Systems Theory: Linear Differential Systems

ELEC 3004: **Systems**: Signals & Controls
Dr. Surya Singh

Lecture 4

elec3004@itee.uq.edu.au
http://robotics.itee.uq.edu.au/~elec3004/

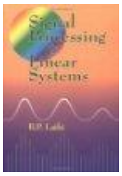
March 8, 2019

http://elec3004.com

Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	**Systems: Linear Differential Systems**
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	Aliasing & Antialiasing
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	Digital Filter (FIR)
7	10-Apr	Digital Windows
	12-Apr	FFT
8	17-Apr	Active Filters & Estimation & Holiday
	19-Apr	**Holiday**
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation/PID
10	8-May	PID & State-Space
	10-May	State-Space Control
11	15-May	Digital Control Design
	17-May	Stability
12	22-May	State Space Control System Design
	24-May	Shaping the Dynamic Response
13	29-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:

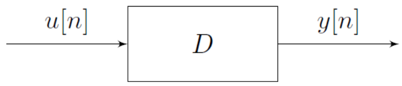


B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 2:
  Time-Domain Analysis of Continuous-Time Systems
  - § 2.1 Introduction
  - § 2.3 The Unit Impulse Response
  - § 2.6 System Stability
  - § 2.7 Intuitive Insights into System Behaviour
  - § 2.9 Summary

Systems as Maps

Then a System is a **MATRIX**



$$y = Du.$$

$$\begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[M] \end{bmatrix} = \begin{bmatrix} D_{11} & D12 & \cdots & D_{1N} \\ D_{21} & D_{22} & \cdots & D_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ D_{M1} & D_{M2} & \cdots & D_{MN} \end{bmatrix} \begin{bmatrix} u[1] \\ u[2] \\ \vdots \\ u[N] \end{bmatrix}.$$

$$y[i] = \sum_j D_{ij} u[j].$$

Linearity: Linear Equations

- Consider system of linear equations:

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ &\vdots \\ y_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{aligned}$$

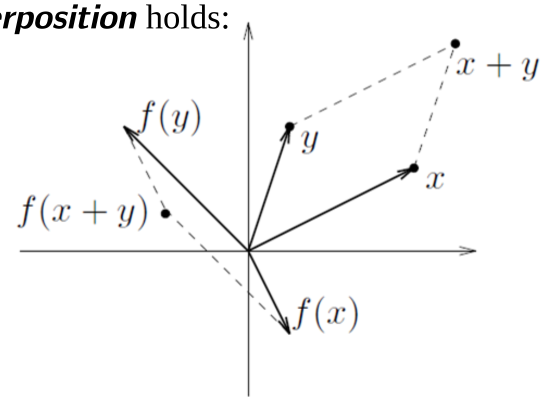
- This can be written in a matrix form as $y = Ax$, where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Source: Boyd, EE263, Slide 2-2

Linearity: Linear Functions

- A function f $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if:
  - $f(x+y) = f(x) + f(y),\ \forall x, y \in \mathbf{R}^n$
  - $f(\alpha x) = \alpha f(x),\ \forall x \in \mathbf{R}^n\ \forall \alpha \in \mathbf{R}$
- That is, ***Superposition*** holds:



Linearity: Linear functions and Matrix Multiplication

Consider a $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$
given by $f(x) = Ax$, where $A \in \mathbb{R}^{m \times n}$

As matrix multiplication function if $\boldsymbol{f}$ is **linear**, we may now say:

- **converse is true**: ***any*** linear function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be written as $f(x) = Ax$, for all $A \in \mathbb{R}^{m \times n}$
- Representation via matrix multiplication is **unique**: for any linear function $\hat{\boldsymbol{f}}$ there is only one matrix $\hat{\boldsymbol{A}}$ for which $\hat{f}(x) = \hat{A}x$ for all x
- $y = Ax$ is a concrete representation of a generic linear function

Linearity: Interpretations

➔ **of $y = Ax$:**

- y is measurement or observation; x is unknown to be determined
- x is an "input" or "stated action"; y is "output" or "result"
  - In controls this "x" is sometimes "separated" into x and u such that $\boldsymbol{x}$ is the state and the $\boldsymbol{u}$ is the action done by the controller
- A function/transformation that maps $\boldsymbol{x} \in \mathbb{R}^n$ into $\boldsymbol{y} \in \mathbb{R}^m$

