Digital Signals &
Sampling Theory

ELEC 3004: **Systems**: Signals & Controls
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Lecture 5

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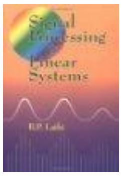
Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	Systems: Linear Differential Systems
3	**13-Mar**	**Sampling Theory & Data Acquisition**
	15-Mar	Aliasing & Antialiasing
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	Digital Filter (FIR)
7	10-Apr	Digital Windows
	12-Apr	FFT
8	17-Apr	Active Filters & Estimation & Holiday
	19-Apr	Holiday
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation/PID
10	8-May	PID & State-Space
	10-May	State-Space Control
11	15-May	Digital Control Design
	17-May	Stability
12	22-May	State Space Control System Design
	24-May	Shaping the Dynamic Response
13	29-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 5:
  Sampling
  - § 5.1 **The Sampling Theorem**
  - § 5.2 Numerical Computation of Fourier Transform: The Discrete Fourier Transform (DFT)

Also:
- § 4.6 Signal Energy

Digital SIGNALS!

(& Systems)

Signal: A carrier of (desired) information [1]

- Need **NOT** be electrical:
  - Thermometer
  - Clock hands
  - Automobile speedometer

- Need **NOT** always being given
  - – "Abnormal" sounds/operations
  - – Ex: "pitch" or "engine hum" during machining as an indicator for feeds and speeds

Signal: A carrier of (desired) information [2]

- Electrical signals
  - – Voltage
  - – Current

- **Digital signals**
  - – **Convert analog electrical signals to an appropriate digital electrical message**
  - – **Processing by a microcontroller or microprocessor**

Transduction (sensor to an electrical signal)

- Sensor reacts to environment (physics)
- Turn this into an electrical signal:
  - V: voltage source
  - I: current source
- **Measure this signal**
  - Resistance
  - Capacitance
  - Inductance

Digital Signal

- Representation of a signal against a discrete set
- The set is fixed in by computing hardware

$$s \in \mathbb{Z}$$

- Can be scaled or normalized … but is limited

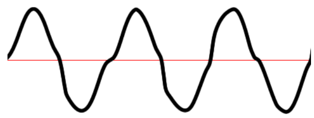
$$s \in \mathbb{Z}(0, \ldots, 2^{16})$$

- Time is also discretized

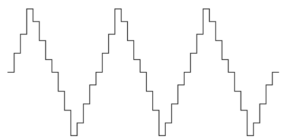
$$s' \in \frac{\mathbb{Z}(0,\ldots,2^{16})}{2^{16}}$$

Analog vs Digital

- *Analog Signal*: An analog or analogue signal is any variable signal **continuous** in both time and amplitude



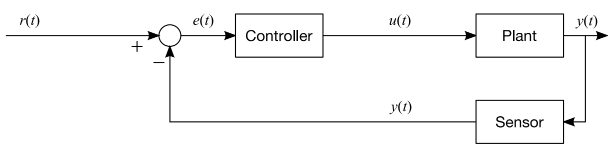
- *Digital Signal*: A digital signal is a signal that is both **discrete** and quantized



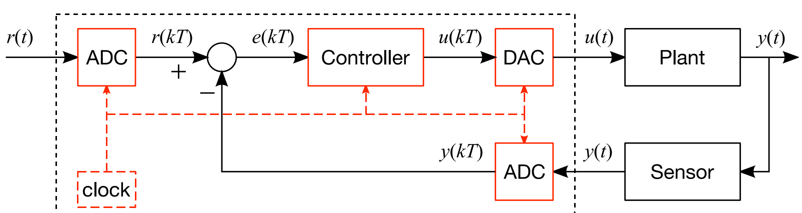
E.g. Music stored in a CD: 44,100 Samples per second and 16 bits to represent amplitude

Digital Systems

- Continuous:



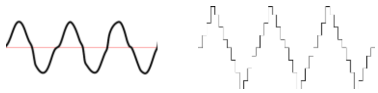
- Digital:



→ Digital Systems

Better SNR

- We trade-off "**certainty in time**" for "**signal noise/uncertainty**"
- Analog: ∞ time resolution
  - *Digital* has fixed time steps



- This avoids the noise and uncertainty in component values that affect analogue signal processing.

Better Processing

- **Digital microprocessors** are in a range of objects, from obvious (e.g. phone) to disposable (e.g. Go cards). (what doesn't have one?)

Compared to antilog computing (op-amp):

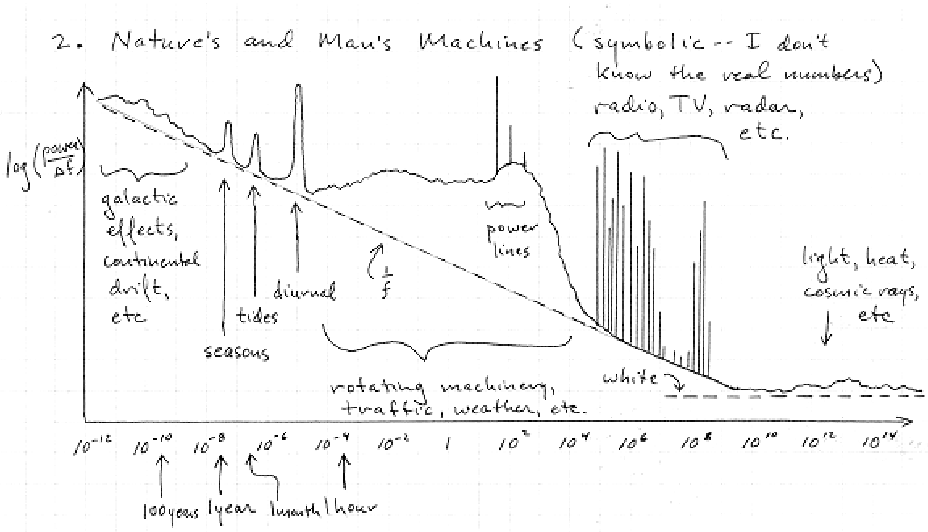
- **Accuracy**: digital signals are usually represented using 12 bits or more.
- **Reliability**: The ALU is stable over time.
- **Flexibility**: limited only programming ability!
- **Cost**: advances in technology make microcontrollers economical even for small, low cost applications. (Raspberry Pi 3: US$35)

Digital Signals & Systems

WHY?

Noise!

