http://elec3004.com

Shaping the Dynamic Response

ELEC 3004: **Systems**: Signals & Controls
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Lecture 22

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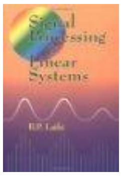
Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	Systems: Linear Differential Systems
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	Aliasing & Antialiasing
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	PS 1: Q & A
7	10-Apr	Digital Windows
	12-Apr	Digital Filter (FIR)
8	17-Apr	Active Filters & Estimation
	19-Apr	Holiday
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation & PID Control
10	8-May	State-Space Control
	10-May	Guest Lecture: FFT
11	15-May	Advanced PID & & FFT Processes
	17-May	State Space Control System Design
12	**22-May**	**Shaping the Dynamic Response**
	24-May	Stability and Examples
13	29-May	System Identification & Information Theory & Information Space
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

G. Franklin,
J. Powell,
M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990

Today

➔ **State-space** ← **[A stately idea! ☺]**

- **FPW**
  - **Chapter 4:** Discrete Equivalents to Continuous
  - Transfer Functions: The Digital Filter

- **FPW**
  - Chapter 5

Next Time

Even More Online Reading Materials


CHAPTER SIX

SHAPING THE DYNAMIC RESPONSE

6.1 INTRODUCTION


CHAPTER THREE

DYNAMICS OF LINEAR SYSTEMS

3.1 DIFFERENTIAL EQUATIONS REVISITED

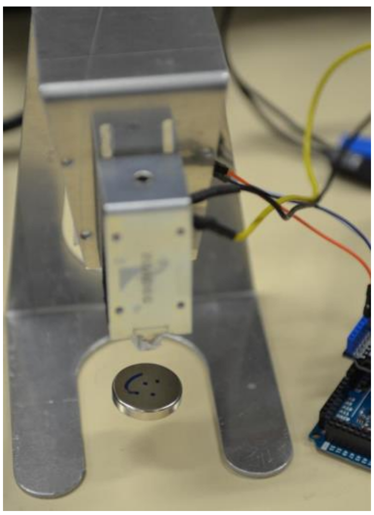
- Friedland, *Control System Design* Ch. 6 and 3

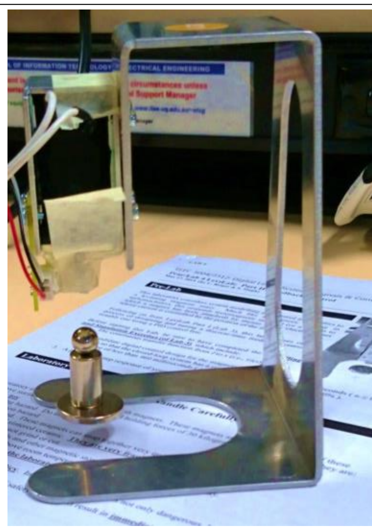
➔ http://robotics.itee.uq.edu.au/~elec3004/tutes.html

Advanced PID –
Integrator Wind-Up
(Non-Linear Effects [e.g. saturation])

& the Levi-Lab

This WEEK: Lab 5 – LeviLab *Encore!*



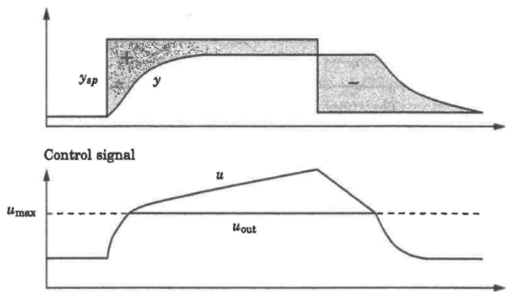


- We will run a combined Lab/Tutorial Session ☺

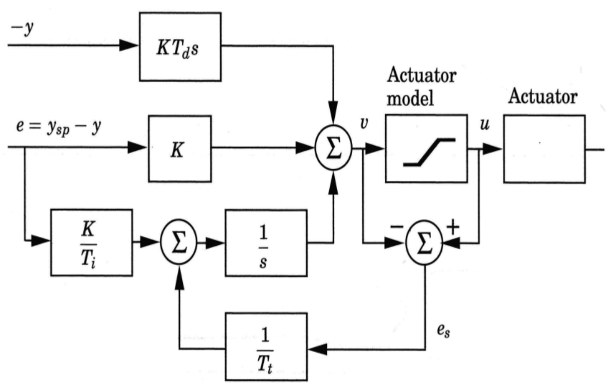
Integrator Wind-Up

Wind-Up:

- A non-linear effect: motor limitations (speed, hysteresis, etc.) / **saturation**
- When this happens the feedback loop is broken and the system runs as an open loop because the actuator will remain at its limit independently of the process output.
- If a controller with integrating action is used, the error may continue to be integrated if the algorithm. is not properly designed. This means that the integral term may become very large or, colloquially, it "winds up."
- Ex:



An Anti Wind-Up Mechanism:



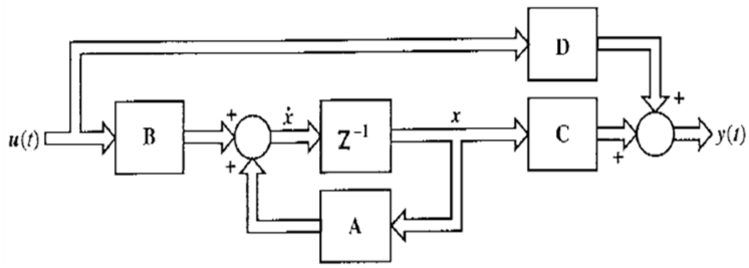
Shaping the Dynamic Response: SISO (Friedland Chapter 6)

Last Week: Solving State-Space:

$$\dot{x} = f(x, u, t)$$

$$\to \dot{x} = Ax + Bu$$

$$\to y = Cx + Du$$



Last Week: State-space Control Design

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:

$$u = -\mathbf{K}\boldsymbol{x}$$

such that $\det(z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K}) = \alpha_c(z)$

where $\boldsymbol{\alpha_c(z)}$ is the desired control characteristic equation

Q:How to get K?

A: *Pole-Placement*
(i.e., engineer $\alpha_c(z)$)

Pole Placement

Pole placement: Big idea:

- Arbitrarily select the desired root locations of the closed-loop system and see if the approach will work.

- AKA: full state feedback
  ∵ enough parameters to influence all the closed-loop poles

- Finding the elements of $\boldsymbol{k}$ so that the roots are in the desired locations. Unlike classical design, where we iterated on parameters in the compensator (hoping) to find acceptable root locations, the full state feedback, pole-placement approach guarantees success and allows us to arbitrarily pick any root locations, providing that n roots are specified for an n^{th}-order system.

Pole Placement

- Given:

$$z_i = \beta_1, \beta_2, \beta_3, \ldots$$

- This gives the desired control-characteristic equation as:

$$a_c(z) = (z - \beta_1)(z - \beta_2)(z - \beta_3) \ldots =$$

- Now we "just solve" for **K** and "bingo" ☺

➢ Recall that the modal form is given by:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)} = b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n} \Leftrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & 0 \\ & -p_2 & & \\ & & \ddots & \\ 0 & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u \quad y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

Pole Placement

- Start with a simple feedback control law ("controller")

$$u = -\mathbf{K}\mathbf{x} = -[K_1 K_2 \ldots] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

- It's actually a regulator
  ∵ it does not allow for a reference input to the system.
  (there is no "reference" **r** (r = 0))

- Substitute in the difference equation

$$x(k+1) = \Phi x(k) - \Gamma K x(k)$$

- $\mathcal{Z}$ Transform:

$$(zI - \Phi + \Gamma K)X(z) = 0$$

➔Characteristic Eqn: $\det|zI - \Phi + \Gamma K| = 0$

⇨ A polynomial of ***z***, that we can expand & *solve for* ***K***

Pole Placement Example (FPW p. 241)