➔ **of A (or a_{ij}):**

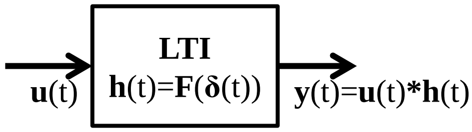
- a_{ij} is a gain factor from j^{th} input (x_j) to i^{th} output (y_i)
- i^{th} row of A concerns i^{th} **output** *("row-out to sea")*
- j^{th} column of A concerns j^{th} **input** *("col-in to land")*
- $a_{34} = 0$ means 3rd output (y_3) doesn't depend on 4th input (x_4)
- $|a_{34}| \gg |a_{3j}|$ for $\boldsymbol{j} \neq \boldsymbol{4}$ means $\boldsymbol{y_3}$ ***depends*** mainly on x_4
- $|a_{34}| \gg |a_{i4}|$ for $i \neq 3$ means x_4 ***affects*** mainly y_3
- *If A* is **diagonal**, *then* i^{th} output depends only on i^{th} input
- If A is lower triangular [i.e., $a_{ij} = 0$ for $i < j$], then the y_i only depends on $x_1, \ldots, x_i$

➔ **Nothing tells you something:**

- The sparsity pattern of A [i.e, zero/nonzero entries], shows which x_j affect which y_i
- `Matlab`: **`spy(A)`** [or just try **`spy`**]

Linear Differential Systems

Linear Time Invariant



- Linear & Time-invariant (of course - tautology!)
- Impulse response: **h**(t)=**F**(**δ**(t))
- Why?
  - Since it is linear the output response (**y**) to any input (**x**) is:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$
$$y(t) = F\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] \overset{linear}{\rightarrow} \int_{-\infty}^{\infty} x(\tau)\,F[\delta(t-\tau)]\,d\tau$$
$$h(t-\tau) \overset{TI}{\equiv} F[\delta(t-\tau)]$$
$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

- The output of any continuous-time LTI system is the **convolution** of input **u**(t) with the impulse response **F**(**δ**(t)) of the system.

Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

$$a_0 y + a_1\frac{dy}{dt} + \cdots + a_n\frac{d^n y}{dt^n} = b_0 x + b_1\frac{dx}{dt} + \cdots + b_m\frac{d^m x}{dt^m}$$

Laplace:

$$a_0 Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

- Total response = Zero-input response + Zero-state response

Initial conditions (Zero-input response) **External Input** (Zero-state response)

Linear Systems and ODE's

- Linear system described by differential equation

$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

- Which using Laplace Transforms can be written as

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s) Y(s) = B(s) X(s)$$

where *A(s)* and *B(s)* are polynomials in *s*

First Order Systems

First order systems

$$ay' + by = 0 \quad \text{(with } a \neq 0\text{)}$$

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

$$Ty' + y = 0 \quad \text{or} \quad y' + ry = 0$$

where

- $T = a/b$ is a *time* (units: seconds)
- $r = b/a = 1/T$ is a *rate* (units: 1/sec)

First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT+1} = \frac{y(0)}{s+1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, y *decays* exponentially

- T gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half $(0.693 = \log 2)$
- $4.6T$ gives time to decay by 0.01 $(4.6 = \log 100)$

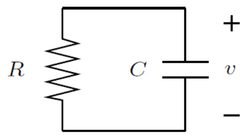
if $T < 0$, y *grows* exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100

First Order Systems

Examples

simple RC circuit:



circuit equation: $RCv' + v = 0$

solution: $v(t) = v(0)e^{-t/(RC)}$

population dynamics:

- $y(t)$ is population of some bacteria at time t
- growth (or decay if negative) rate is $y' = by - dy$ where b is birth rate, d is death rate
- $y(t) = y(0)e^{(b-d)t}$ (grows if $b > d$; decays if $b < d$)

Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a\underbrace{(s^2Y(s) - sy(0) - y'(0))}_{\mathcal{L}(y'')} + b\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