There is so much Noise …



Note: this picture illustrates the concepts but it is not quantitatively precise

Source: Prof. M. Siegel, CMU

Noise: "Unwanted" Signals: Carrying Errant **Information**

- Cross-coupled measurements
- Cross-talk (at a restaurant or even a lecture)
- A bright sunny day obstructing picture subject
- Strong radio station near weak one
- observation-to-observation variation
  - Measurement fluctuates (ex: student)
  - Instrument fluctuates (ex: quiz !)
- Unanticipated effects / variation (Temperature)
- **One man's noise might be another man's signal**

Noise: Fundamental Natural Sources

- Voltage (EMF) – Capacitive & Inductive Pickup
- Johnson Noise – Thermal / Brownian
- $\frac{1}{f}$ (Hooge Noise)

  $$V_j = \sqrt{4k_bTR}$$

- Shot noise (interval-to-interval statistical count)

  $$V_f = \sqrt{\frac{\alpha V_R^2}{Nf}}$$

SNR : Signal to Noise Ratio

$$V = V_s + V_n$$

Magnitude: $\bar{V}^2 = \bar{V_s}^2 + \bar{V_n}^2 + \bar{V_s}\bar{V_n}$

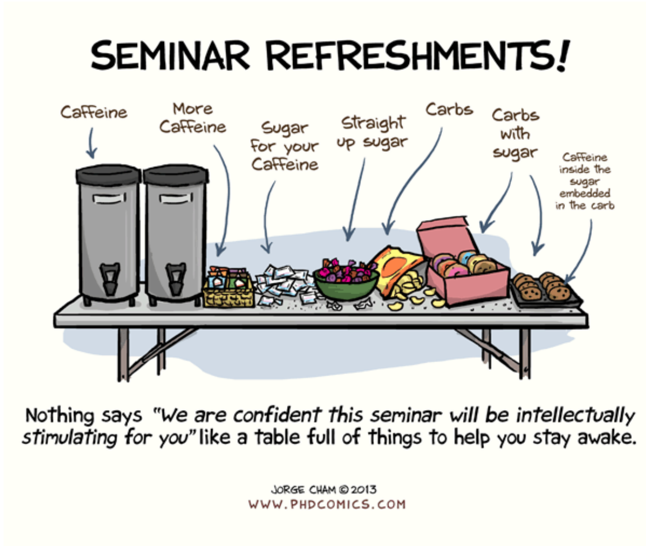
$$\frac{S}{N} = \frac{V_s^2}{V_n^2}$$

in dB: $10 \log \left(\frac{\bar{V_s}^2}{\bar{V_n}^2} \right) = 20 \log \left(\frac{V_s^{rms}}{V_n^{rms}} \right)$

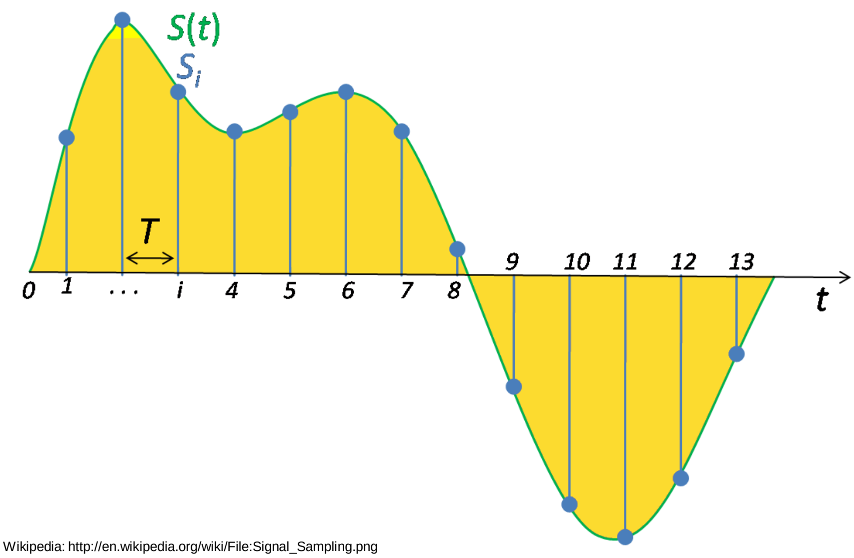
A theory for all this …

Sampling!

Not this type of sampling … ☺

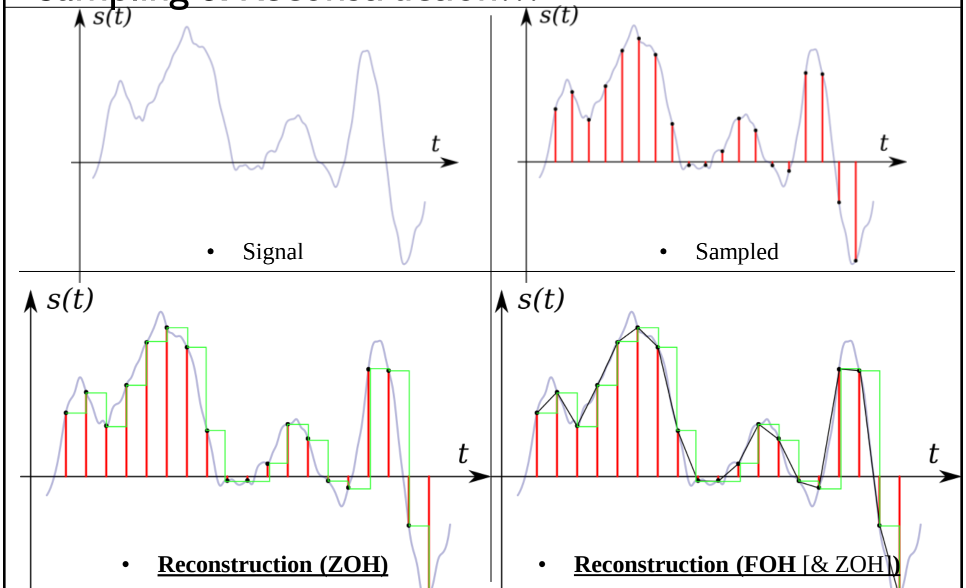


This type of sampling…



Source: Wikipedia: http://en.wikipedia.org/wiki/File:Signal_Sampling.png

Sampling & Reconstruction…



Based of: https://github.com/mammenx/synesthesia_zen/wiki/Audio-Technica

Sampling Theorem

- The Nyquist criterion states:

To prevent aliasing, a ***bandlimited*** *signal of bandwidth* w_B *rad/s must be sampled at a rate greater than* $2w_B$ *rad/s*

