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Signals as Vectors
Systems as Maps

ELEC 3004: **Systems**: Signals & Controls

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Lecture 3

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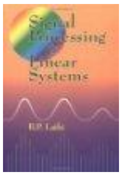
Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	**6-Mar**	**Systems as Maps & Signals as Vectors**
	8-Mar	Systems: Linear Differential Systems
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	Aliasing & Antialiasing
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	Digital Filter (FIR)
7	10-Apr	Digital Windows
	12-Apr	FFT
8	17-Apr	Active Filters & Estimation & Holiday
	19-Apr	**Holiday**
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation/PID
10	8-May	PID & State-Space
	10-May	State-Space Control
11	15-May	Digital Control Design
	17-May	Stability
12	22-May	State Space Control System Design
	24-May	Shaping the Dynamic Response
13	29-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
[TK5102.9.L38 1998](TK5102.9.L38 1998)

- Chapter 1:
  Introduction to Signals and Systems
  - § 1.7 Classification of Systems

- Chapter 3:
  Signal Representation By Fourier Series
  - § 3.1 Signals and Vectors
  - § 3.3 Signal Representation by Orthogonal Signal Set

Linearity (Superposition) Recap (from Lecture 2)

Linear Systems: Superposition

- **<u>Given</u>** input $x_1(t)$ produces output $y_1(t)$
  and input $x_2(t)$ produces output $y_2(t)$

- **<u>Then</u>**: The linearly combined input

$$x(t) = ax_1(t) + bx_2(t)$$

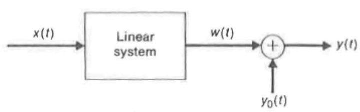
  must produce the linearly combined output

$$y(t) = ay_1(t) + by_2(t)$$

  for arbitrary a and b

- **<u>Generalizing:</u>**
  – Input: $x(t) = \sum_k a_k x_k(t)$ | Output: $y(t) = \sum_k a_k y_k(t)$

Linear Systems: Superposition Consequences



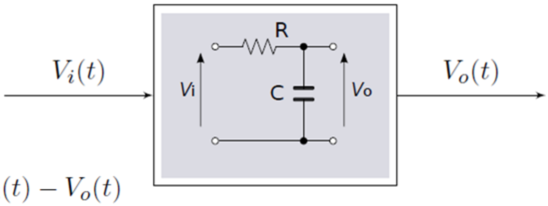
<u>Consequences:</u>

- Zero input for all time yields a zero output.
  – This follows readily by setting $a = 0$, then $0 \cdot x(t) = 0$

- For an **invertible system**,
  Zero output for all time means yields that there was zero input

- DC output/Bias ➔ **<u>Incrementally linear</u>**
  - Ex: $y(t) = [2x(t)] + [1]$
  - Set offset to be added offset [Ex: $y_0(t)$=1]

Example: Is it Linear?



$$C\frac{dV_o(t)}{dt} = \frac{V_i(t) - V_o(t)}{R}$$

$$C\left(V_O(s)\cdot s\right) = \left(\frac{1}{R}\right)\left(V_i(s) - V_O(s)\right)$$

$$(RCs+1)\left(V_O(s)\right) = V_i(s)$$

$$V_O(s) = \left(\frac{1}{RCs+1}\right)V_i(s)$$

$$[y] \overset{?}{=} [A]\,[x]$$

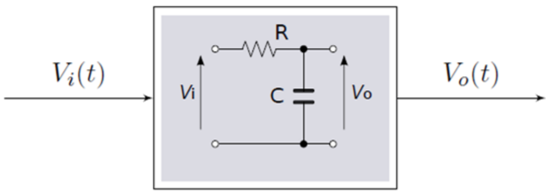
$$[V_O(s)] \overset{\checkmark}{=} \left[\left(\frac{1}{RCs+1}\right)\right][V_i(s)]$$

$\therefore$ (Therefore) :

- Voltage (or current) superposition may be employed ☺
- Zero output, means Zero input (?)

