http://elec3004.com

Stability and Examples

ELEC 3004: **Systems**: Signals & Controls
Dr. Surya Singh

Lecture 23

elec3004@itee.uq.edu.au

http://robotics.itee.uq.edu.au/~elec3004/

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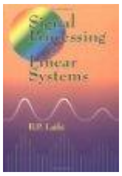
Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	Systems: Linear Differential Systems
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	Aliasing & Antialiasing
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	PS 1: Q & A
7	10-Apr	Digital Windows
	12-Apr	Digital Filter (FIR)
8	17-Apr	Active Filters & Estimation
	19-Apr	Holiday
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation & PID Control
10	8-May	State-Space Control
	10-May	Guest Lecture: FFT
11	15-May	Advanced PID & & FFT Processes
	17-May	State Space Control System Design
12	22-May	Shaping the Dynamic Response
	24-May	**Stability and Examples**
13	29-May	System Identification & Information Theory & Information Space
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

G. Franklin,
J. Powell,
M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990

Today

➔ **State-space** 🡐 **[A stately idea! ☺]**

- **FPW**
  - **Chapter 4:**
    Discrete Equivalents to Continuous
  - Transfer Functions: The Digital Filter

- **FPW**
  - Chapter 5

Next Time

Even More Online Reading Materials


CHAPTER SIX
SHAPING THE DYNAMIC RESPONSE

6.1 INTRODUCTION


CHAPTER THREE
DYNAMICS OF LINEAR SYSTEMS

3.1 DIFFERENTIAL EQUATIONS REVISITED

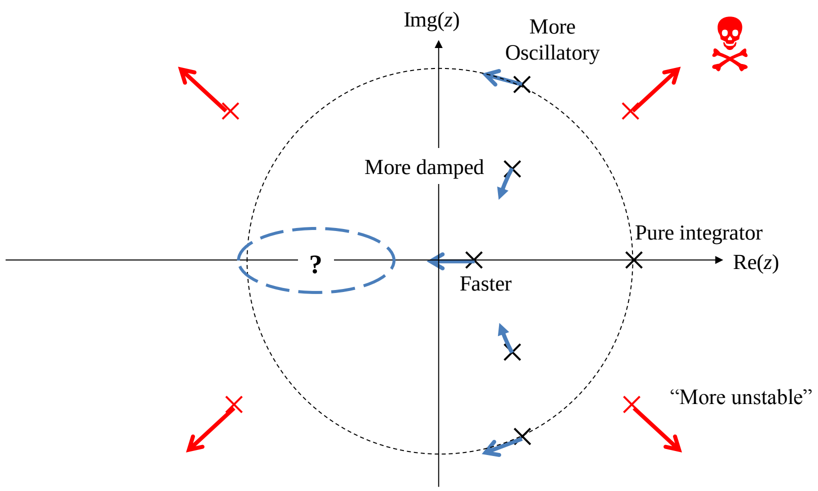
- Friedland, *Control System Design* Ch. 6 and 3

➔ http://robotics.itee.uq.edu.au/~elec3004/tutes.html

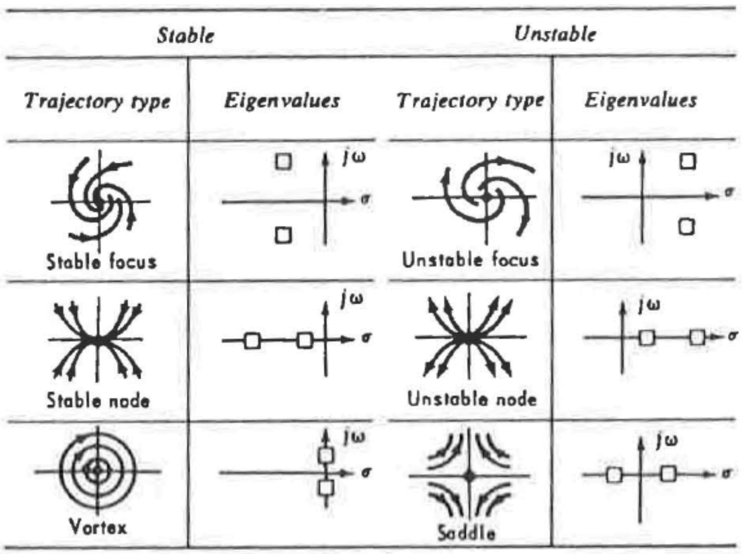
Stability

Recall dynamic responses

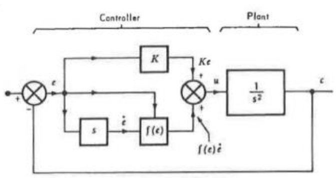
- Ditto the z-plane:



Various Types of Singularities (2nd order systems)



Stability of a 2nd order regulator



$$u = Ke + f(e)\dot{e}$$

state equations let $e = x_1$ and $\dot{e} = x_2$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Kx_1 - f(x_1)\ x_2$$

assume for simplicity that $K = 1$.

$$0 = x_2^0$$

$$0 = -x_1^0 - f(x_1^0)x_2^0$$

The Jacobian matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -f(0) \end{bmatrix}$$

- The linear behavior of the system in the close neighborhood of the origin is described by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - f(0)x_2$$

- AND, the characteristic equation is:

$$s[s + f(0)] + 1 = 0$$

with the eigenvalues

$$\lambda_1 = -\frac{1}{2}f(0) + \sqrt{\frac{1}{4}f^2(0) - 1}$$

$$\lambda_2 = -\frac{1}{2}f(0) - \sqrt{\frac{1}{4}f^2(0) - 1}$$

Controllability

Controllability

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where $\mathbf{x}$ = state vector (n-vector)
$\mathbf{u}$ = control vector (r-vector)
$\mathbf{y}$ = output vector (m-vector) $(m \leq n)$
$\mathbf{A}$ = $n \times n$ matrix
$\mathbf{B}$ = $n \times r$ matrix
$\mathbf{C}$ = $m \times n$ matrix

is completely output controllable if and only if the composite $m \times nr$ matrix $\mathbf{P}$, where

$$\mathbf{P} = \left[\mathbf{CB} \mid \mathbf{CAB} \mid \mathbf{CA}^2\mathbf{B} \mid \cdots \mid \mathbf{CA}^{n-1}\mathbf{B}\right]$$

is of rank m. (Notice that complete state controllability is neither necessary nor sufficient for complete output controllability.)

Controllability matrix

- To convert an arbitrary state representation in **F**, **G**, **H** and J to control canonical form **A**, **B**, **C** and D, the "controllability matrix"

$$\mathcal{C} = [\mathbf{G} \quad \mathbf{FG} \quad \mathbf{F}^2\mathbf{G} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}]$$

must be nonsingular.

Why is it called the "controllability" matrix?

Controllability matrix

- If you can write it in CCF, then the system equations must be linearly independent.
- Transformation by any nonsingular matrix preserves the controllability of the system.
- Thus, a nonsingular controllability matrix means $\boldsymbol{x}$ can be driven to any value.

Controllability Example

- Is this fully controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- We see that vectors **B** and **AB** are **not linearly independent** and
- The rank of the matrix **[B | AB]** is 1 < **m** (m=2)
- ∴ **the system is not completely state controllable.**

- In fact, elimination of x_2 from the given problem yields:

$$\ddot{x}_1 + 1.5\dot{x}_1 - 2.5x_1 = \dot{u} + 2.5u \longrightarrow \frac{X_1(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$$

- Notice that cancellation of the factor (**s** + 2.5) occurs in the numerator and denominator of the transfer function. Because of this cancellation, this system is **not** completely state controllable and it's unstable system (s=1, RHP!). Remember that stability and controllability are quite different things. **There are many systems that are unstable, but are completely state controllable**.