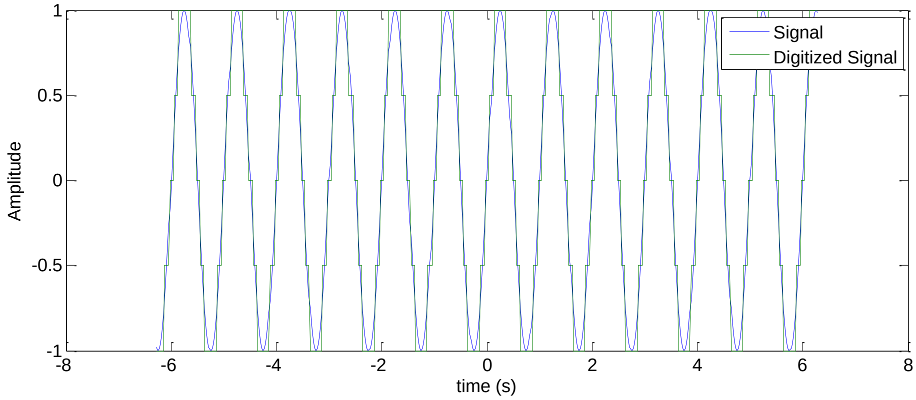
$$w_s > 2w_B$$

Note: this is a > sign not a ≥

Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

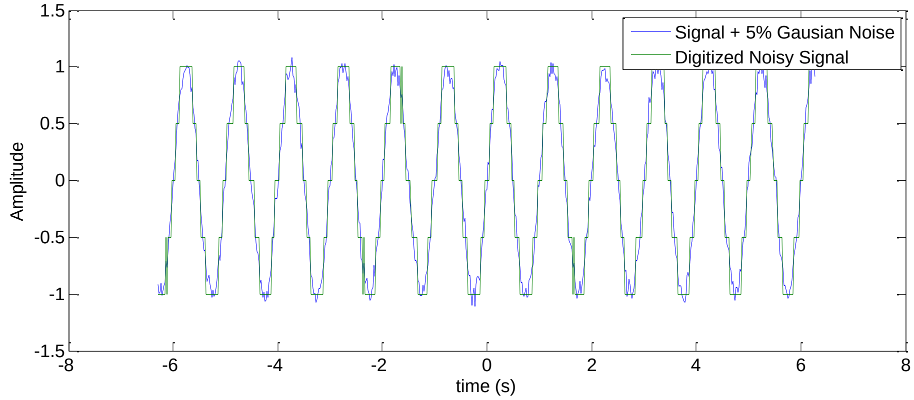
Discrete Time Signal

- Image a signal…



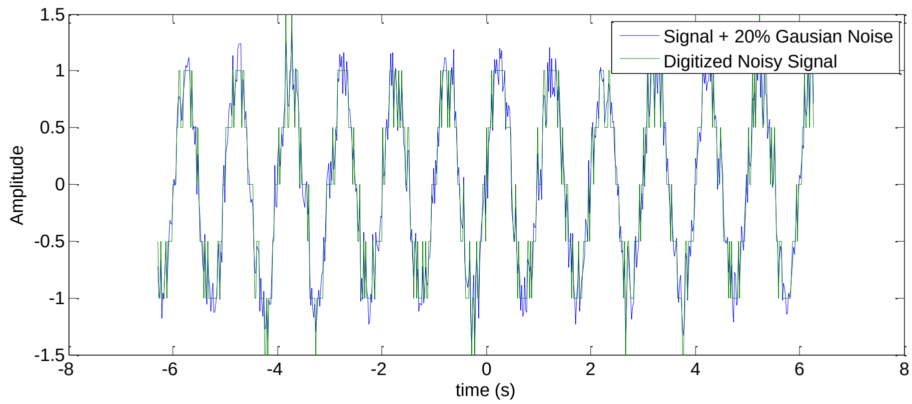
Discrete Time Signals

- Digitization helps beat the Noise!

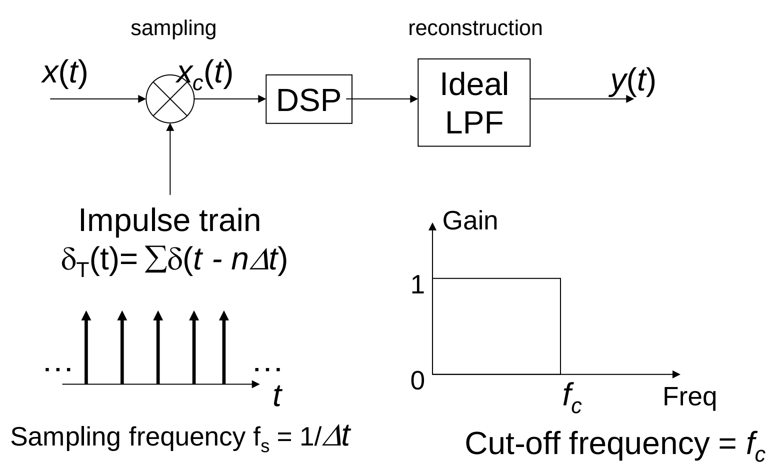


Discrete Time Signals

- But only so much…



Mathematics of Sampling and Reconstruction

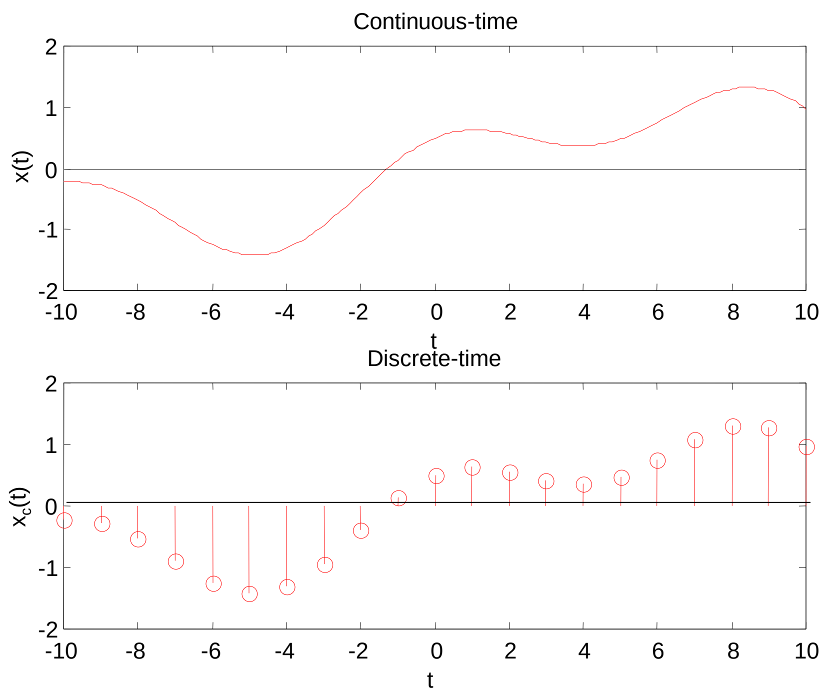


Mathematical Model of Sampling

- x(t) multiplied by impulse train δT(t)

$$\begin{aligned} x_c(t) &= x(t)\delta_T(t) \\ &= x(t)\left[\delta(t)+\delta(t-\Delta t)+\delta(t-2\Delta t)+\cdots\right] \\ &= \sum_n x(n\Delta t)\delta(t-n\Delta t) \end{aligned}$$

- $x_c(t)$ is a train of impulses of height $x(t)|_{t=n\Delta t}$



Signal Manipulations

- Shifting

$$y(n) = x(n - n_0)$$

- Reversal

$$y(n) = x(-n)$$

- Time Scaling

(Down Sampling)

$$y(M) = x(Mn)$$

(Up Sampling)

$$y(n) = x\left(\frac{n}{N}\right)$$

Back to the Sampling Theorem

- The Nyquist criterion states:

To prevent aliasing, a ***bandlimited*** *signal of bandwidth w_B rad/s must be sampled at a rate greater than $2w_B$ rad/s*