Source: ELEC6.003 (s.3-42)

Example: Is it First-Order?



$$\overbrace{ay' + by = w}^{a\left(\frac{d}{dt}(y(t))\right)+by(t)=w(t)}$$

$$Ty' + y = \left(\frac{1}{b}\right)w,\ T \equiv \frac{a}{b}$$

$$\Rightarrow T\left(Ys - y(0)\right) + Y = \left(\frac{1}{b}\right)W(s)$$

If V_i =0 : $\overbrace{aYs + bY = 0}^{[aY(s)\cdot s+bY(s)=0]}$

- "Autonomous System"
- Natural (or unforced) response

$$(RC)\left(V_O\cdot s - V_O(0)\right) + V_O = 0$$

- ➔ T=a/b=RC
- Time Constant: $\tau = \frac{1}{T} = \frac{1}{RC}$
- Solution: $V_O(t) = [V_O(0)]\,e^{-\frac{t}{RC}}$

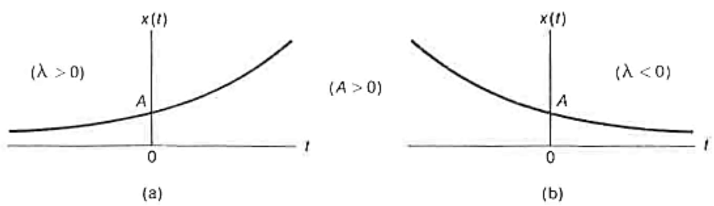
Signals as Vectors

Complex Exponential Signals

$$x(t) = Ae^{\lambda t}$$

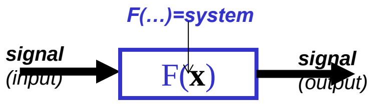
- A and λ are generally complex numbers.

- If A and λ are, in fact, real-valued numbers, $x(t)$ is itself real-valued and is called a **real exponential**

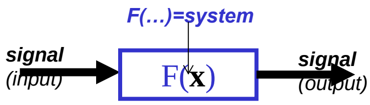


Signals as Vectors

- Back to the beginning!



Signals as Vectors



- There is a perfect analogy between signals and vectors …

Signals are vectors!

- A vector can be represented as a sum of its components in a variety of ways, depending upon the choice of coordinate system. A signal can also be represented as a sum of its components in a variety of ways.

Signals as Vectors

- Represent them as Column Vectors

$$x = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ \vdots \\ x[N] \end{bmatrix}.$$

Signals as Vectors

- Can represent phenomena of interest in terms of signals
- Natural vector space structure (addition/substraction/norms)
- Can use norms to describe and quantify properties of signals

Signals as vectors

Signals can take real or complex values.

In both cases, a natural vector space structure:

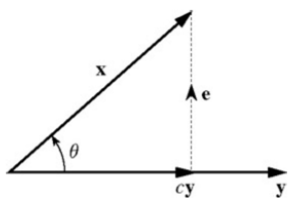
- Can add two signals: $x_1[n] + x_2[n]$
- Can multiply a signal by a scalar number: $C \cdot x[n]$
- Form linear combinations: $C_1 \cdot x_1[n] + C_2 \cdot x_2[n]$

Various Types

- Audio signal (sound pressure on microphone)
- B/W video signal (light intensity on
- photosensor)
- Voltage/current in a circuit (measure with
- multimeter)
- Car speed (from tachometer)
- Robot arm position (from rotary encoder)
- Daily prices of books / air tickets / stocks
- Hourly glucose level in blood (from glucose monitor)
- Heart rate (from heart rate sensor)

Vector Refresher

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos \theta \qquad (6.46)$$



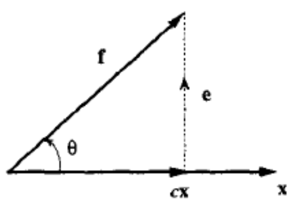
- Length: $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$
- Decomposition: $\mathbf{x} = c_1\mathbf{y} + \mathbf{e}_1 = c_2\mathbf{y} + \mathbf{e}_2$
- Dot Product of ⊥ is 0: $\mathbf{x} \cdot \mathbf{y} = 0$

Vectors [2]

- Magnitude and Direction

$$f \cdot x = |f||x| \cos(\theta)$$

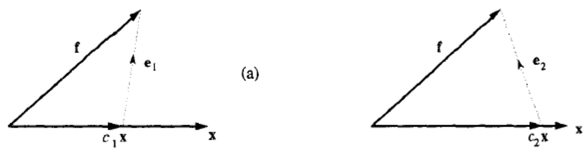
- Component (projection) of a vector along another vector



