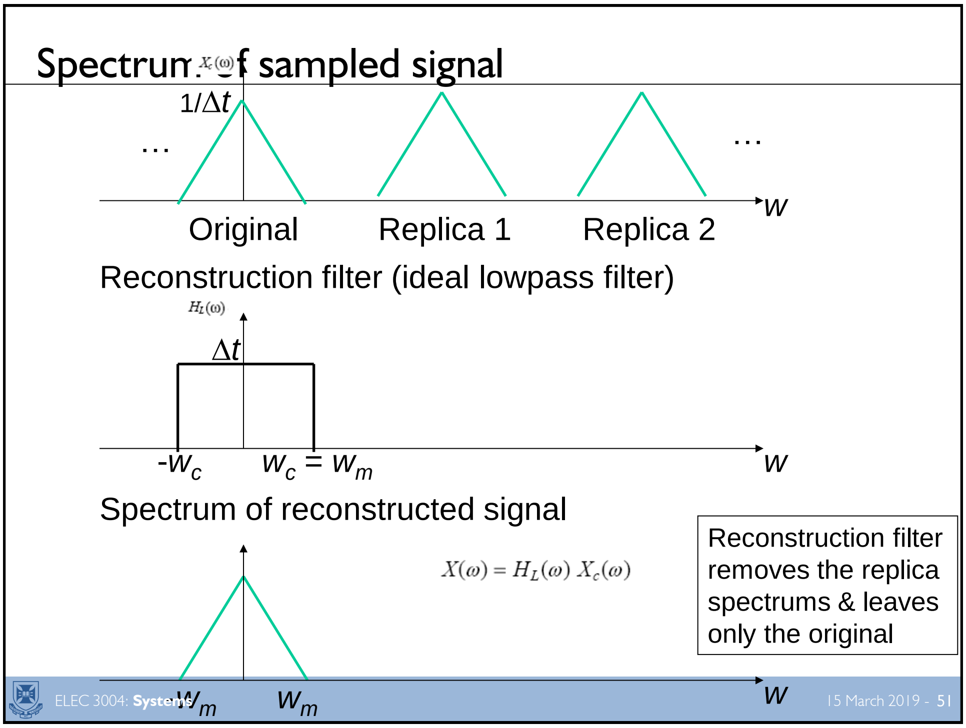
Fourier transform of impulse train $\delta_T(\omega/2\pi)$ (sampling signal)



Fourier transform of sampled signal



Original spectrum convolved with spectrum of impulse train


Spectrum of sampled signal
$X_c(\omega)$
$1/\Delta t$
…
…
w
Original
Replica 1
Replica 2
Reconstruction filter (ideal lowpass filter)
$H_L(\omega)$
Δt
$-w_c$
$w_c = w_m$
w
Spectrum of reconstructed signal
$X(\omega) = H_L(\omega)\, X_c(\omega)$
Reconstruction filter removes the replica spectrums & leaves only the original
$-w_m$
w_m
w
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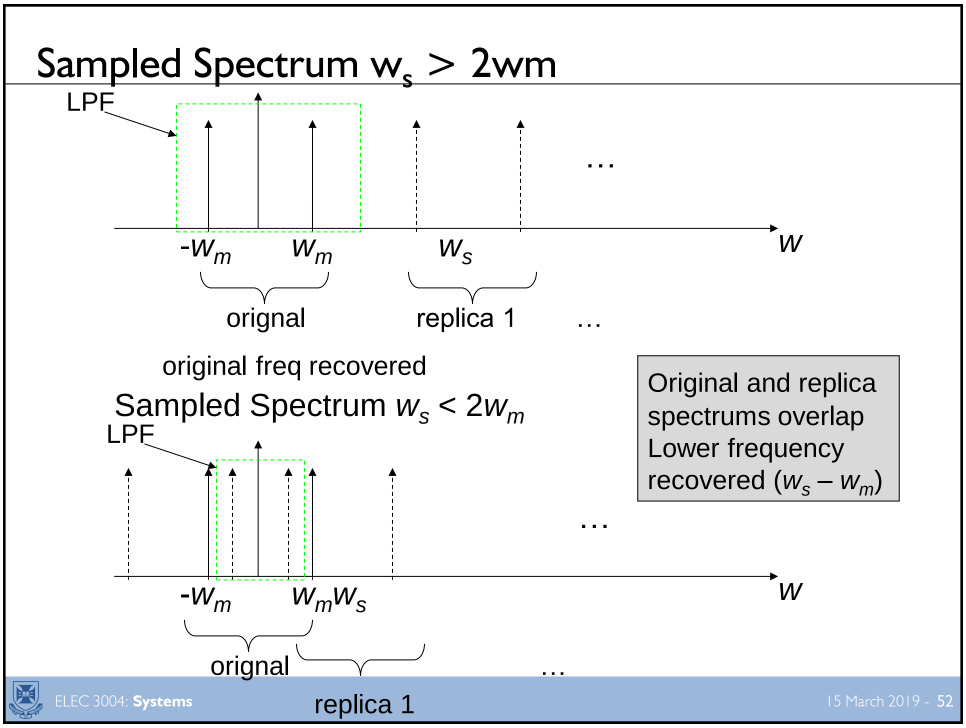
Sampled Spectrum w_s > 2wm
LPF
…
w
$-w_m$
w_m
w_s
orignal
replica 1
…
original freq recovered
Sampled Spectrum $w_s < 2w_m$
LPF
Original and replica spectrums overlap
Lower frequency recovered ($w_s - w_m$)
…
w
$-w_m$
$w_m w_s$
orignal
replica 1
…
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Taking Advantage of the Folding