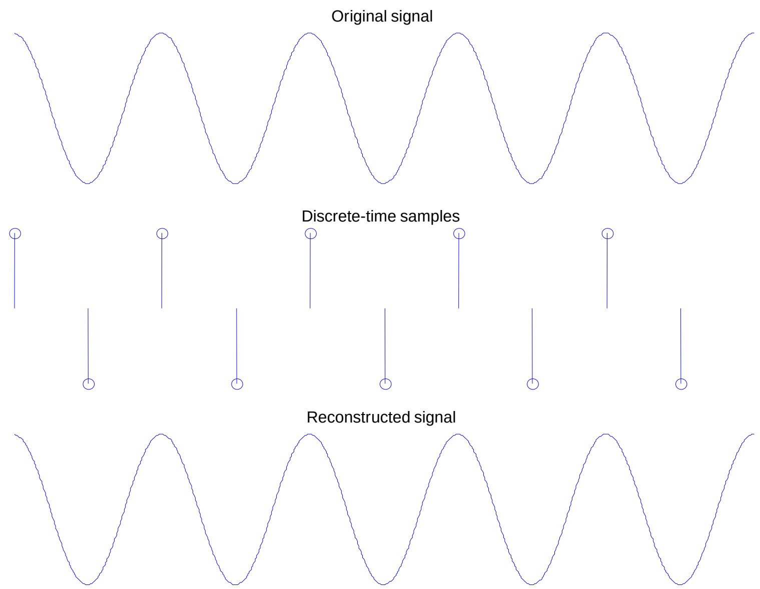
$$-w_s > 2w_B$$

Note: this is a > sign not a ≥

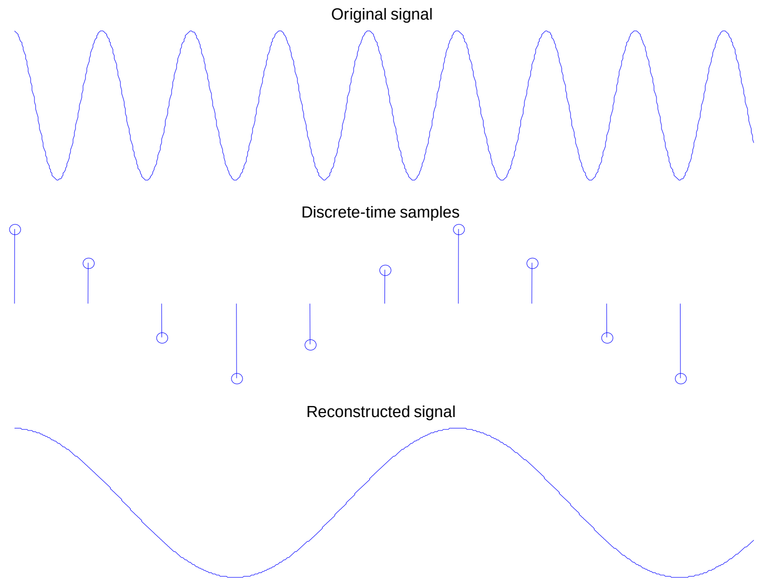
Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

Time Domain Analysis of Sampling

- Frequency domain analysis of sampling is very useful to understand
  - sampling ($X(w) * \sum \delta(w - 2\pi n/\Delta t)$)
  - reconstruction (lowpass filter removes replicas)
  - aliasing (if $w_s \leq 2w_B$)
- Time domain analysis can also illustrate the concepts
  - sampling a sinewave of increasing frequency
  - sampling images of a rotating wheel

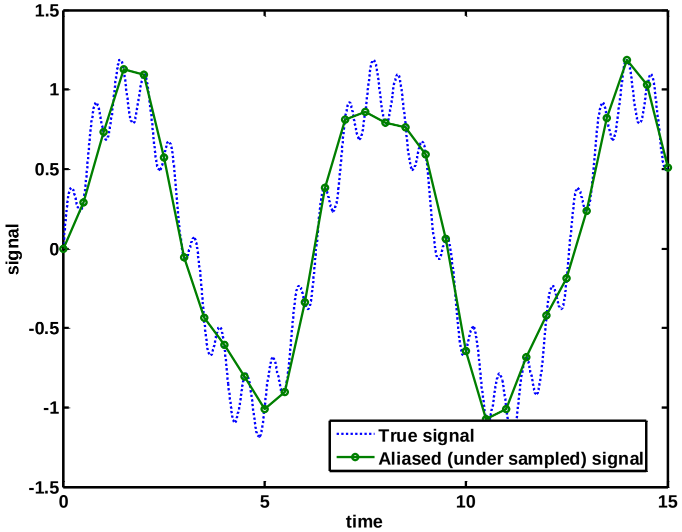


A signal of the original frequency is reconstructed

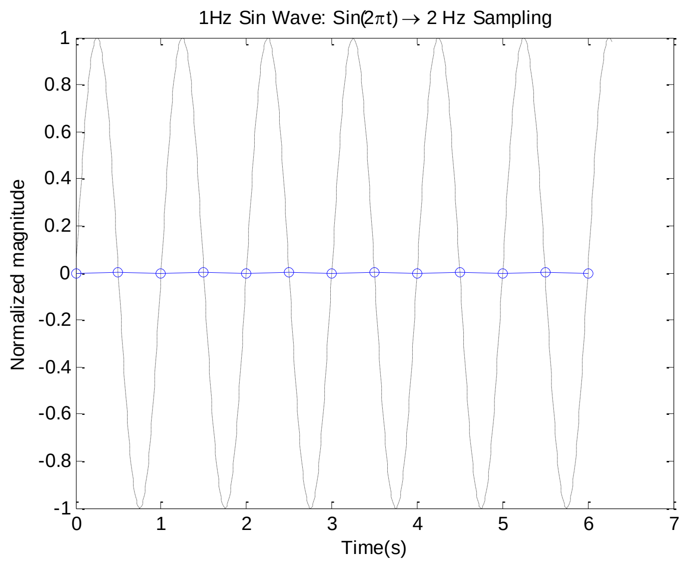


A signal with a reduced frequency is recovered, i.e., the signal is aliased to a lower frequency (we recover a replica)

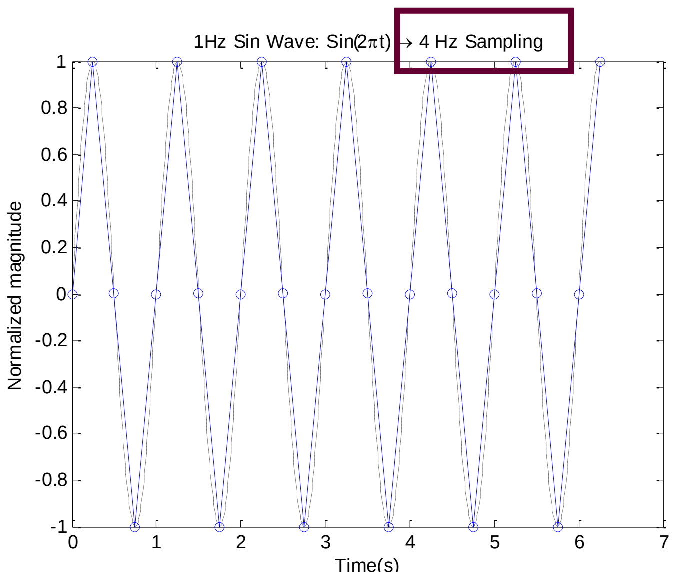
Sampling < Nyquist → Aliasing



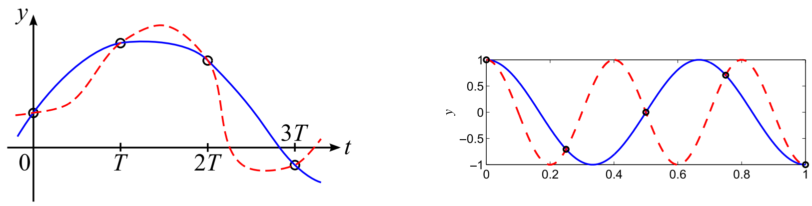
Nyquist is not enough …



A little more than Nyquist is not enough …



Nyquist Sampling Theorem and Aliasing



- A signal $y(t)$ is uniquely defined by its samples $y(kT)$ if the sampling frequency is more than twice the bandwidth of $y(t)$.