$\mathbf{f} = c\mathbf{x} + \mathbf{e}$ ← Error Vector

Vectors [3]

- ∞ bases given $\vec{\mathbf{x}}$



- Which is the best one?

$$\mathbf{f} \simeq c\mathbf{x}$$

$$c|\mathbf{x}| = |\mathbf{f}| \cos \theta$$

$$c|\mathbf{x}|^2 = |\mathbf{f}||\mathbf{x}| \cos \theta = \mathbf{f} \cdot \mathbf{x}$$

$$c = \frac{\mathbf{f} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{1}{|\mathbf{x}|^2} \mathbf{f} \cdot \mathbf{x}$$

$$\mathbf{f} \cdot \mathbf{x} = 0$$

- Can I allow more basis vectors than I have dimensions?

Signals **Are** Vectors

- A Vector / Signal can represent a sum of its components

Remember (Lecture 5, Slide 10):

Total response = Zero-input response + Zero-state response

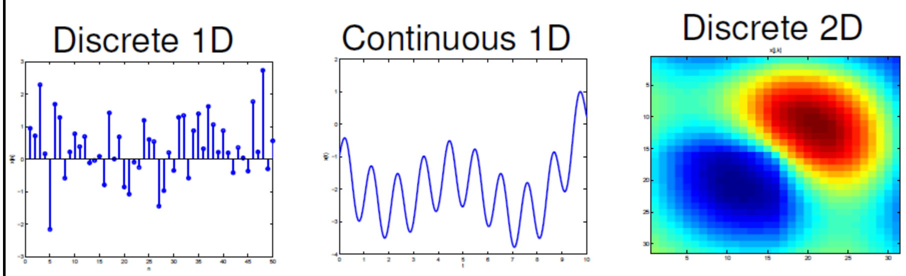
Initial conditions **External Input**

- Vectors are Linear
  - They have **additivity** and **homogeneity**

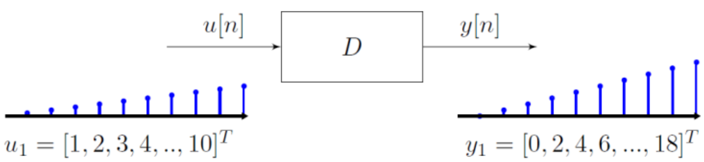
Vectors / Signals Can Be Multidimensional

- A signal is a quantity that varies as a function of an index set

- They can be multidimensional:
  - 1-dim, discrete index (time): x[n]
  - 1-dim, continuous index (time): x(t)
  - 2-dim, discrete (e.g., a B/W or RGB image): x[j; k]
  - 3-dim, video signal (e.g, video): x[j; k; n]



It's Just a Linear Map



- y[n]=2u[n-1] is a linear map
- BUT y[n]=2(u[n]-1) is **NOT** **Why?**

- **Because of homogeneity!**

  T(au)=aT(u)

Norms of signals

Can introduce a notion of signals being ”nearby.”

This is characterized by a metric (or distance function).

$$d(\mathbf{x}, \mathbf{y})$$

If compatible with the vector space structure, we have a norm.

$$\|\mathbf{x} - \mathbf{y}\|$$

Examples of Norms

Can use many different norms, depending on what we want to do.
The following are particularly important:

- ℓ_2 (Euclidean) norm:

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x[k]|^2 \right)^{\frac{1}{2}} \qquad \texttt{norm(x,2)}$$

- ℓ_1 norm:

$$\|x\|_1 = \sum_{k=1}^{n} |x[k]| \qquad \texttt{norm(x,1)}$$

- ℓ_∞ norm:

$$\|x\|_\infty = \max_k |x[k]| \qquad \texttt{norm(x,inf)}$$

What are the differences?