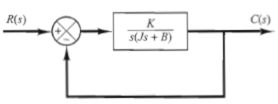
Second Order Systems

so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

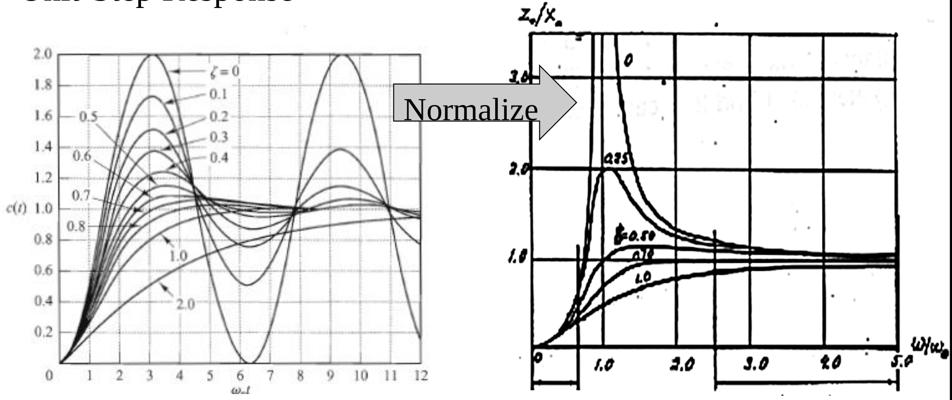
Second Order Response



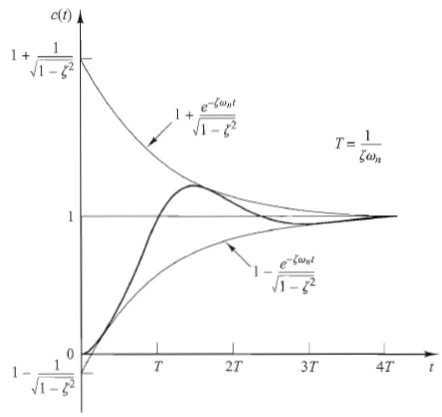
$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

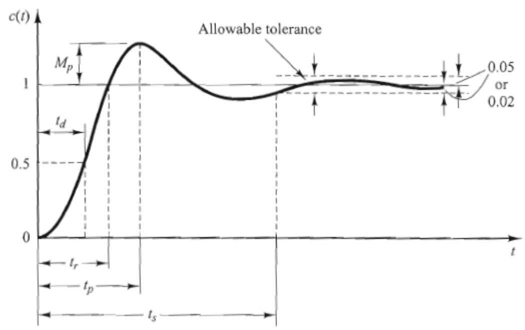
Unit-Step Response



Second Order Response
Envelope Curves



Second Order Response
Unit Step Response Terms



- Delay time, t_d: The time required for the response to reach half the final value
- Rise time, t_r: The time required for the response to rise from 10% to 90%
- Peak time, t_p:The time required for the response to reach the first peak of the overshoot
- Maximum (percent) overshoot, Mp:

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

- Settling time, t_s: The time to be within 2-5% of the final value

Second Order Response
Seeing this on the S-plane

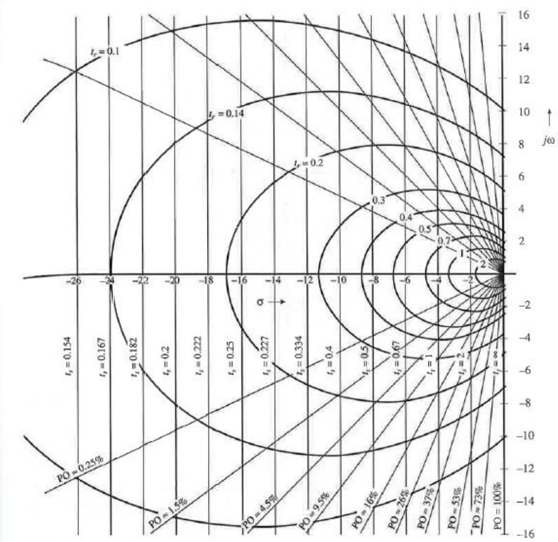
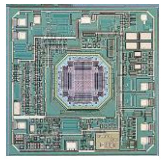


Fig. 6.40 Contours of second-order system pole location for constant PO, constant t_s, and constant t_r in s plane.