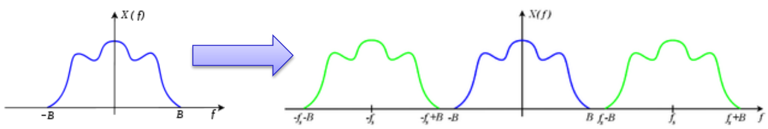
Spectrum Replication

- Sampling with a pulse train ($\delta(t)$)…

$$x(t) = x(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$$

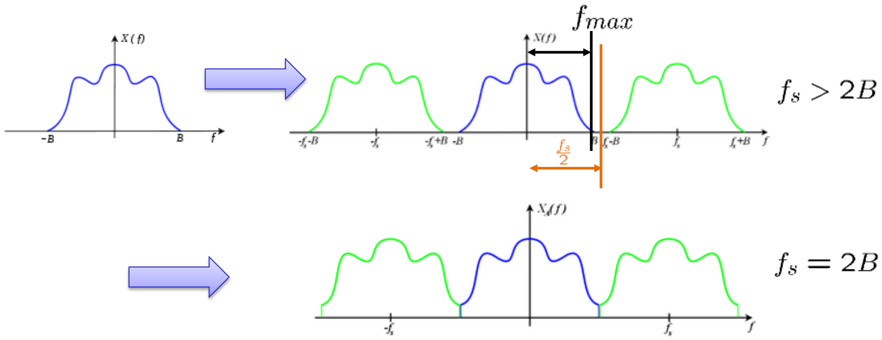
- Gives replication in $X(f)$

$$X\left(f\right) = \frac{1}{T_s} \cdot \sum_{k=-\infty}^{+\infty} X\left(f - \frac{k}{T_s}\right)$$



$f_s > 2B$

Spectrum Replication & Nyquist



- This suggests a limit:
  - Analog signal spectrum $X(f)$ runs up to f_{max}Hz
  - Spectrum replicas are separated by $f_s = \frac{1}{T_s}$ Hz

Sampling Frequency

- In this example it was possible to recover the original signal from the discrete-time samples
- But is this always the case?
- Consider an example where the sampling frequency w_s is reduced
  - i.e., Δt is increased

Frequency Domain Analysis of Sampling

- Consider the case where the DSP performs no filtering operations
  - i.e., only passes xc(t) to the reconstruction filter
- To understand we need to look at the frequency domain
- Sampling: we know
  - multiplication in time ≡ convolution in frequency
  - F{x(t)} = X(w)
  - F{δT(t)} = ∑δ(w - 2πn/Δt),
  - i.e., an impulse train in the frequency domain

Frequency Domain Analysis of Sampling

- In the frequency domain we have

$$X_c(w) = \frac{1}{2\pi}\left(X(w) * \frac{2\pi}{\Delta t}\sum_n \delta\left(w - \frac{2\pi n}{\Delta t}\right)\right)$$

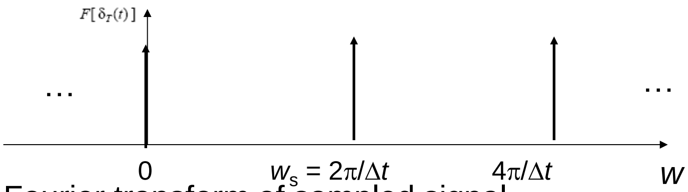
$$= \frac{1}{\Delta t}\sum_n X\left(w - \frac{2\pi n}{\Delta t}\right)$$

Remember convolution with an impulse? Same idea for an impulse train

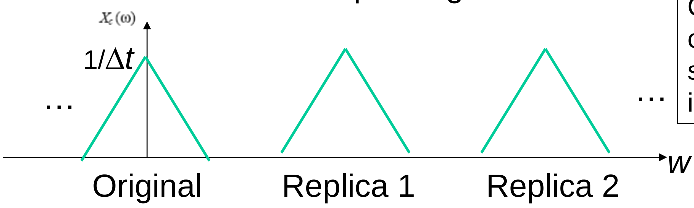
- Let's look at an example
  - where X(w) is triangular function
  - with maximum frequency w_m rad/s
  - being sampled by an impulse train, of frequency w_s rad/s

Fourier transform of original signal X(ω) (signal spectrum)

Fourier transform of impulse train $\delta_T(\omega/2\pi)$ (sampling signal)

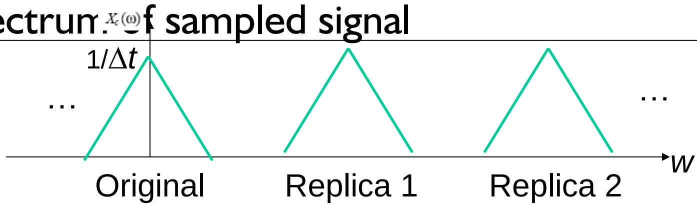


Fourier transform of sampled signal

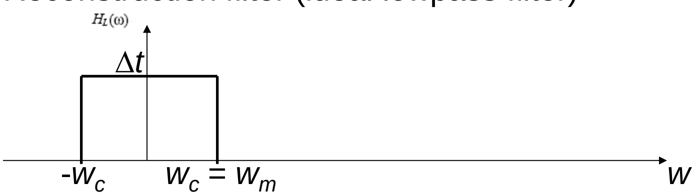


Original spectrum convolved with spectrum of impulse train

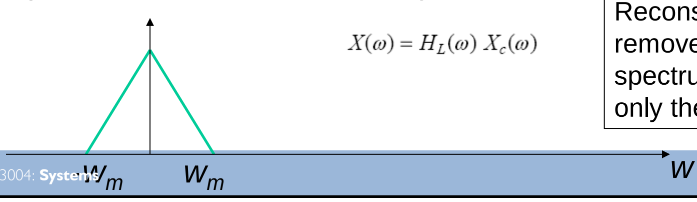
Spectrum of sampled signal



Reconstruction filter (ideal lowpass filter)

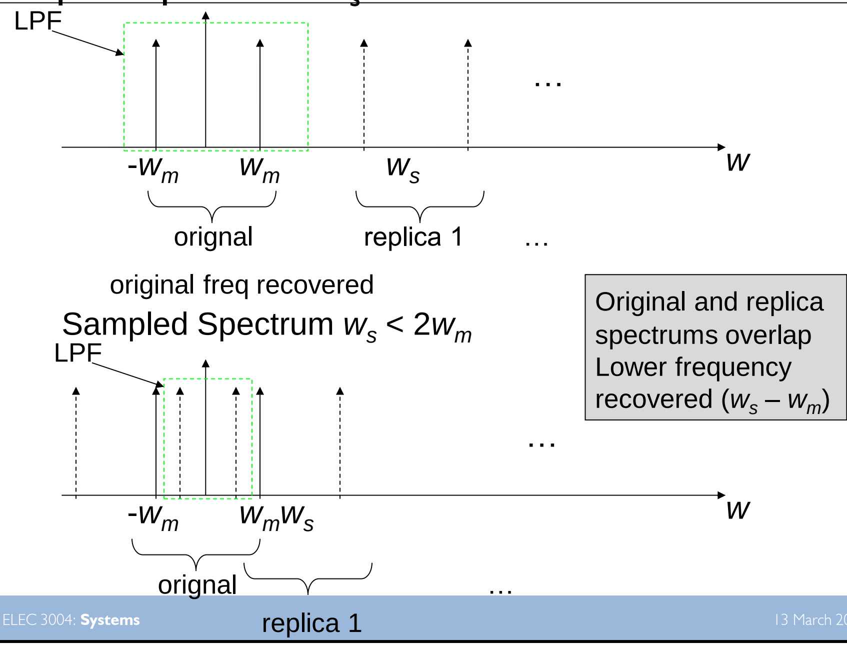


Spectrum of reconstructed signal



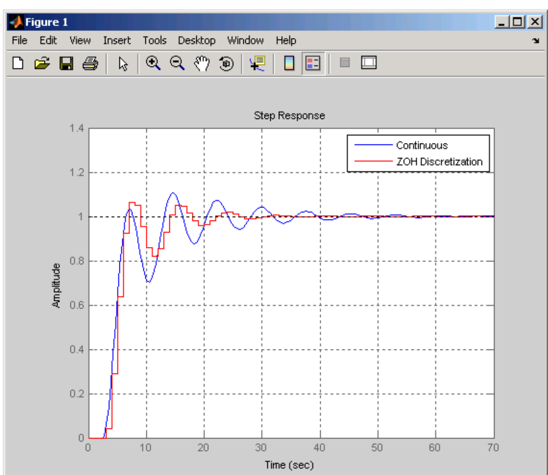
Reconstruction filter removes the replica spectrums & leaves only the original

Sampled Spectrum w_s > 2wm



Discrete Time Signals

- Can make control tricky!