5.1 The Sampling Theorem

We now show that a real signal whose spectrum is bandlimited to B Hz $[F(\omega) = 0 \text{ for } |\omega| > 2\pi B]$ can be reconstructed exactly (without any error) from its samples taken uniformly at a rate $\mathcal{F}_s > 2B$ samples per second. In other words, the minimum sampling frequency is $\mathcal{F}_s = 2B$ Hz.†

To prove the sampling theorem, consider a signal $f(t)$ (Fig. 5.1a) whose spectrum is bandlimited to B Hz (Fig. 5.1b).‡ For convenience, spectra are shown as functions of ω as well as of $\mathcal{F}$ (Hz). Sampling $f(t)$ at a rate of $\mathcal{F}_s$ Hz ($\mathcal{F}_s$ samples per second) can be accomplished by multiplying $f(t)$ by an impulse train $\delta_T(t)$(Fig. 5.1c), consisting of unit impulses repeating periodically every T seconds, where $T = 1/\mathcal{F}_s$. The result is the sampled signal $\overline{f}(t)$ resented in Fig. 5.1d. The sampled signal consists of impulses spaced every T seconds (the sampling interval). The nth impulse, located at $t = nT$, has a strength $f(nT)$, the value of $f(t)$ at $t = nT$.

$$\overline{f}(t) = f(t)\delta_T(t) = \sum_n f(nT)\delta(t - nT) \qquad (5.1)$$

†The theorem stated here (and proved subsequently) applies to lowpass signals. A bandpass signal whose spectrum exists over a frequency band $\mathcal{F}_c - \frac{B}{2} < |\mathcal{F}| < \mathcal{F}_c + \frac{B}{2}$ has a bandwidth of B Hz. Such a signal is uniquely determined by $2B$ samples per second. In general, the sampling scheme is a bit more complex in this case. It uses two interlaced sampling trains, each at a rate of B samples per second (known as second-order sampling). See, for example, the references.[1,2]

‡The spectrum $F(\omega)$ in Fig. 5.1b is shown as real, for convenience. However, our arguments are valid for complex $F(\omega)$ as well.

Next Time…

- **Digital Systems**

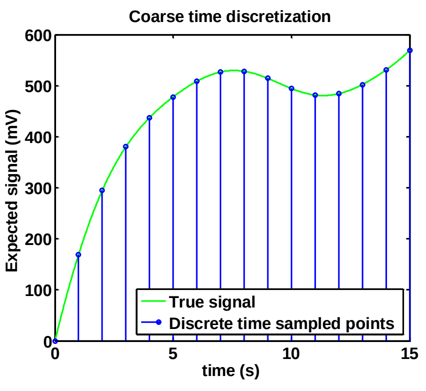
- Review:
  - Chapter 8 of Lathi

- A signal has many signals ☺
  [Unless it's bandlimited. Then there is the one ω]