Controllability Example II

- TF ➔ CCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} u$$

$$y = [0.8 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad\qquad y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution. Consider the system defined by Equations (9–120) and (9–121). The rank of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$$

is 2. Hence, the system is completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$$

is 1. Hence the system is not observable.

Next consider the system defined by Equations (9–122) and (9–123). The rank of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$$

is 1. Hence, the system is not completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$$

is 2. Hence, the system is observable.

The apparent difference in the controllability and observability of the same system is caused by the fact that the original system has a pole-zero cancellation in the transfer function. Referring to Equation (2–29), for $D = 0$ we have

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

If we use Equations (9–120) and (9–121), then

$$G(s) = [0.8 \quad 1] \begin{bmatrix} s & -1 \\ 0.4 & s + 1.3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + 1.3s + 0.4} [0.8 \quad 1] \begin{bmatrix} s + 1.3 & 1 \\ -0.4 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{s + 0.8}{(s + 0.8)(s + 0.5)}$$

[Note that the same transfer function can be obtained by using Equations (9–122) and (9–123).] Clearly, cancellation occurs in this transfer function.

Observability

Observability

- Observability is concerned with the issue of what can be said about the state when one is given measurements of the plant output.

- **Definition**: The state $x0 \neq 0$ is said to be unobservable if, given $x(0) = x0$, and $u[k] = 0$ for $k \geq 0$, then $y[k] = 0$ for $k \geq 0$. The system is said to be completely observable if there exists no nonzero initial state that it is unobservable.

Observability and Detectability

- Consider again the state space model

$$\delta x[k] = \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k]$$
$$y[k] = \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]$$

- In general, the dimension of the observed output, y, can be less than the dimension of the state, x.

- However, one might conjecture that, if one observed the output over some nonvanishing time interval, then this might tell us something about the state.

- The associated properties are called observability

Observability criteria for LTI systems

- Given the *n*-dimensional LTI dynamical equation:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Then for the system to be observable for any t_0 in $[0, +\infty)$: All the columns of $\boldsymbol{Ce^{At}}$ *must be* are linearly independent on $[t_0, +\infty)$.

Recall: Ce^{At}: Matrix Exponential

Observability criteria for LTI systems [2]

We test for this by:

- Forming a $nq \times n$ observability matrix
- and checking its rank:

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

Example

- Consider the following state space model:

$$\mathbf{A} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

- Then

$$\mathbf{\Gamma}_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -4 & -2 \end{bmatrix}$$

- Hence, $rank(\Gamma_0[A, C]) = 2$,
  and the **system is completely observable.**

Example [2]: $A_{(1,1)} = -3 \rightarrow A_{(1,1)} = -1$

- Consider

$$\mathbf{A} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

- Here

$$\mathbf{\Gamma}_o[\mathbf{A}, \mathbf{C}] = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix}$$

- Hence, $rank(\Gamma_0[A, C]) = 1 < 2$,
  and **the system is <u>not</u> completely observable**

Controllability ↔ Observability Duality

- There is a remarkable similarity between the results for controllability and Observability.

- We can formalize this *Duality* as follows:

  Consider a state space model described by the 4-tuple (**A**, **B**, **C**, **D**).
  Then the system is completely controllable *if and only if* the dual system ($\mathbf{A}^T$, $\mathbf{C}^T$, $\mathbf{B}^T$, $\mathbf{D}^T$) is completely observable.

Break ☺

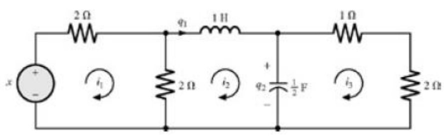
State Space Control Examples

Example 1:

State-Space

in First Order Problems

A Quick Example



1. The inductor current q_1 and the capacitor voltage $q2$ as the state variables.

2. $$q_1 = i_2$$
$$\tfrac{1}{2}\dot{q}_2 = i_2 - i_3$$

3. $$4i_1 - 2i_2 = x$$
$$2(i_2 - i_1) + \dot{q}_1 + q_2 = 0$$
$$-q_2 + 3i_3 = 0$$

$$\dot{q}_1 = 2(i_1 - i_2) - q_2$$
$$\dot{q}_1 = -q_1 - q_2 + \tfrac{1}{2}x$$
$$\dot{q}_2 = 2q_1 - \tfrac{2}{3}q_2$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} x$$

See also: Fig. 13.2, Lathi p. 789

A Systematic Procedure for Determining State Eqs.

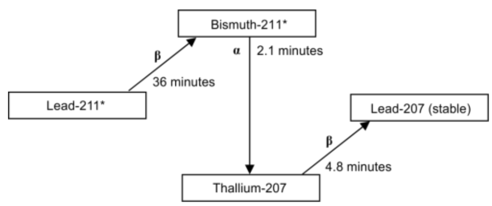
1. Choose all independent capacitor voltages and inductor currents to be the state variables.

2. Choose a set of loop currents; express the state variables and their first derivatives in terms of these loop currents.

3. Write the loop equations and eliminate all variables other than state variables (and their first derivatives) from the equations derived in Steps 2 and 3.

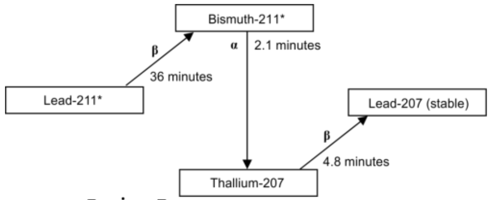
See also: Lathi § 13.2-1 (p. 788)

Another Example



- $\frac{dN1(t)}{dt} = -\lambda_1 N1(t)$
- $\frac{dN2(t)}{dt} = -\lambda_2 N2(t) + \lambda_1 N1(t)$
- $\frac{dN3(t)}{dt} = -\lambda_3 N3(t) + \lambda_2 N2(t)$
- $\frac{dN4(t)}{dt} = \lambda_3 N3(t)$

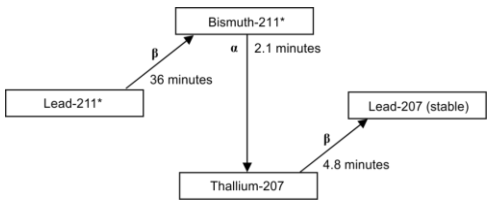
Another Example



$$X = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \rightarrow \dot{X} = \begin{bmatrix} \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}_3 \\ \dot{N}_4 \end{bmatrix}$$

$$\dot{X} = FX \rightarrow \begin{bmatrix} \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}_3 \\ \dot{N}_4 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 & 0 \\ 0 & \lambda_2 & -\lambda_3 & 0 \\ 0 & 0 & \lambda_3 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$$