Discrete Time Signals

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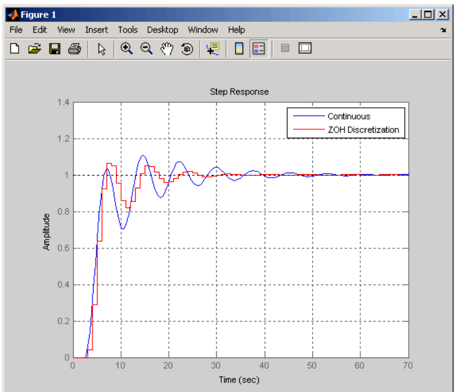
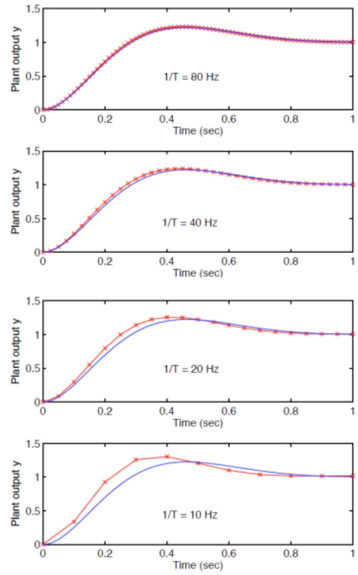




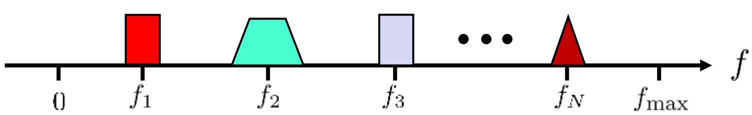
Violating Nyquist? Compressed Sensing

- Not so fast…
  "Exploits" the observation that most signals are sparse

- Why?
  - Note that the Maximum Achievable Rate comes from the Karhunen-Loeve Decomposition or DFT Decomposition
  - ➔ This assumes a "dense" signal…

$$C = \frac{1}{2}\int \log\left(\nu \frac{|H(f)|^2}{S_\eta(f)}\right) df$$

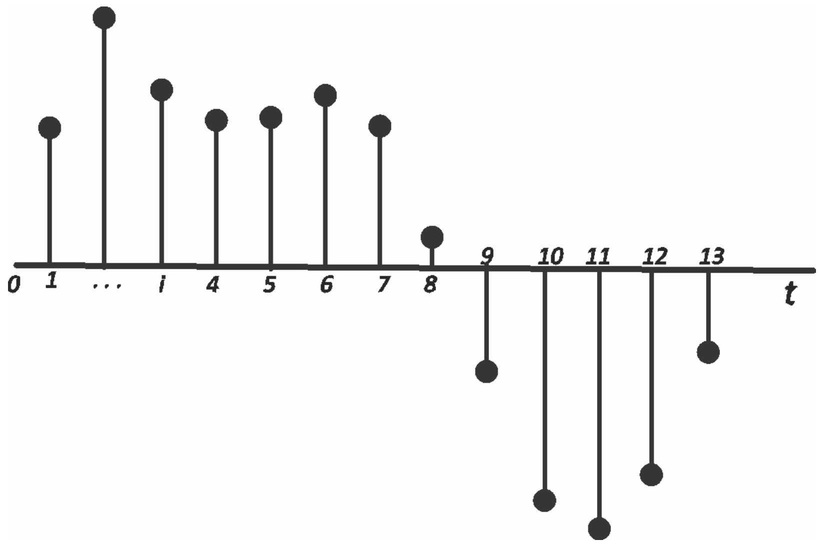
- Note:
  - Analog Compressed Sensing – Xampling [MishaliEldar'10]
  - Multi-band receivers at sub-Nyquist sampling rates
  - Can be used in low-complexity cognitive radios



BREAK

RECONSTRUCTION

Reconstruction



Sampling and Reconstruction Theory and Practice

- Signal is bandlimited to bandwidth WB
  - Problem: real signals are not bandlimited
    - Therefore, require (non-ideal) anti-aliasing filter
- Signal multiplied by ideal impulse train
  - problems: sample pulses have finite width
  - and not ⊗ in practice, but sample & hold circuit
- Samples discrete-time, continuous valued
  - Problem: require discrete values for DSP
    - Therefore, require A/D converter (quantisation)
- Ideal lowpass reconstruction (‘sinc’ interpolation)
  - problems: ideal lowpass filter not available
    - Therefore, use D/A converter and practical lowpass filter

Time Domain Analysis of Reconstruction

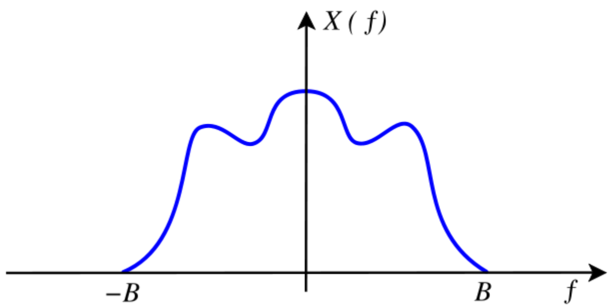
- Frequency domain: multiply by ideal LPF
  - ideal LPF: 'rect' function (gain Δt, cut off w_c)
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with 'sinc' function
  - as $F^{-1}\{\Delta t \ \text{rect}(w/w_c)\} = \Delta t \, w_c \, \text{sinc}(w_c t/\pi)$
  - i.e., weighted sinc on every sample
- Normally, $w_c = w_s/2$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \, \text{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$

Reconstruction

- Whittaker–Shannon interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \text{sinc}\left(\frac{t-nT}{T}\right)$$



Why *sinc*?