Properties of norms

For any norm $\|\cdot\|$, and any signal $\mathbf{x}$, we have:

1. Linearity: if C is a scalar,

$$\|C \cdot \mathbf{x}\| = |C| \cdot \|\mathbf{x}\|$$

2. Subadditivity (triangle inequality):

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$$

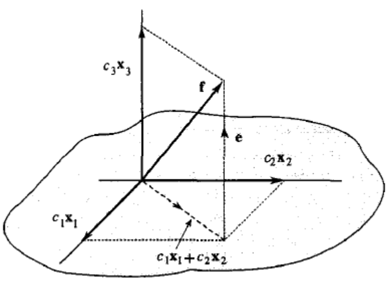
Can use norms:

- To detect whether a signal is (approximately) zero.
- To compare two signals, and determine if they are "close."

$$\|\mathbf{x} - \mathbf{y}\| \approx 0$$

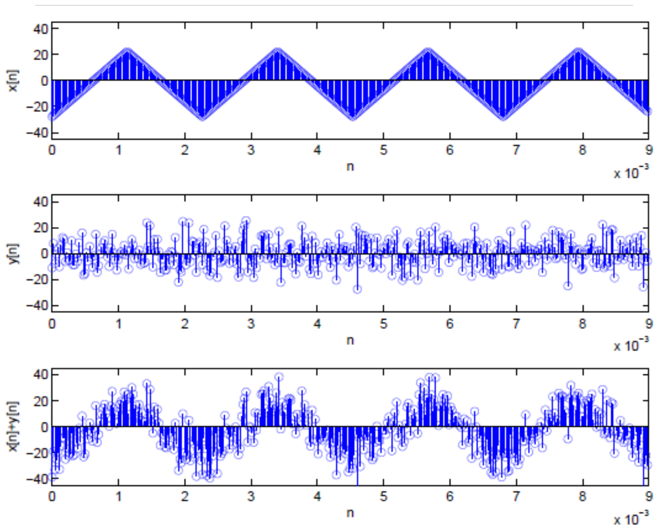
Signal representation by Orthogonal Signal Set

- **Orthogonal Vector Space**



➔ A signal may be thought of as having components.

Linear combinations of signals



Application Example: Active Noise Cancellation

A "noise" signal, that we want to get rid of.

- At subject location, signal is

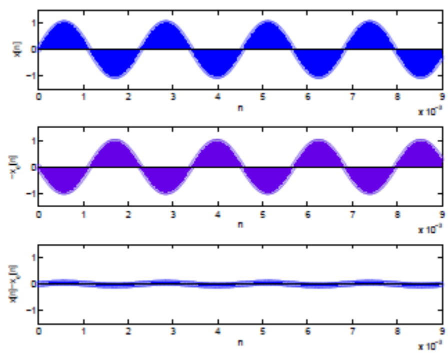
$$x[n]$$

- Microphone picks up signal

$$x_c[n]$$

- Subtract the two signals:

$$y(t) = x(t) - x_c(t)$$



Notice careful synchronization is needed!

Component of a Signal

$$f(t) \simeq cx(t) \qquad t_1 \le t \le t_2$$

$$c = \frac{\int_{t_1}^{t_2} f(t)x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt} = \frac{1}{E_x}\int_{t_1}^{t_2} f(t)x(t)\,dt$$

$$\int_{t_1}^{t_2} f(t)x(t)\,dt = 0$$

- Let's take an example:

$$f(t) \simeq c\sin t \qquad 0 \le t \le 2\pi$$

$$x(t) = \sin t \quad \text{and} \quad E_x = \int_0^{2\pi} \sin^2(t)\,dt = \pi$$

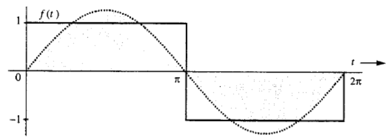


Fig. 3.3 Approximation of square signal in terms of a single sinusoid.

Thus

$$f(t) \simeq \frac{4}{\pi}\sin t \qquad (3.14)$$

Basis Spaces of a Signal

$$\int_{t_1}^{t_2} x_m(t)x_n(t)\,dt = \begin{cases} 0 & m \ne n \\ E_n & m = n \end{cases}$$

$$\begin{aligned} f(t) &\simeq c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t) \\ &= \sum_{n=1}^{N} c_nx_n(t) \end{aligned}$$

$$e(t) = f(t) - \sum_{n=1}^{N} c_nx_n(t)$$

$$\begin{aligned} c_n &= \frac{\int_{t_1}^{t_2} f(t)x_n(t)\,dt}{\int_{t_1}^{t_2} {x_n}^2(t)\,dt} \\ &= \frac{1}{E_n}\int_{t_1}^{t_2} f(t)x_n(t)\,dt \qquad n = 1, 2, \ldots, N \end{aligned}$$

$$\begin{aligned} f(t) &= c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + \cdots \\ &= \sum_{n=1}^{\infty} c_nx_n(t) \qquad t_1 \le t \le t_2 \end{aligned}$$