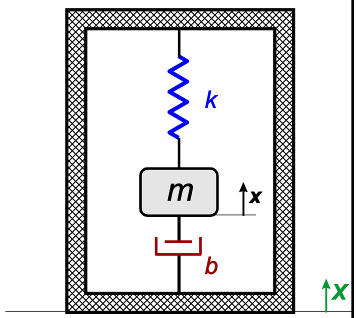
More Examples ☺
(Elaborating from Lecture 2)

Another 2nd Order System: Accelerometer or Mass Spring Damper (MSD)



- General accelerometer:
  - Linear spring (k) (0^{th} order w/r/t o)
  - Viscous damper (b) (1^{st} order)
  - Proof mass (m) (2^{nd} order)

➔ Electrical system analogy:
  - resistor (R) : damper (b)
  - inductance (L) : spring (k)
  - capacitance (C) : mass (m)



Measuring Acceleration: Sense *a* by measuring spring motion *Z*

- Start with Newton's 2nd Law:

$$ma = F$$

- Substitute:

$$m\frac{d^2x}{dt^2} = k(X-x) + b\frac{d(X-x)}{dt}$$

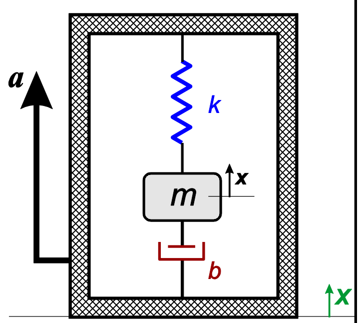
$$\mathbf{Z} \equiv (X-x) \rightarrow x = X - \mathbf{Z}$$

$$\Rightarrow m\frac{d^2X}{dt^2} = m\frac{d^2\mathbf{Z}}{dt^2} + k\mathbf{Z} + b\frac{d\mathbf{Z}}{dt}$$

The "displacement" measured by the unit (the motion of *m* relative the accelerometer frame)

- Solve ODE:

$$X(t) = X_0 e^{i\omega t} \qquad \mathbf{Z}(t) = Z_0 e^{i\omega t}$$



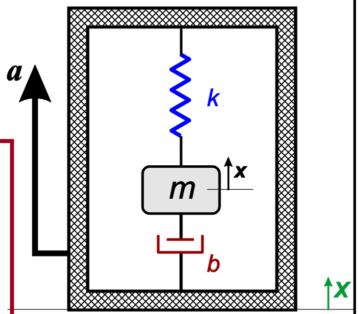
Measuring Acceleration [2]

- Substitute candidate solutions:

$$m\frac{d^2\left(X_0e^{i\omega t}\right)}{dt^2} = m\frac{d^2\left(Z_0e^{i\omega t}\right)}{dt^2} + k\left(Z_0e^{i\omega t}\right) + b\frac{d\left(Z_0e^{i\omega t}\right)}{dt}$$
$$-m\omega^2X_0e^{i\omega t} = -m\omega^2Z_0e^{i\omega t} + kZ_0e^{i\omega t} + (i\omega)\,bZ_0e^{i\omega t}$$

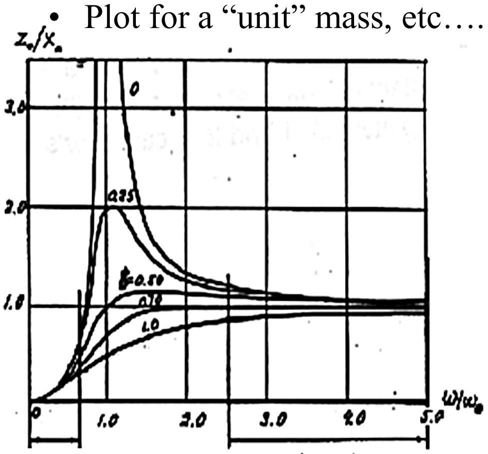
- Define Natural Frequency (ω_0) & Simplify for Z_0 (the spring displacement "magnitude"):

$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$
$$Z_0 = \frac{m\omega^2X_0}{m\omega^2 - k - i\omega b} = \frac{X_0}{\sqrt{1 - \frac{\omega_0{}^2}{\omega^2} - \frac{b^2}{m^2\omega^2}}}$$



Acceleration: 2nd Order System

- Plot for a "unit" mass, etc….



- For $\omega \ll \omega_0$:

$$Z_0 \approx \frac{\omega^2X_0}{\omega_0^2} = \frac{a}{\omega_0^2}$$
$$\rightarrow a = Z_0\omega_0^2$$

➔ **it's an Accelerometer**

- For $\omega \sim \omega_0$
  - As: b→ 0, Z → ∞
  - Sensitivity ↑
- For $\omega \gg \omega_0$:

$$Z_0 \approx X_0$$

➔ **it's a Seismometer**

Equivalence Across Domains

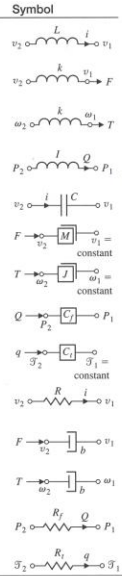
Table 2.1 Summary of Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Source: Dorf & Bishop, *Modern Control Systems*, 12th Ed., p. 73