Another Example



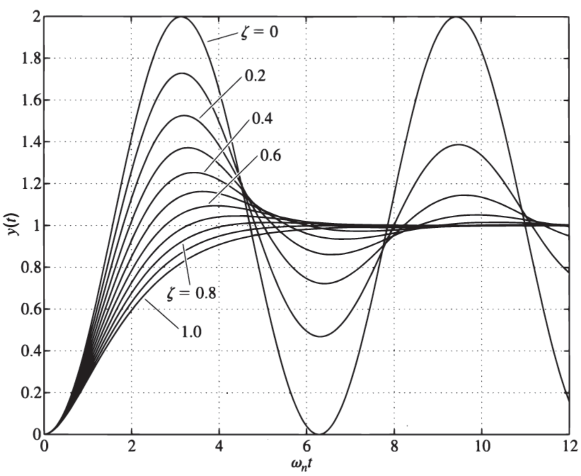
- $N_1(t)=N_1(0)\exp(-\lambda_1 t)$
- $N2(t) = N2(0)exp(-\lambda_2 t) - N1(0)\frac{\lambda_1}{\lambda_2-\lambda_1}(exp(-\lambda_2 t) - exp(-\lambda_1 t))$
- $N3(t) = \lambda_1\lambda_2 N1(0)\left[\frac{exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} + \frac{exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right]$
- $N4(t) = \lambda_1\lambda_2\lambda_3 N1(0)\left[\frac{exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(-\lambda_1)} + \frac{exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(-\lambda_2)} + \frac{exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(-\lambda_3)} + \frac{1}{(\lambda_1\lambda_2\lambda_3)}\right]$

Example 2:
State-Space
in Second Order Problems

Featuring:
C — to → D *Recap*

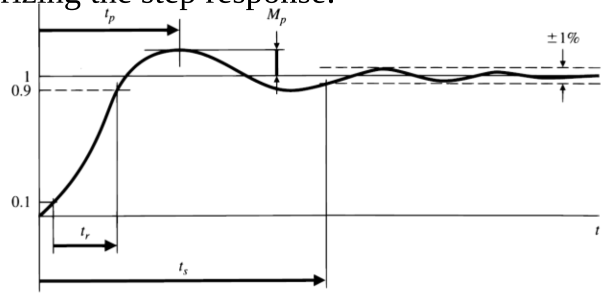
2nd Order System Response

- Response of a 2nd order system to increasing levels of damping:



2^{nd} Order System Specifications

Characterizing the step response:

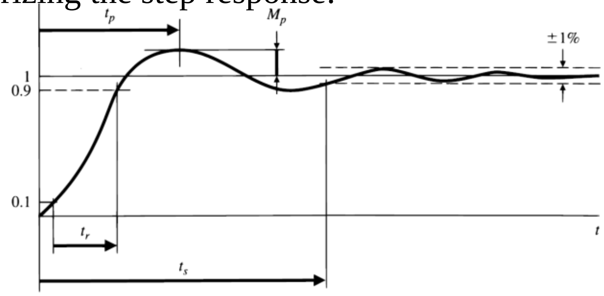


- Rise time (10% → 90%): $t_r \approx \frac{1.8}{\omega_0}$
- Overshoot: $M_p \approx \frac{e^{-\pi\zeta}}{\sqrt{1-\zeta^2}}$
- Settling time (to 1%): $t_s = \frac{4.6}{\zeta\omega_0}$
- Steady state error to unit step: e_{ss}
- Phase margin:

$$\phi_{PM} \approx 100\zeta$$

2^{nd} Order System Specifications

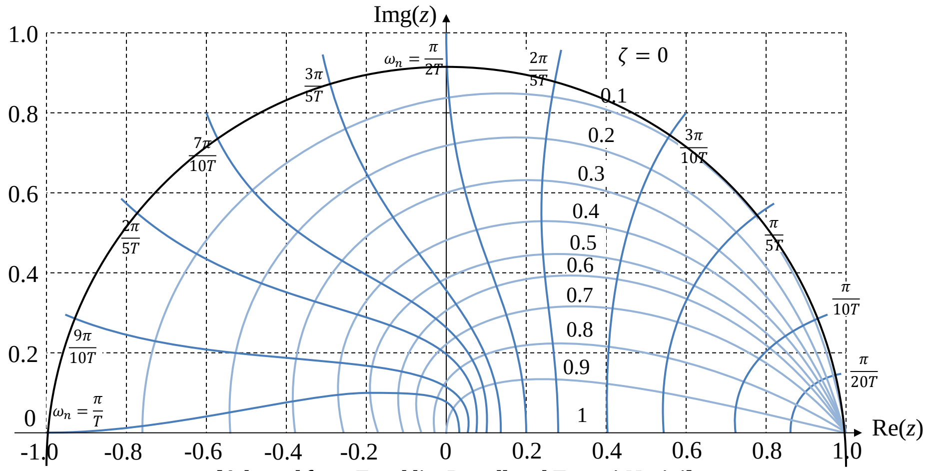
Characterizing the step response:



- Rise time (10% → 90%) & Overshoot:
  t_r, M_p → ζ, ω_0 : Locations of dominant poles
- Settling time (to 1%):
  t_s → radius of poles: $|z| < 0.01^{\frac{T}{t_s}}$
- Steady state error to unit step:
  e_{ss} → final value theorem $e_{ss} = \lim_{z \to 1} \{(z-1)\, F(z)\}$

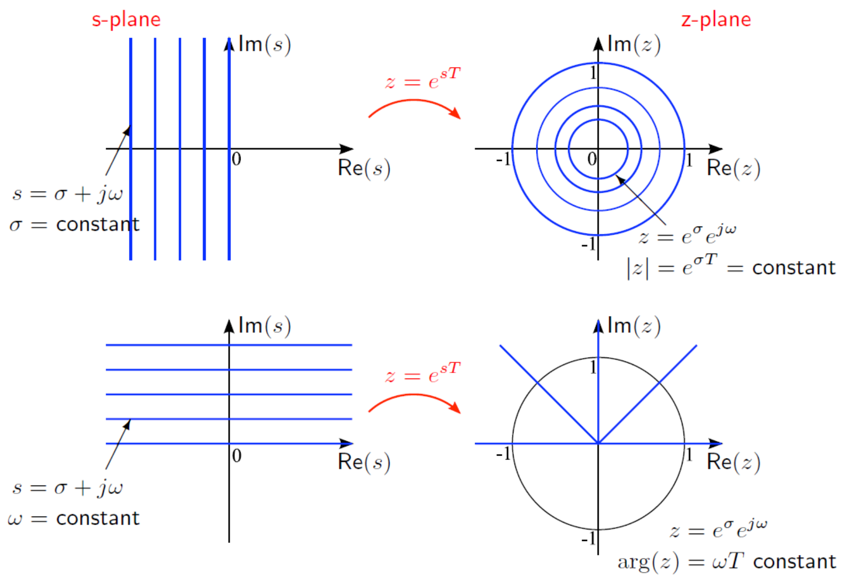
Damping and natural frequency

$$z = e^{sT} \text{ where } s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



[Adapted from Franklin, Powell and Emami-Naeini]

Recall: S-Plane to z-Plane [1/2]



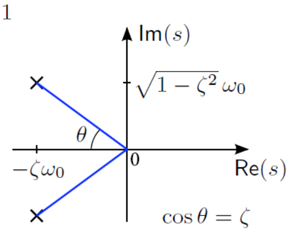
Recall: S-Plane to z-Plane [2/2]

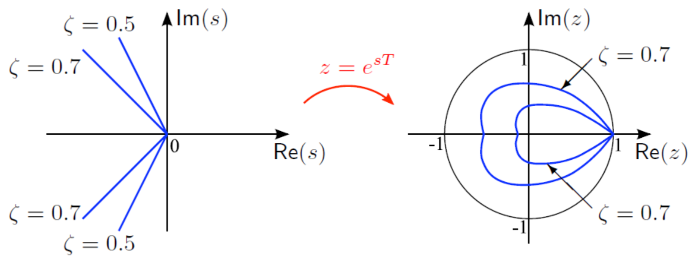
Pole locations for constant damping ratio $\zeta < 1$

$$s^2 + \zeta\omega_0 s + \omega_0^2 = 0$$

$$\Downarrow$$

$$s = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\,\omega_0$$





$s = -\zeta\omega_0 + j\sqrt{1-\zeta^2}\,\omega_0$: $\zeta =$ constant $\quad z = e^{-\zeta\omega_0 T} e^{-j\sqrt{1-\zeta^2}\omega_0 T}$