Basis Spaces of a Signal

$$f(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) + \cdots$$
$$= \sum_{n=1}^{\infty} c_n x_n(t) \qquad t_1 \leq t \leq t_2$$

- Observe that the error energy Ee generally decreases as N, the number of terms, is increased because the term $C_k^2 E_k$ is nonnegative. Hence, it is possible that the error Energy → 0 as $N \to \infty$. When this happens, the orthogonal signal set is said to be complete.

- In this case, it's no more an approximation but an equality

This (and Other) Basis for $\mathbb{R}^n$

- A fundamental idea of linear algebra
- One basis maybe better suited for a particular problem

For vectors $w_1, \ldots, w_n$ to be a basis for $\mathbb{R}^{\boldsymbol{n}}$, this means:

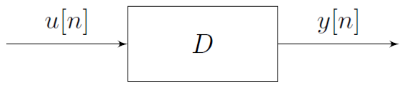
1. The w_i's are linearly independent
2. A $n \times n$ matrix W with these columns is invertible
3. Every vector v in $\mathbb{R}^{\boldsymbol{n}}$ can be written in exactly one was as a combination of the w_i's

$$\boldsymbol{v} = c_1 \boldsymbol{w}_1 + c_2 \boldsymbol{w}_2 + \cdots c_n \boldsymbol{w}_n$$

BREAK

Systems as Maps

Then a System is a **MATRIX**

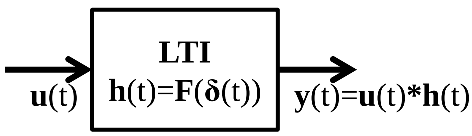


$$y = Du.$$

$$\begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[M] \end{bmatrix} = \begin{bmatrix} D_{11} & D12 & \cdots & D_{1N} \\ D_{21} & D_{22} & \cdots & D_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ D_{M1} & D_{M2} & \cdots & D_{MN} \end{bmatrix} \begin{bmatrix} u[1] \\ u[2] \\ \vdots \\ u[N] \end{bmatrix}.$$

$$y[i] = \sum_j D_{ij} u[j].$$

Linear Time Invariant



- Linear & Time-invariant (of course - tautology!)
- Impulse response: **h**(t)=**F**(**δ**(t))
- Why?
  - Since it is linear the output response (**y**) to any input (**x**) is:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$
$$y(t) = F\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] \overset{linear}{\rightarrow} \int_{-\infty}^{\infty} x(\tau)\,F[\delta(t-\tau)]\,d\tau$$
$$h(t-\tau) \overset{TI}{\equiv} F[\delta(t-\tau)]$$
$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

- The output of any continuous-time LTI system is the **convolution** of input **u**(t) with the impulse response **F**(**δ**(t)) of the system.

Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

Laplace:

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

- Total response = $\underbrace{\text{Zero-input response}}_{\textbf{Initial conditions}}$ + $\underbrace{\text{Zero-state response}}_{\textbf{External Input}}$

Linear Systems and ODE's

- Linear system described by differential equation

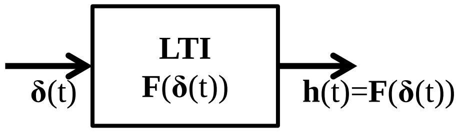
$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

  - Which using Laplace Transforms can be written as

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

where *A(s)* and *B(s)* are polynomials in *s*

Unit Impulse Response



- **δ**(t): Impulsive excitation
- h(t): characteristic mode terms

Ex:

EXAMPLE 2.4

Determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)\, y(t) = Dx(t) \qquad (2.25)$$

This is a second-order system ($N = 2$) having the characteristic polynomial
$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2)$

The characteristic roots of this system are $\lambda = -1$ and $\lambda = -2$. Therefore