Time Domain Analysis of Reconstruction

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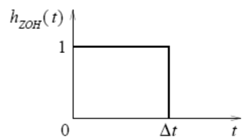
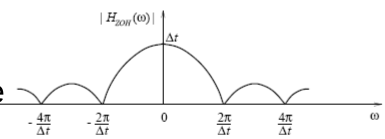
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c\, \mathrm{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$

Zero Order Hold (ZOH)

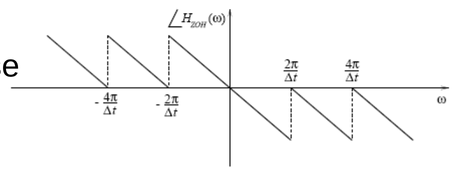
ZOH impulse response



ZOH amplitude response

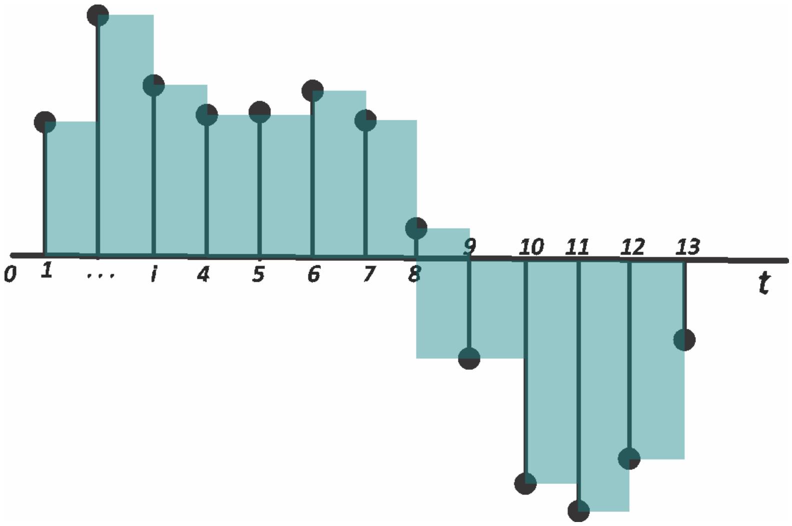


ZOH phase response



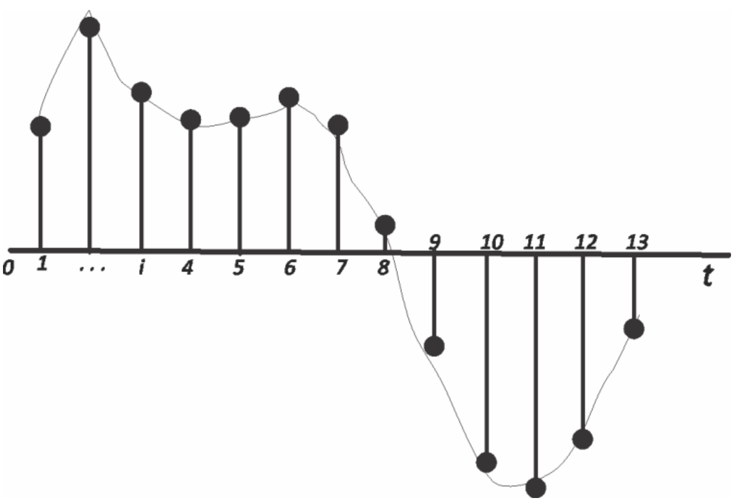
Reconstruction

- Zero-Order Hold [ZOH]

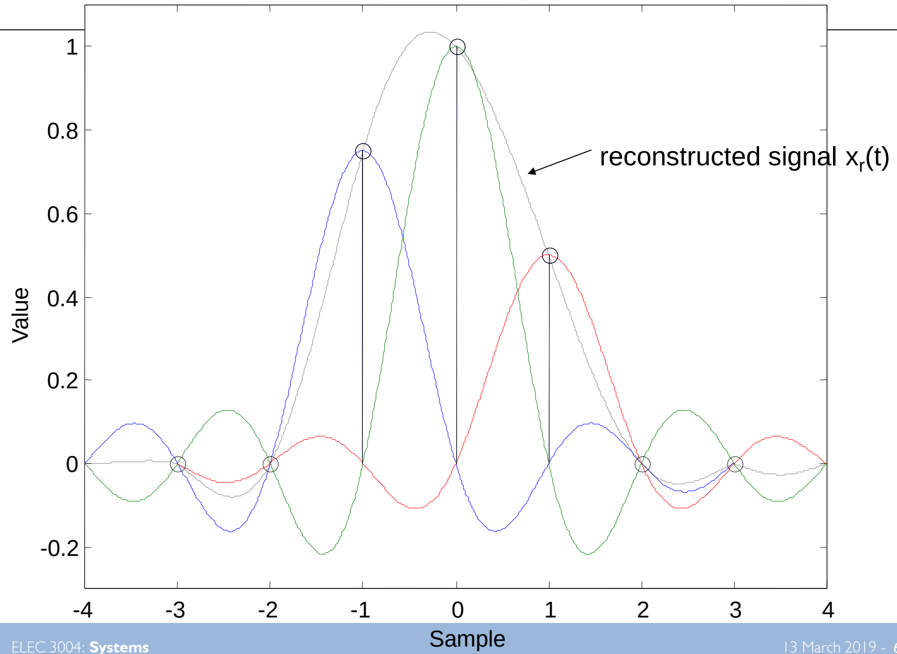


Reconstruction

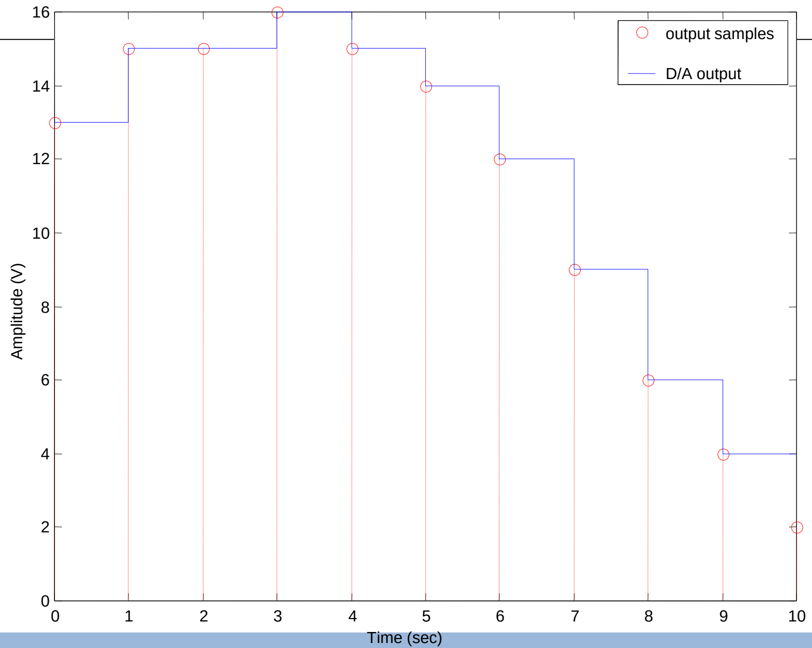
- Whittaker–Shannon interpolation formula