Discrete-time transfer function

take $\mathcal{Z}$-transform of system equations

$$x(t+1) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) + Du(t)$$

yields

$$zX(z) - zx(0) = AX(z) + BU(z), \qquad Y(z) = CX(z) + DU(z)$$

solve for $X(z)$ to get

$$X(z) = (zI - A)^{-1}\underline{z}x(0) + (zI - A)^{-1}BU(z)$$

(note extra z in first term!)

hence

$$Y(z) = H(z)U(z) + C(zI - A)^{-1}zx(0)$$

where $H(z) = C(zI - A)^{-1}B + D$ is the *discrete-time transfer function*

note power series expansion of resolvent:

$$(zI - A)^{-1} = z^{-1}I + z^{-2}A + z^{-3}A^2 + \cdots$$

Source: Boyd, Lecture Notes for EE263, 13-39

Second Order Digital Systems

Consider the z-transform of a decaying exponential signal:

$y(t) = e^{-at}\cos(bt)\,\mathcal{U}(t)$ $\qquad (\mathcal{U}(t) = \text{unit step})$

- sample: $y(kT) = r^k \cos(k\theta)\,\mathcal{U}(kT)$ $\quad$ with $r = e^{-aT}$ & $\theta = bT$
- transform: $Y(z) = \frac{1}{2}\frac{z}{(z - re^{j\theta})} + \frac{1}{2}\frac{z}{(z - re^{-j\theta})}$

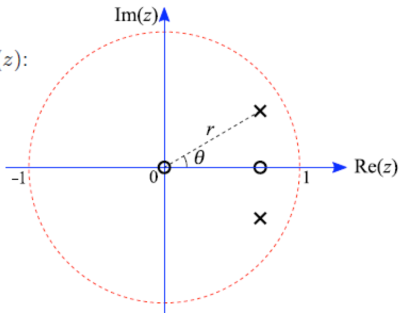
$$= \frac{z(z - r\cos\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

- e.g. y_k is the pulse response of $G(z)$:

$$G(z) = \frac{z(z - r\cos\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

poles: $\begin{cases} z = re^{j\theta} \\ z = re^{-j\theta} \end{cases}$

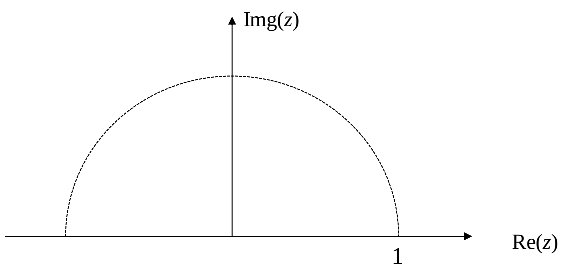
zeros: $\begin{cases} z = 0 \\ z = r\cos\theta \end{cases}$



The z-plane [for all pole systems]

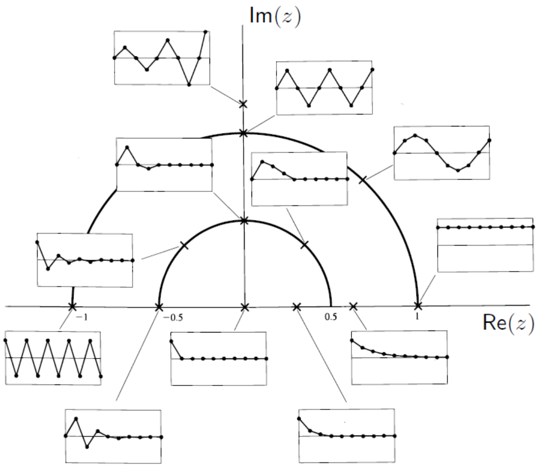
- We can understand system response by pole location in the *z*-plane

[Adapted from Franklin, Powell and Emami-Naeini]



Pole positions in the z-plane

- Poles inside the unit circle are **stable**
- Poles outside the unit circle **unstable**
- Poles on the unit circle are oscillatory
- Real poles at 0 < z < 1 give exponential response
- Higher frequency of oscillation for larger
- Lower apparent damping for larer and r



Response of 2nd order system [1/3]

Responses for varying r:

▷ $r < 1$

$\Downarrow$

exponentially decaying envelope

▷ $r = 1$

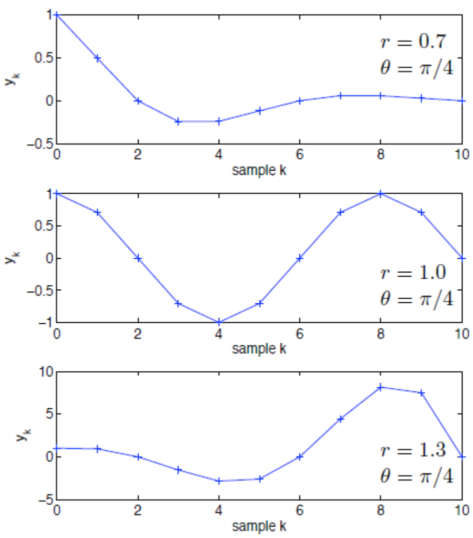
$\Downarrow$

sinusoidal response with $2\pi/\theta$ samples per period

▷ $r > 1$

$\Downarrow$

exponentially increasing envelope



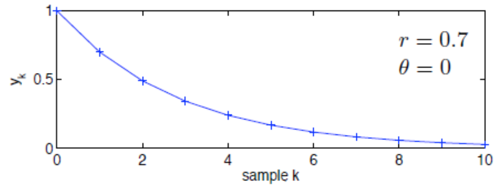
Response of 2nd order system [2/3]

Responses for varying θ:

- $\theta = 0$

  $\Downarrow$

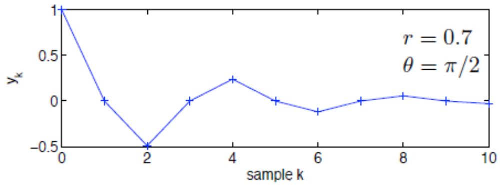
  decaying exponential



- $\theta = \pi/2$

  $\Downarrow$

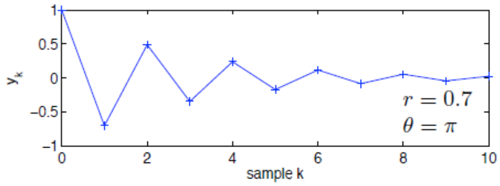
  $2\pi/\theta = 4$ samples per period



- $\theta = \pi$

  $\Downarrow$

  2 samples per period



Response of 2nd order system [3/3]

Some special cases:

- for $\theta = 0$, $Y(z)$ simplifies to:

$$Y(z) = \frac{z}{z - r}$$

$\Longrightarrow$ exponentially decaying response

- when $\theta = 0$ and $r = 1$:

$$Y(z) = \frac{z}{z - 1}$$

$\Longrightarrow$ unit step

- when $r = 0$:

$$Y(z) = 1$$

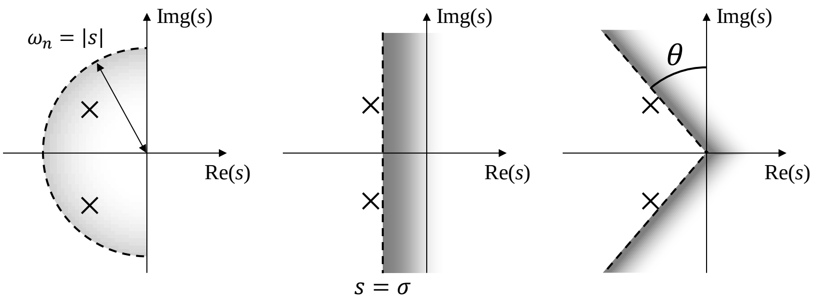
$\Longrightarrow$ unit pulse

- when $\theta = 0$ and $-1 < r < 0$:

samples of alternating signs

Specification bounds

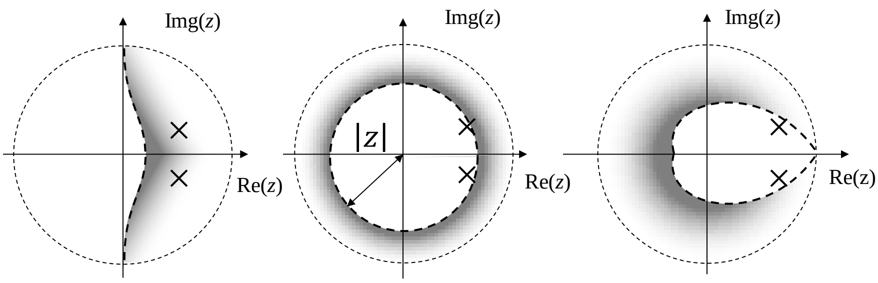
- Recall in the continuous domain, response performance metrics map to the s-plane:



$|s| = \frac{1.8}{t_r}$ $s = \frac{4.6}{t_s}$ $\theta = \sin^{-1}\zeta$

Discrete bounds

- These map to the discrete domain:



$|z| = e^{-t_s T}$

In practice, you'd use Matlab to plot these, and check that the spec is satisfied

Ex: System Specifications → Control Design [1/4]

Design a controller for a system with:

- A continuous transfer function: $G(s) = \dfrac{0.1}{s(s+0.1)}$
- A discrete ZOH sampler
- Sampling time (T_s): T_s= 1s
- Controller:

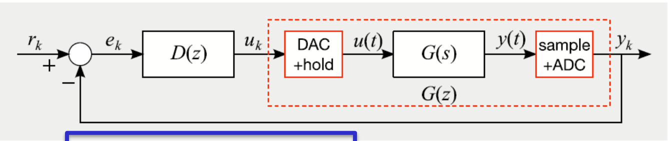
$$u_k = -0.5u_{k-1} + 13\left(e_k - 0.88e_{k-1}\right)$$

The closed loop system is required to have:

- $M_p < 16\%$
- $t_s < 10$ s
- $e_{ss} < 1$

Ex: System Specifications → Control Design [2/4]

1. (a) Find the pulse transfer function of $G(s)$ plus the ZOH



$$\boxed{G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}} = \frac{(z-1)}{z}\mathcal{Z}\left\{\frac{0.1}{s^2(s+0.1)}\right\}$$

e.g. look up $\mathcal{Z}\{a/s^2(s+a)\}$ in tables:

$$G(z) = \frac{(z-1)}{z}\,\frac{z\left((0.1 - 1 + e^{-0.1})z + (1 - e^{-0.1} - 0.1e^{-0.1})\right)}{0.1(z-1)^2(z-e^{-0.1})}$$

$$= \frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)}$$

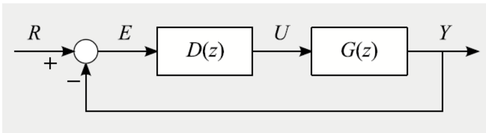
(b) Find the controller transfer function (using z = shift operator):

$$\frac{U(z)}{E(z)} = D(z) = 13\frac{(1-0.88z^{-1})}{(1+0.5z^{-1})} = 13\frac{(z-0.88)}{(z+0.5)}$$

Ex: System Specifications → Control Design [3/4]

2. Check the steady state error e_{ss} when $r_k =$ unit ramp

$$e_{ss} = \lim_{k\to\infty} e_k = \lim_{z\to 1}(z-1)E(z)$$



$$\frac{E(z)}{R(z)} = \frac{1}{1+D(z)G(z)}$$

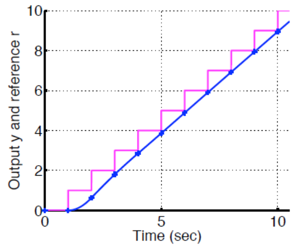
$$R(z) = \frac{Tz}{(z-1)^2}$$

so $$e_{ss} = \lim_{z\to 1}\left\{(z-1)\frac{Tz}{(z-1)^2}\frac{1}{1+D(z)G(z)}\right\} = \lim_{z\to 1}\frac{T}{(z-1)D(z)G(z)}$$

$$= \lim_{z\to 1}\frac{T}{(z-1)\dfrac{0.0484(z+0.9672)}{(z-1)(z-0.9048)}D(1)}$$

$$= \frac{1-0.9048}{0.0484(1+0.9672)D(1)} = 0.96$$

$\Longrightarrow\ e_{ss} < 1$ (as required)



Ex: System Specifications → Control Design [4/4]

3. Step response: overshoot $M_p < 16\% \implies \zeta > 0.5$

settling time $t_s < 10 \implies |z| < 0.01^{1/10} = 0.63$

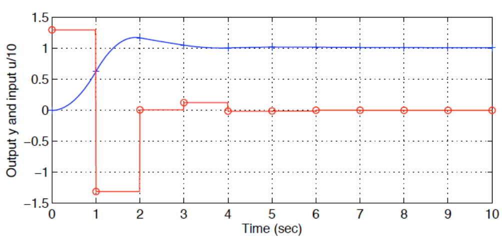
The closed loop poles are the roots of $1 + D(z)G(z) = 0$, i.e.

$$1 + 13\frac{(z-0.88)}{(z+0.5)}\frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)} = 0$$

$$\implies z = 0.88,\ -0.050 \pm j0.304$$

But the pole at $z = 0.88$ is cancelled by controller zero at $z = 0.88$, and

$$z = -0.050 \pm j0.304 = re^{\pm j\theta} \implies \begin{cases} r = 0.31,\ \theta = 1.73 \\ \zeta = 0.56 \end{cases}$$



all specs satisfied!

Another Example: PID control

- Consider a system parameterised by three states:
  - x_1, x_2, x_3
  - where $x_2 = \dot{x}_1$ and $x_3 = \dot{x}_2$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix} \boldsymbol{x} - \mathbf{K}u$$

$$y = [0 \quad 1 \quad 0]\boldsymbol{x} + 0u$$

x_2 is the output state of the system;
x_1 is the value of the integral;
x_3 is the velocity.