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t} \qquad (2.26a)$$

Differentiation of this equation yields

$$\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \qquad (2.26b)$$

The initial conditions are [see Eq. (2.24b) for $N = 2$]
$\dot{y}_n(0) = 1$ and $y_n(0) = 0$

Setting $t = 0$ in Eqs. (2.26a) and (2.26b), and substituting the initial conditions just given, we obtain
$0 = c_1 + c_2$

$1 = -c_1 - 2c_2$

Solution of these two simultaneous equations yields
$c_1 = 1$ and $c_2 = -1$

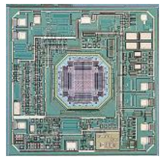
Therefore
$y_n(t) = e^{-t} - e^{-2t}$

Moreover, according to Eq. (2.25), $P(D) = D$, so that
$P(D)y_n(t) = Dy_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}$

Also in this case, $b_0 = 0$ [the second-order term is absent in $P(D)$]. Therefore
$h(t) = [P(D)y_n(t)]u(t) = (-e^{-t} + 2e^{-2t})u(t)$

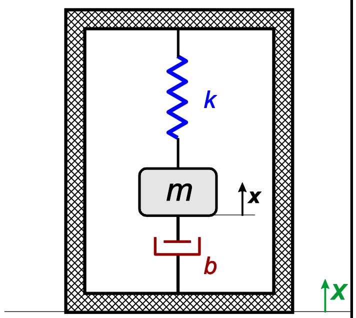
More Examples ☺
(Elaborating from Lecture 2)

Another 2nd Order System: Accelerometer or Mass Spring Damper (MSD)



- General accelerometer:
  - Linear spring (k) (0^{th} order w/r/t o)
  - Viscous damper (b) (1^{st} order)
  - Proof mass (m) (2^{nd} order)

➔ Electrical system analogy:
  - resistor (R) : damper (b)
  - inductance (L) : spring (k)
  - capacitance (C) : mass (m)



Measuring Acceleration: Sense *a* by measuring spring motion *Z*

- Start with Newton's 2^{nd} Law:

$$ma = F$$

- Substitute:

$$m\frac{d^2x}{dt^2} = k(X-x) + b\frac{d(X-x)}{dt}$$

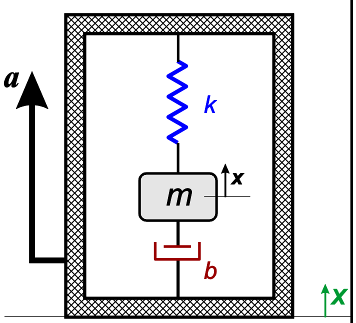
$$\mathbf{Z} \equiv (X-x) \rightarrow x = X - \mathbf{Z}$$

$$\Rightarrow m\frac{d^2X}{dt^2} = m\frac{d^2\mathbf{Z}}{dt^2} + k\mathbf{Z} + b\frac{d\mathbf{Z}}{dt}$$

The "displacement" measured by the unit (the motion of *m* relative the accelerometer frame)

- Solve ODE:

$$X(t) = X_0 e^{i\omega t} \qquad \mathbf{Z}(t) = Z_0 e^{i\omega t}$$



Measuring Acceleration [2]

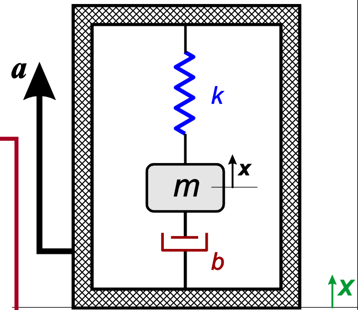
- Substitute candidate solutions:

$$m\frac{d^2\left(X_0e^{i\omega t}\right)}{dt^2} = m\frac{d^2\left(Z_0e^{i\omega t}\right)}{dt^2}+k\left(Z_0e^{i\omega t}\right)+b\frac{d\left(Z_0e^{i\omega t}\right)}{dt}$$
$$-m\omega^2X_0e^{i\omega t} = -m\omega^2Z_0e^{i\omega t}+kZ_0e^{i\omega t}+(i\omega)\,bZ_0e^{i\omega t}$$

- Define Natural Frequency (ω_0)
  & Simplify for Z_0
  (the spring displacement "magnitude"):