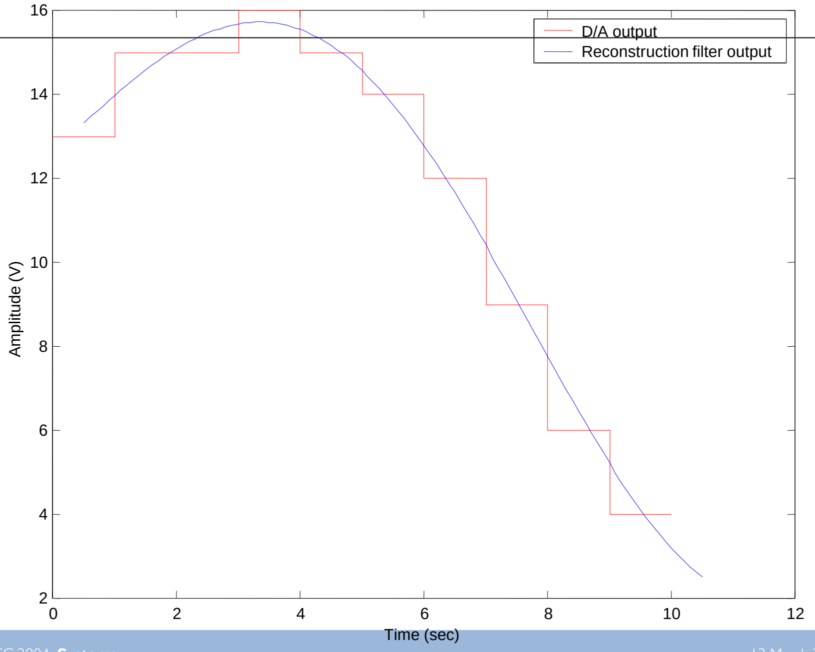
Ideal *sinc* Interpolation of sample values [0 0 0.75 1 0.5 0 0]



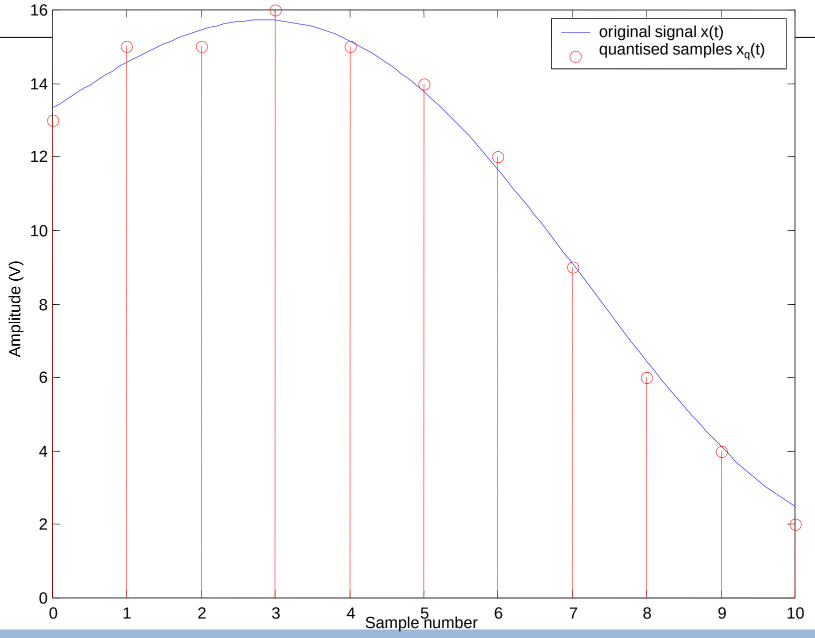
'staircase' output from D/A converter (ZOH)



Smooth output from reconstruction filter



Example: error due to signal quantisation

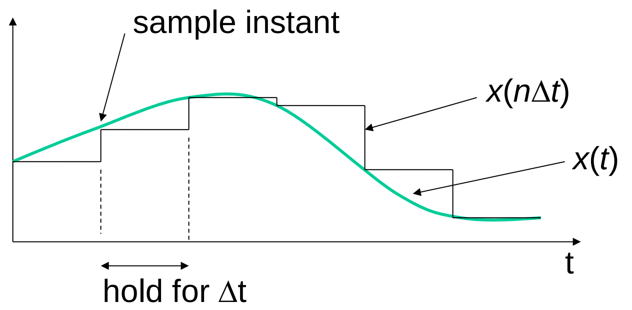


Finite Width Sampling

- Impulse train sampling not realisable
  - sample pulses have finite width (say nanosecs)
- This produces two effects,
- Impulse train has sinc envelope in frequency domain
  - impulse train is square wave with small duty cycle
  - Reduces amplitude of replica spectrums
    - smaller replicas to remove with reconstruction filter ☺
- Averaging of signal during sample time
  - effective low pass filter of original signal
    - can reduce aliasing, but can reduce fidelity ☹
    - negligible with most S/H ☺

Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every Δt seconds
  2. holds that value constant until next sample
- Produces 'staircase' waveform, $x(n\Delta t)$



Practical Reconstruction

Two stage process:

1. Digital to analogue converter (D/A)
   - zero order hold filter
   - produces 'staircase' analogue output
2. Reconstruction filter
   - non-ideal filter: $w_c = w_s/2$
   - further reduces replica spectrums
   - usually 4th – 6th order e.g., Butterworth
     - for acceptable phase response

D/A Converter

- Analogue output y(t) is
  - convolution of output samples y(nΔt) with $h_{ZOH}(t)$

$$y(t) = \sum_n y(n\Delta t) h_{ZOH}(t - n\Delta t)$$

$$h_{ZOH}(t) = \begin{cases} 1, & 0 \le t < \Delta t \\ 0, & \text{otherwise} \end{cases}$$

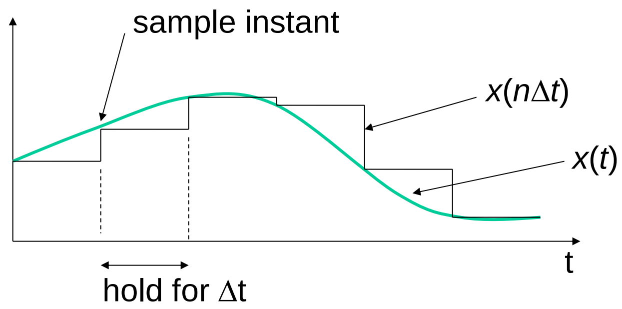
$$H_{ZOH}(w) = \Delta t \exp\left(\frac{-jw\Delta t}{2}\right) \frac{\sin(w\Delta t / 2)}{w\Delta t / 2}$$

D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional reconstruction filter required

Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every Δt seconds
  2. holds that value constant until next sample

- Produces 'staircase' waveform, $x(n\Delta t)$



Practical Reconstruction

Two stage process:

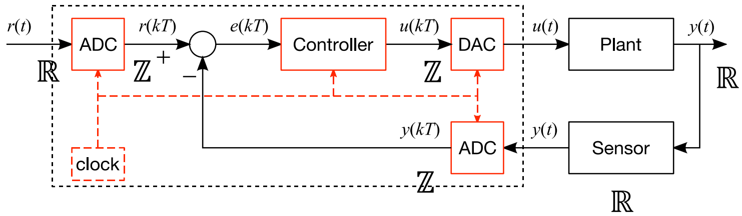
- Digital to analogue converter (D/A)
  - zero order hold filter
  - produces 'staircase' analogue output

- Reconstruction filter
  - non-ideal filter: $\omega_c = \frac{\omega_s}{2}$
  - further reduces replica spectrums
  - usually 4th – 6th order e.g., Butterworth
    - for acceptable phase response

⊣⊢ Recap: ⊣⊢ Digital Systems:

- Implies something "something discrete" or …
  that a mapping exists to an "integer set" $s \in \mathbb{Z}$

- Often the "state-space" and "time" are discretised.
  *(But they **both** need no be)*



- Why?
  - Beat the noise (e.g., more signal "sharing")
  - Leverage time-keeping (oscillator) precision

Next Time…

- Aliasing and Anti-Aliasing

- Review:
  - Chapter 5 of Lathi

- A signal has many signals ☺
  [Unless it's bandlimited]