Example: PID control [2]

- We can choose **K** to move the eigenvalues of the system as desired:

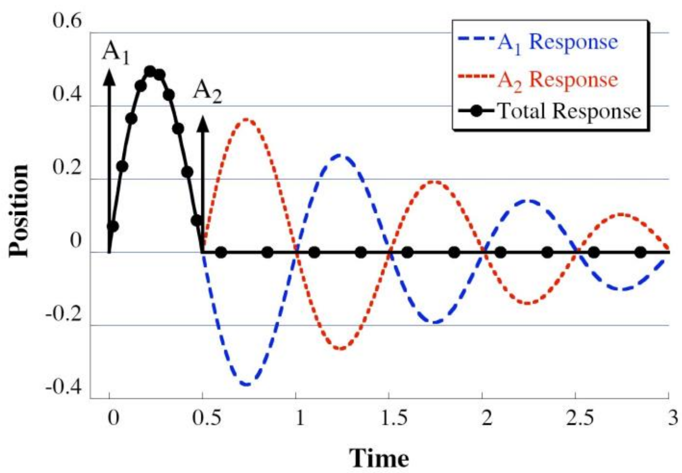
$$\det \begin{bmatrix} 1 - K_1 & & \\ & 1 - K_2 & \\ & & -2 - K_3 \end{bmatrix} = \mathbf{0}$$

All of these eigenvalues must be positive.

It's straightforward to see how adding derivative gain K_3 can stabilise the system.

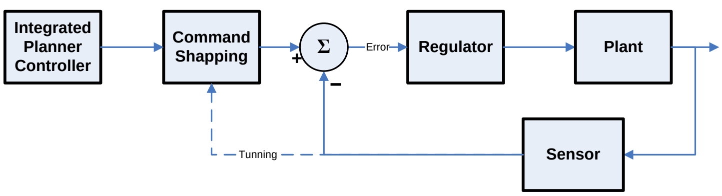
Example 3: Command Shaping

Command Shaping

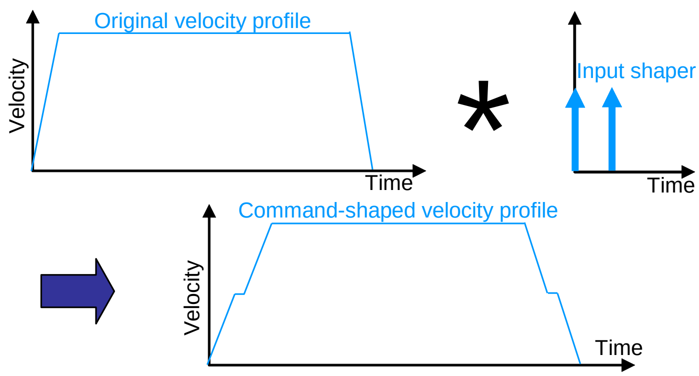


Experiments: Scanning Over Obstacle

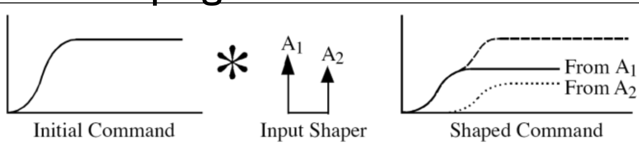
Robust Control:
Command Shaping for Vibration Reduction



Command Shaping



Command Shaping



- Zero Vibration (ZV)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{1+K} & \frac{K}{1+K} \\ 0 & \frac{T_d}{2} \end{bmatrix} \qquad K = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

- Zero Vibration and Derivative (ZVD)

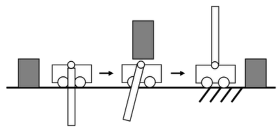
$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{(1+K)^2} & \frac{2K}{(1+K)^2} & \frac{K^2}{(1+K)^2} \\ 0 & \frac{T_d}{2} & T_d \end{bmatrix}$$

Experiments: Command Shaping

Example 4: Inverted Pendulum

Digital Control

Wikipedia, Cart and pole



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$
$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

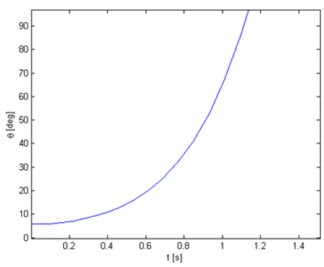
$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

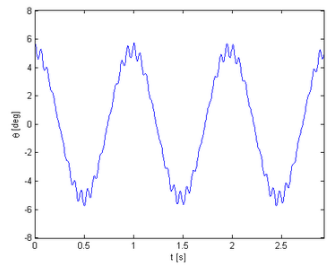
and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

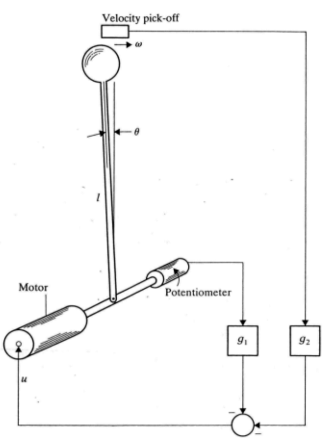
substituting L in these equations and simplifying leads to the equations that describe the motion o

$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$
$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$





Inverted Pendulum



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$
$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

substituting L in these equations and simplifying leads to the equations that describe the motion o

$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$
$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$

Inverted Pendulum – Equations of Motion

- The equations of motion of an inverted pendulum (under a small angle approximation) may be linearized as:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \ddot{\theta} = Q^2\theta + Pu$$

Where:

$$Q^2 = \left(\frac{M+m}{Ml}\right)g$$
$$P = \frac{1}{Ml}.$$

If we further assume unity ***Ml*** ($Ml \approx 1$), then $P \approx 1$

Inverted Pendulum –State Space

- We then select a state-vector as:

$$\boldsymbol{x} = \begin{bmatrix}\theta \\ \omega\end{bmatrix}, \text{ hence } \dot{\boldsymbol{x}} = \begin{bmatrix}\dot{\theta} \\ \dot{\omega}\end{bmatrix} = \begin{bmatrix}\omega \\ \dot{\omega}\end{bmatrix}$$

- Hence giving a state-space model as:

$$A = \begin{bmatrix}0 & 1 \\ Q^2 & 0\end{bmatrix}, B = \begin{bmatrix}0 \\ 1\end{bmatrix}$$

- The resolvent of which is:

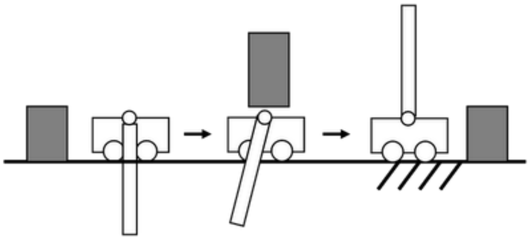
$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix}s & -1 \\ -Q^2 & s\end{bmatrix}^{-1} = \frac{1}{s^2 - Q^2}\begin{bmatrix}s & 1 \\ Q^2 & s\end{bmatrix}$$

- And a state-transition matrix as:

$$\Phi(t) = \begin{bmatrix}\cosh Qt & \dfrac{\sinh Qt}{Q} \\ Q\ sinh\ Qt & \cosh Qt\end{bmatrix}$$

Cart & Pole in State-Space With Obstacles?