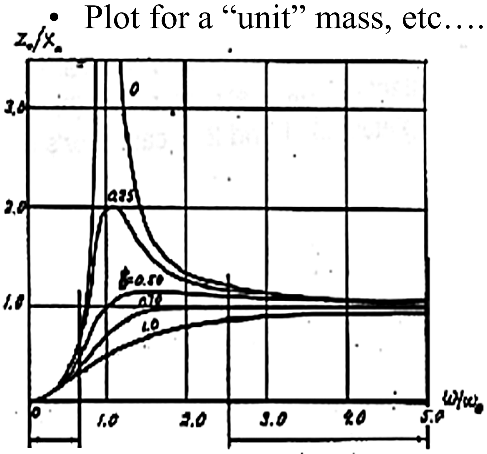
$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

$$Z_0 = \frac{m\omega^2X_0}{m\omega^2-k-i\omega b} = \boxed{\frac{X_0}{\sqrt{1-\frac{\omega_0{}^2}{\omega^2}-\frac{b^2}{m^2\omega^2}}}}$$



Acceleration: 2nd Order System

- Plot for a "unit" mass, etc….



Accelerometer Seismometer

- For $\omega \ll \omega_0$:

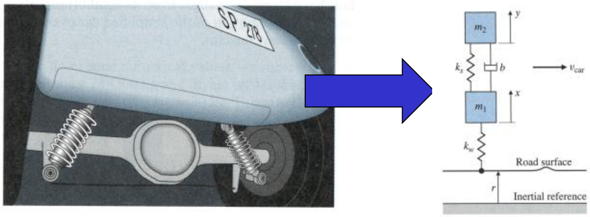
$$Z_0 \approx \frac{\omega^2X_0}{\omega_0^2} = \frac{a}{\omega_0^2}$$
$$\rightarrow a = Z_0\omega_0^2$$

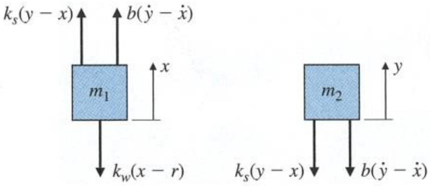
➔ **it's an Accelerometer**

- For $\omega \sim \omega_0$
  – As: b➔ 0, Z ➔ ∞
  – Sensitivity ↑
- For $\omega \gg \omega_0$:
  $$Z_0 \approx X_0$$

➔ **it's a Seismometer**

Cascades of Linear Systems: Ex $_6$: Quarter-Car Model





REF: FPE, Feedback Control of Dynamic Systems, 6th Ed, p.25

Example: Quarter-Car Model (2)

$$\ddot{x} + \frac{b}{m_1}(\dot{x}-\dot{y}) + \frac{k_s}{m_1}(x-y) + \frac{k_w}{m_1}x = \frac{k_w}{m_1}r,$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y}-\dot{x}) + \frac{k_s}{m_2}(y-x) = 0.$$

$$s^2X(s) + s\frac{b}{m_1}(X(s)-Y(s)) + \frac{k_s}{m_1}(X(s)-Y(s)) + \frac{k_w}{m_1}X(s) = \frac{k_w}{m_1}R(s),$$

$$s^2Y(s) + s\frac{b}{m_2}(Y(s)-X(s)) + \frac{k_s}{m_2}(Y(s)-X(s)) = 0,$$

$$\boxed{\frac{Y(s)}{R(s)} = \frac{\dfrac{k_w b}{m_1 m_2}\left(s+\dfrac{k_s}{b}\right)}{s^4 + \left(\dfrac{b}{m_1}+\dfrac{b}{m_2}\right)s^3 + \left(\dfrac{k_s}{m_1}+\dfrac{k_s}{m_2}+\dfrac{k_w}{m_1}\right)s^2 + \left(\dfrac{k_w b}{m_1 m_2}\right)s + \dfrac{k_w k_s}{m_1 m_2}}.}$$

Next Time…

- We'll talk about Other System Properties ☺
- We will introduce this via the lens of:
  "Systems as Maps. Signals as Vectors"
- Review:
  - Phasers, complex numbers, polar to rectangular, and general functional forms.
  - Chapter B and Chapter 1 of Lathi
    (particularly the first sections on signals & classification thereof)
- Register on Platypus
- Try the practise assignment