Table 2.2 Summary of Governing Differential Equations for Ideal Elements

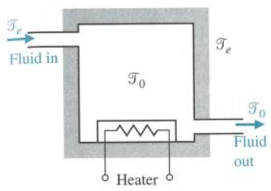
Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\frac{dv_{21}}{dt}$	$E = \frac{1}{2}Cv_{21}{}^2$	
	Translational mass	$F = M\frac{dv_2}{dt}$	$E = \frac{1}{2}Mv_2{}^2$	
	Rotational mass	$T = J\frac{d\omega_2}{dt}$	$E = \frac{1}{2}J\omega_2{}^2$	
	Fluid capacitance	$Q = C_f\frac{dP_{21}}{dt}$	$E = \frac{1}{2}C_fP_{21}{}^2$	
	Thermal capacitance	$q = C_t\frac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \frac{1}{R}v_{21}$	$\mathcal{P} = \frac{1}{R}v_{21}{}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}{}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}{}^2$	
	Fluid resistance	$Q = \frac{1}{R_f}P_{21}$	$\mathcal{P} = \frac{1}{R_f}P_{21}{}^2$	
	Thermal resistance	$q = \frac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t}\mathcal{T}_{21}$	



Source: Dorf & Bishop, *Modern Control Systems*, 12th Ed., p. 74

Thermal Systems

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_0 - \mathcal{T}_e$ = temperature difference due to thermal process

C_t = thermal capacitance

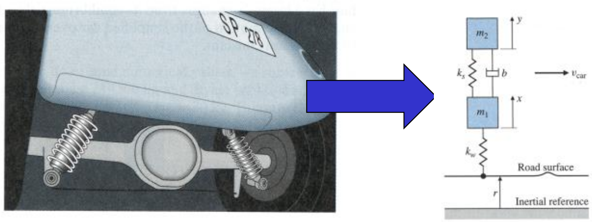
Q = fluid flow rate = constant

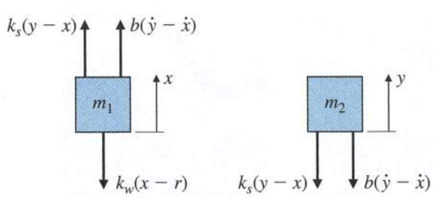
S = specific heat of water

R_t = thermal resistance of insulation

$q(s)$ = transform of rate of heat flow of heating element

Cascades of Linear Systems: Ex $_6$: Quarter-Car Model





REF: FPE, Feedback Control of Dynamic Systems, 6th Ed, p.25

Example: Quarter-Car Model (2)

$$\ddot{x} + \frac{b}{m_1}(\dot{x} - \dot{y}) + \frac{k_s}{m_1}(x - y) + \frac{k_w}{m_1}x = \frac{k_w}{m_1}r,$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y} - \dot{x}) + \frac{k_s}{m_2}(y - x) = 0.$$

$$s^2X(s) + s\frac{b}{m_1}(X(s) - Y(s)) + \frac{k_s}{m_1}(X(s) - Y(s)) + \frac{k_w}{m_1}X(s) = \frac{k_w}{m_1}R(s),$$

$$s^2Y(s) + s\frac{b}{m_2}(Y(s) - X(s)) + \frac{k_s}{m_2}(Y(s) - X(s)) = 0,$$

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}.$$

Economics: Cost of Production

Materials, parts, labour, etc. (***inputs***) are combined to make a number of products (***outputs***):

- x_j: price per unit of production input j
- a_{ij}: input j required to manufacture one unit of product i
- y_i: production cost per unit of product i
- For $y = Ax$:
  - i^{th} row of A is bill of materials for unit of product i
- Production inputs needed:
  - q_i is quantity of product i to be produced
  - r_j is total quantity of production input j needed

$\therefore\ r = A^T q$

& Total production cost is:

$$r^T x = (A^T q)^T x = q^T A x$$

Source: Boyd, EE263, Slide 2-18

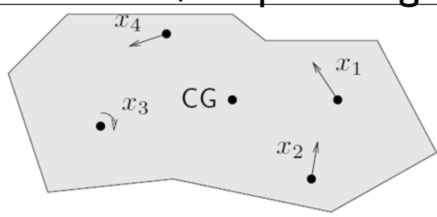
Estimation (or inversion)

$$y = Ax$$

- y_i is i^{th} measurement or sensor reading (which we have)
- x_j is j^{th} parameter to be estimated or determined
- a_{ij} is sensitivity of i^{th} sensor to j^{th} parameter
- sample problems:
  - find x, given y
  - find all x's that result in y (i.e., all x's consistent with measurements)
  - if there is no x such that $y = Ax$, find x s.t. $y \approx Ax$
    (i.e., if the sensor readings are inconsistent, find x which is almost consistent)