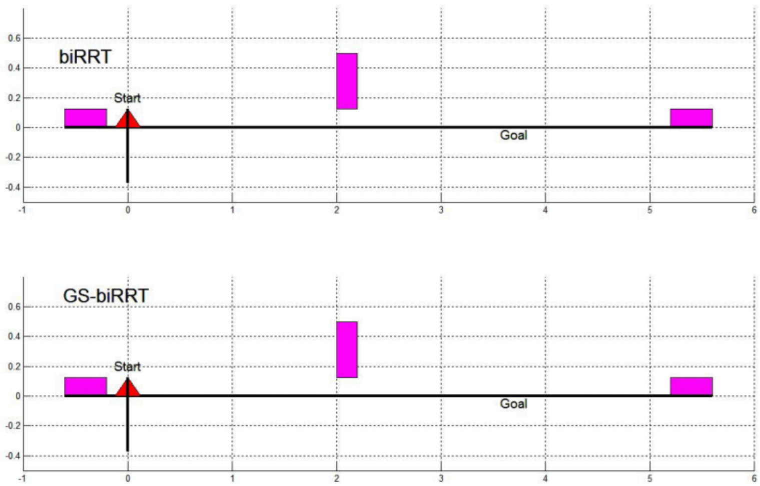
Swing-up is a little more than stabilization…



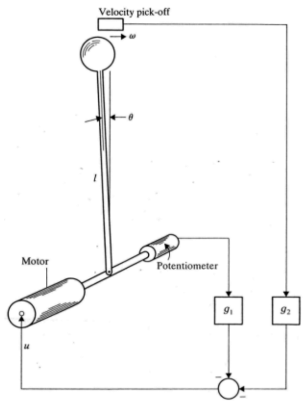
See also: METR4202 – Tutorial 11:
http://robotics.itee.uq.edu.au/~metr4202/tpl/t11-Week11-pendulum.pdf

Cart & Pole in State-Space

Swing-up is a little more than stabilization…



Inverted Pendulum [Extended Reading…]



Example 6B Stabilization of an inverted pendulum An inverted pendulum can readily be stabilized by a closed-loop feedback system, just as a person of moderate dexterity can do it.

A possible control system implementation is shown in Fig. 6.3, for a pendulum constrained to rotate about a shaft at its bottom point. The actuator is a dc motor. The angular position of the pendulum, being equal to the position of the shaft to which it is attached, is measured by means of a potentiometer. The angular velocity in this case can be measured by a "velocity pick-off" at the top of the pendulum. Such a device could consist of a coil of wire

Source: Friedland, *Control System Design*, **Chapter 6, p. 232**

Inverted Pendulum [2]

in a magnetic field created by a small permanent magnet in the pendulum bob. The induced voltage in the coil is proportional to the linear velocity of the bob as it passes the coil. And since the bob is at a fixed distance from the pivot point the linear velocity is proportional to the angular velocity. The angular velocity could of course also be measured by means of a tachometer on the dc motor shaft.

As determined in Prob. 2.2, the dynamic equations governing the inverted pendulum in which the point of attachment does not translate is given by

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \Omega^2\theta - \alpha\omega + \beta u \tag{6B.1}$$

where α and β are given in Example 6A, with the inertia J being the total reflected inertia:

$$J = J_m + ml^2$$

where m is the pendulum bob mass and l is the distance of the bob from the pivot. The natural frequency Ω is given by

$$\Omega^2 = \frac{mgl}{J + ml^2} = \frac{g}{l + J/ml}$$

(Note that the motor inertia J_m affects the natural frequency.)

Since the linearization is valid only when the pendulum is nearly vertical, we shall assume that the control objective is to maintain $\theta = 0$. Thus we have a simple regulator problem.

The matrices A and b for this problem are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

Source: Friedland, *Control System Design*, Chapter 6

Inverted Pendulum [3]

The open-loop characteristic polynomial is

$$|sI - A| = \begin{vmatrix} s & -1 \\ -\Omega^2 & s+\alpha \end{vmatrix} = s^2 + \alpha s - \Omega^2$$

Thus

$$a_1 = \alpha$$

$$a_2 = -\Omega^2$$

The open-loop system is unstable, of course.

The controllability test matrix and the W matrix are given respectively by

$$Q = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \qquad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

(which are the same as they were for the instrument servo). And

$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Thus the gain matrix required for pole placement using (6.34), is

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} (\bar{a}_1 - \alpha) \\ \bar{a}_2 + \Omega^2 \end{bmatrix} = \begin{bmatrix} (\bar{a}_2 + \Omega^2)/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix}$$

Example 6C Control of spring-coupled masses The dynamics of a pair of spring-coupled masses, shown in Fig. 3.7(a), were shown in Example 3I to have the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K/\bar{M} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Inverted Pendulum [4]

The system has the characteristic polynomial

$$D(s) = s^4 + (K/\bar{M})s^2$$

Hence $\quad a_1 = a_3 = a_4 = 0, \qquad a_2 = K/\bar{M}.$

The controllability test and W matrices are given, respectively, by

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -K/\bar{M} \\ 1 & 0 & -K/\bar{M} & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 1 & 0 & K/\bar{M} & 0 \\ 0 & 1 & 0 & K/\bar{M} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (6C.1)$$

Multiplying we find that

$$QW = (QW)' = (QW)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad (6C.2)$$

(This rather simple result is not really as surprising as it may at first seem. Note that A is in the first companion form but using the right-to-left numbering convention. If the left-to-right numbering convention were used the A matrix would already be in the companion form of (6.11) and would not require transformation. The transformation matrix T given by (6C.2) has the effect of changing the state variable numbering order from left-to-right to right-to-left, and vice versa.)

The gain matrix g is thus given by

$$g = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 - K/\bar{M} \\ \bar{a}_3 \\ \bar{a}_4 \end{bmatrix} = \begin{bmatrix} \bar{a}_4 \\ \bar{a}_3 \\ \bar{a}_2 - K/\bar{M} \\ \bar{a}_1 \end{bmatrix}$$

A suitable pole "constellation" for the closed-loop process might be a Butterworth pattern as discussed in Sec. 6.5. To achieve this pattern the characteristic polynomial should be of the form

$$\bar{D}(s) = s^4 + (1+\sqrt{3})\Omega s^3 + (2+\sqrt{3})\Omega^2 s^2 + (1+\sqrt{3})\Omega^3 s + \Omega^4$$

Thus

$$\bar{a}_1 = (1+\sqrt{3})\Omega$$
$$\bar{a}_2 = (2+\sqrt{3})\Omega^2$$
$$\bar{a}_3 = (1+\sqrt{3})\Omega^3$$
$$\bar{a}_4 = \Omega^4$$

Thus the gain matrix g is given by

$$g = \begin{bmatrix} \Omega^4 \\ (1+\sqrt{3})\Omega^3 \\ (2+\sqrt{3})\Omega^2 - K/\bar{M} \\ (1+\sqrt{3})\Omega \end{bmatrix}$$

Source: Friedland, *Control System Design*, Chapter 6

Next Time…

- **System Identification**
- **Information Theory**

- **Last Lecture:** Exam Review!
  Including:
  – Key slides from all lectures
  – Key charts from all lectures
  – Exam Hints!

- Almost Done! ☺

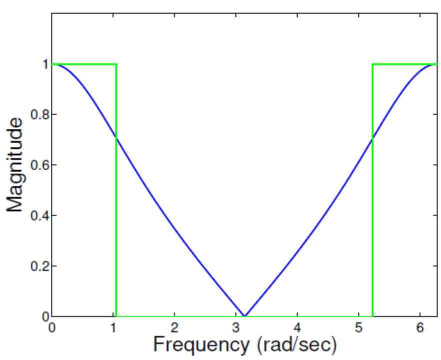
Encore State Space Example 5:

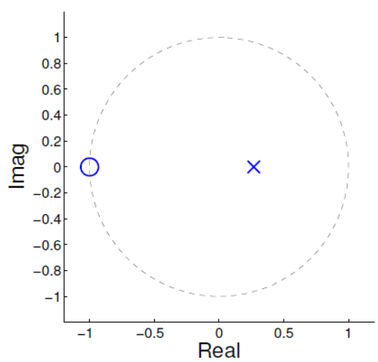
Can you use this for more than Control?