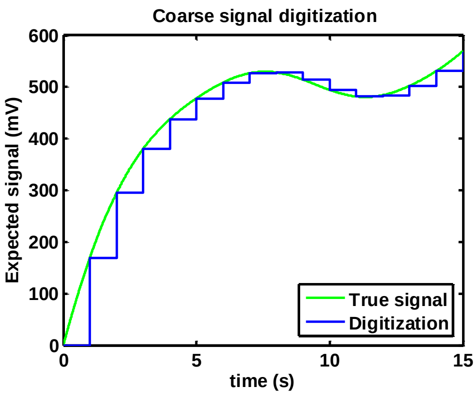
Data Acquisition (A/D Conversion)

Representation of Signal

- Time Discretization



- Digitization



Quantisation

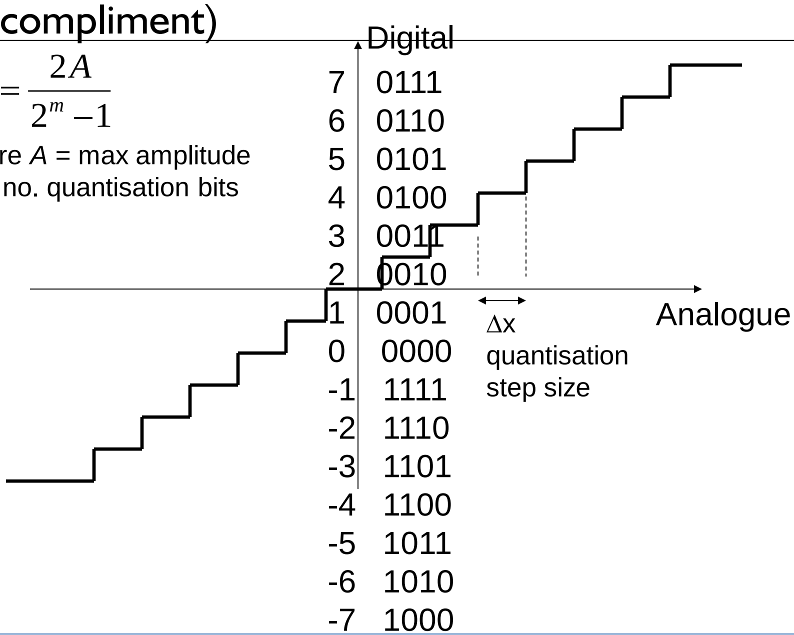
- Analogue to digital converter (A/D)
  - Calculates nearest binary number to $x(n\Delta t)$
    - $x_q[n] = q(x(n\Delta t))$, where q() is non-linear rounding fctn
  - output modeled as $x_q[n] = x(n\Delta t) + e[n]$
- Approximation process
  - therefore, loss of information (unrecoverable)
  - known as 'quantisation noise' ($e[n]$)
  - error reduced as number of bits in A/D increased
    - i.e., Δx, quantisation step size reduces

$$|e[n]| \leq \frac{\Delta x}{2}$$

Input-output for 4-bit quantiser (two's compliment)

$$\Delta x = \frac{2A}{2^m - 1}$$

where A = max amplitude
m = no. quantisation bits



Signal to Quantisation Noise

- To estimate SQNR we assume
  - $e[n]$ is uncorrelated to signal and is a
  - uniform random process
- assumptions not always correct!
  - not the only assumptions we could make…
- Also known a 'Dynamic range' (R_D)
  - expressed in decibels (dB)
  - ratio of power of largest signal to smallest (noise)

$$R_D = 10\log_{10}\left(\frac{P_{signal}}{P_{noise}}\right)$$

Dynamic Range

Need to estimate:

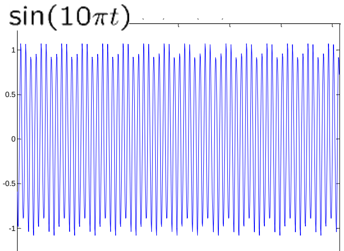
1. Noise power
   - uniform random process: $P_{noise} = \Delta x^2/12$
2. Signal power
   - (at least) two possible assumptions
   1. sinusoidal: $P_{signal} = A^2/2$
   2. zero mean Gaussian process: $P_{signal} = \sigma^2$
      - Note: as $\sigma \approx A/3$: $P_{signal} \approx A^2/9$
      - where σ^2 = variance, A = signal amplitude

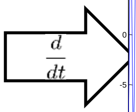
1 extra bit halves Δx
i.e., 20log10(1/2) = 6dB

Regardless of assumptions: R_D increases by 6dB for every bit that is added to the quantiser

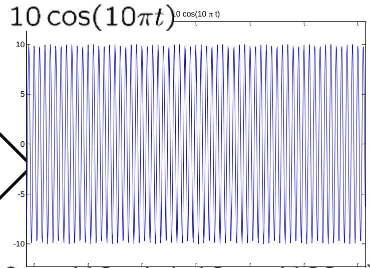
Derivatives magnify noise!