Source: Boyd, EE263, Slide 2-26

Mechanics: Total force/torque on rigid body



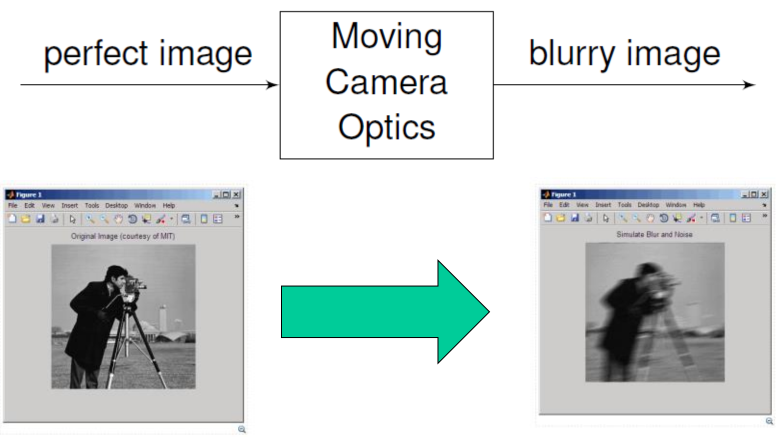
- x_j is external force/torque applied at some point/direction/axis
- $y \in \mathbf{R}^6$ is resulting total force & torque on body
  (y_1, y_2, y_3 are x-, y-, z- components of total force,
  y_4, y_5, y_6 are x-, y-, z- components of total torque)
- we have $y = Ax$
- A depends on geometry
  (of applied forces and torques with respect to center of gravity CG)
- jth column gives resulting force & torque for unit force/torque j

Source: Boyd, EE263, Slide 2-9

Next Time…

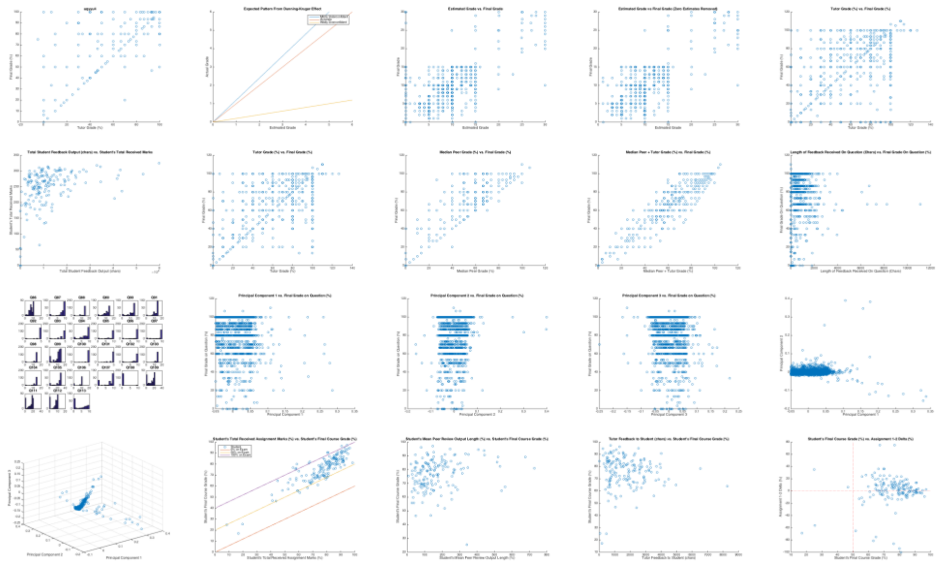
- We will talk about sampling
- Please complete the "practice assignment" **before** starting Problem Set 1
- Thank you!

Matlab Fun: Deblurring



- Matlab: **deconvwnr**

Platypus Fun: All Sorts of "Analytics"



Platypus Fun: `less` is `more` (`more` is `less`?)

- Longer answers better?
- Total Student Feedback Output Characters vs. Total Student Marks Received