YES!

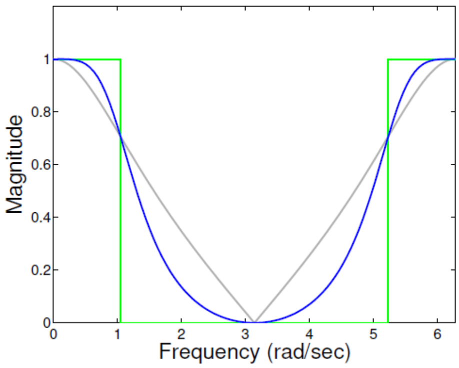
Discrete Time Butterworth Filters

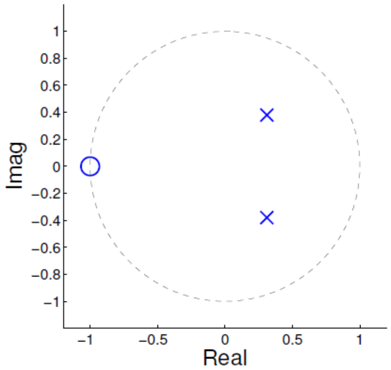
“Maximally-flat filter”. Sacrifice sharpness to have flat response in pass band and stop band.



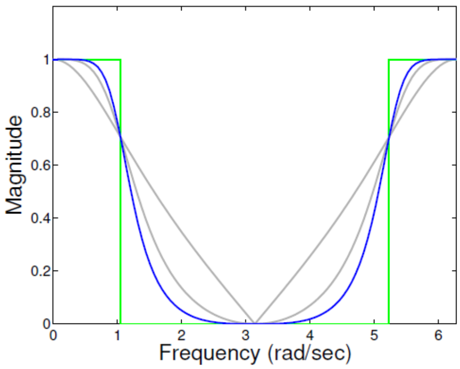


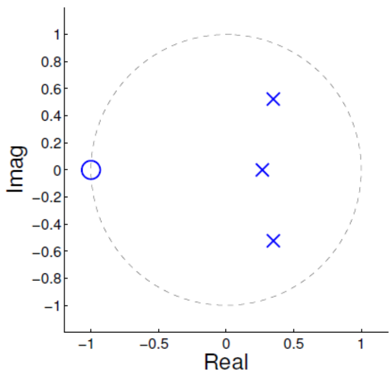
“Maximally-flat filter”. Sacrifice sharpness to have flat response in pass band and stop band.





“Maximally-flat filter”. Sacrifice sharpness to have flat response in pass band and stop band.





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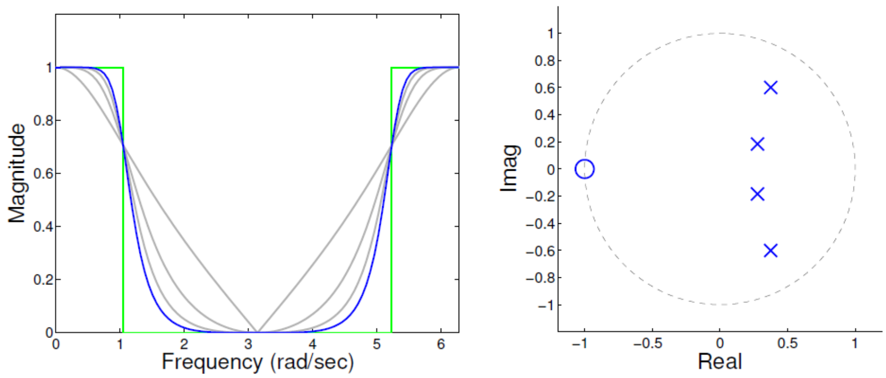

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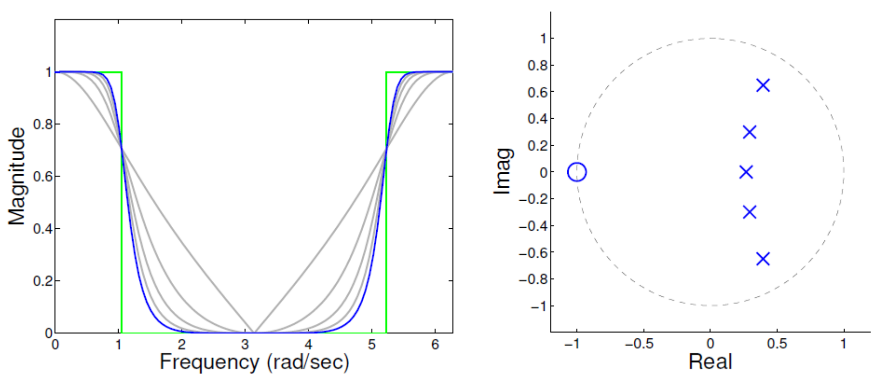

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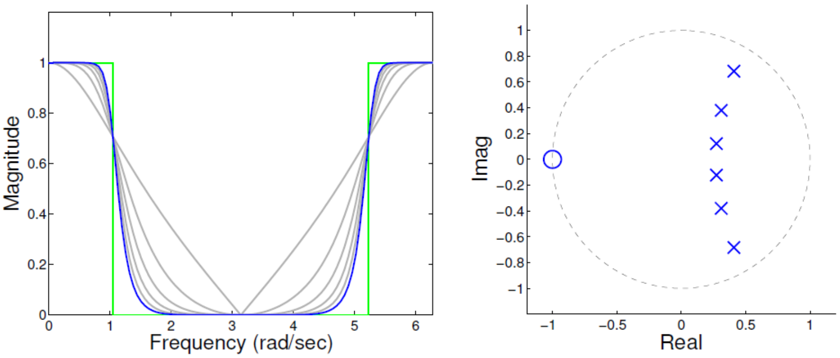

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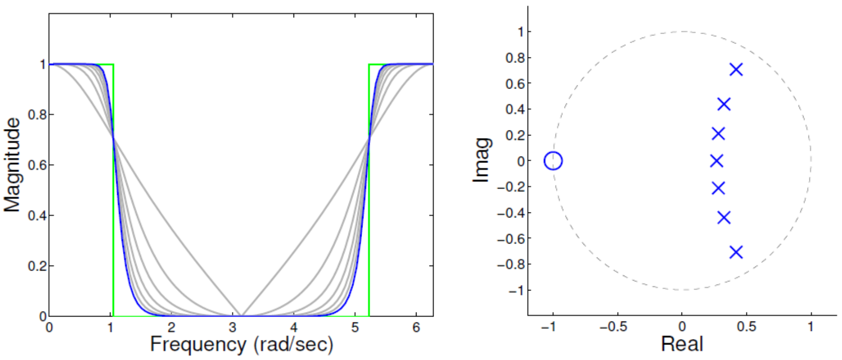

How?

- Constrained Least-Squares …

One formulation: Given $x[0]$

$$\underset{u[0],u[1],\ldots,u[N]}{\text{minimize}} \quad ||\vec{u}||^2, \quad \text{where } \vec{u} = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N] \end{bmatrix}$$

$$\text{subject to} \quad x[N] = 0.$$

Note that

$$x[n] = A^n x[0] + \sum_{k=0}^{n-1} A^{(n-1-k)} B u[k],$$

so this problem can be written as

$$\underset{x_{ls}}{\text{minimize}} \, ||A_{ls} x_{ls} - b_{ls}||^2 \quad \text{subject to} \quad C_{ls} x_{ls} = D_{ls}.$$

Break ☺