- $\sin(10\pi t)$

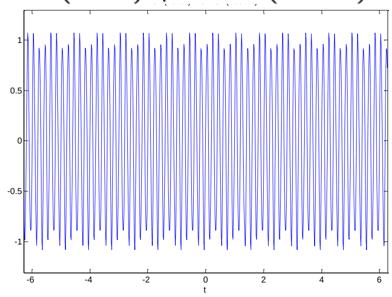




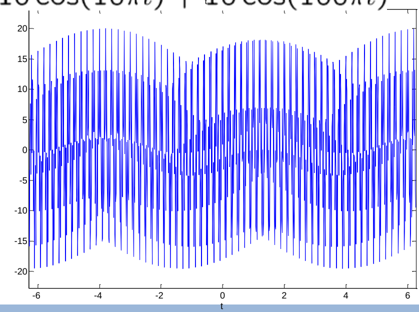
- $10\cos(10\pi t)$



- $\sin(10\pi t) + 0.1\sin(100\pi t)$



- $10\cos(10\pi t) + 10\cos(100\pi t)$



D/A Converter

- Analogue output y(t) is
  - convolution of output samples $y(n\Delta t)$ with $h_{ZOH}(t)$

$$y(t) = \sum_n y(n\Delta t) h_{ZOH}(t - n\Delta t)$$

$$h_{ZOH}(t) = \begin{cases} 1, & 0 \le t < \Delta t \\ 0, & \text{otherwise} \end{cases}$$

$$H_{ZOH}(w) = \Delta t \exp\left(\frac{-jw\Delta t}{2}\right) \frac{\sin(w\Delta t/2)}{w\Delta t/2}$$

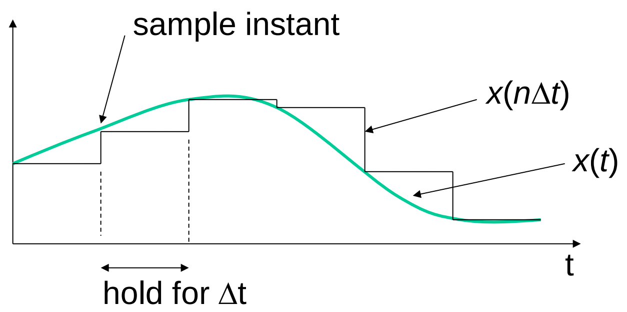
D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional reconstruction filter required

Finite Width Sampling

- Impulse train sampling not realisable
  - sample pulses have finite width (say nanosecs)
- This produces two effects,
- Impulse train has sinc envelope in frequency domain
  - impulse train is square wave with small duty cycle
  - Reduces amplitude of replica spectrums
    - smaller replicas to remove with reconstruction filter ☺
- Averaging of signal during sample time
  - effective low pass filter of original signal
    - can reduce aliasing, but can reduce fidelity ☹
    - negligible with most S/H ☺

Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every Δt seconds
  2. holds that value constant until next sample
- Produces 'staircase' waveform, $x(n\Delta t)$



Practical Reconstruction

Two stage process:

- Digital to analogue converter (D/A)
  - zero order hold filter
  - produces ‘staircase’ analogue output

- Reconstruction filter
  - non-ideal filter: $\omega_c = \frac{\omega_s}{2}$
  - further reduces replica spectrums
  - usually 4th – 6th order e.g., Butterworth
    - for acceptable phase response

Summary

- Theoretical model of Sampling
  - bandlimited signal (wB)
  - multiplication by ideal impulse train (ws > 2wB)
    - convolution of frequency spectrums (creates replicas)
  - Ideal lowpass filter to remove replica spectrums
    - wc = ws /2
    - Sinc interpolation
- Practical systems
  - Anti-aliasing filter (wc < ws /2)
  - A/D (S/H and quantisation)
  - D/A (ZOH)
  - Reconstruction filter (wc = ws /2)

Don’t confuse theory and practice!

Next Time…

- **Z-Transform**

- Review:
  - Chapter 5 of Lathi

- A signal has many signals ☺
  [Even if it bandlimited]