Example 6.1: Suppose we want to design a control law for the satellite attitude-control system described by (2.45) with $\boldsymbol{x} = [\boldsymbol{x}_1 \;\; \boldsymbol{x}_2]$. Example 2.13 showed that the discrete model for this system is

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Gamma} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}.$$

"Kinematic" State-Transition Matrix

We want to pick z-plane roots of the closed-loop characteristic equation so that the equivalent s-plane roots have a damping ratio of $\zeta = 0.5$ and real part of $s = -1.8$ rad/sec (i.e., $s = -1.8 \pm j3.12$ rad/sec). Using $z = e^{sT}$ with a sample period of $T = 0.1$ sec, we find that $z = 0.8 \pm j0.25$, as shown in Fig. 6.1. The desired characteristic equation is then

$$z^2 - 1.6z + 0.70 = 0, \tag{6.9}$$

and the evaluation of (6.7) for any control law $\mathbf{K}$ leads to

$$\det \left| z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} [K_1 \quad K_2] \right| = 0$$

or

$$z^2 + (TK_2 + (T^2/2)K_1 - 2)z + (T^2/2)K_1 - TK_2 + 1 = 0. \tag{6.10}$$

Pole Placement Example (FPW p. 241)

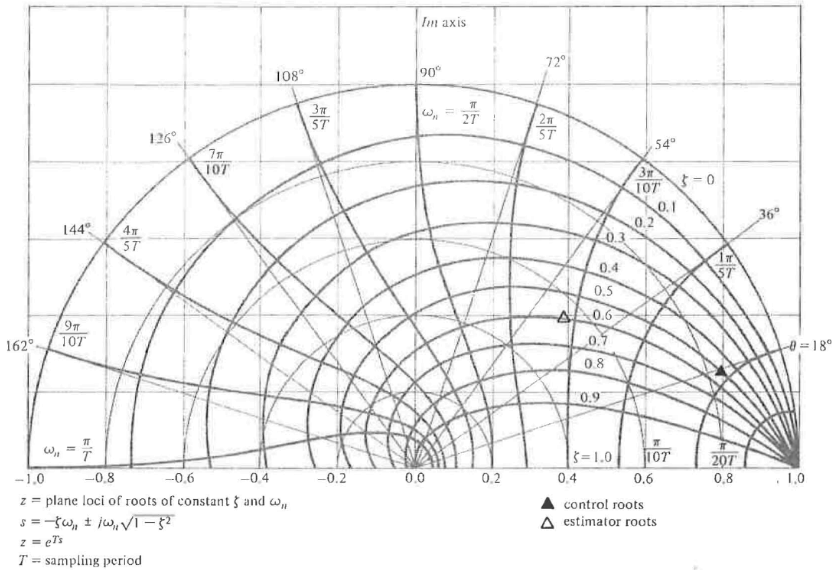
Equating coefficients in (6.9) and (6.10) with like powers of z, we obtain two simultaneous equations in the two unknown elements of $\mathbf{K}$:

$$TK_2 + (T^2/2)K_1 - 2 = -1.6,$$
$$(T^2/2)K_1 - TK_2 + 1 = 0.70,$$

which are easily solved for the coefficients and evaluated for $T = 0.1$ sec:

$$K_1 = \frac{0.10}{T^2} = 10, \qquad K_2 = \frac{0.35}{T} = 3.5.$$

Pole Placement: Graphing K_1 and K_2



Approach II: Ackermann's Formula (FPW p. 245)

- Gains maybe approximated with:

$$\boxed{\mathbf{K} = [0 \ldots 0 \quad 1][\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \quad \mathbf{\Phi}^2\mathbf{\Gamma} \ldots \mathbf{\Phi}^{n-1}\mathbf{\Gamma}]^{-1}\alpha_c(\mathbf{\Phi}),}$$

- Where: C = controllability matrix, n is the order of the system (or number of state elements) and α_c:

$$\mathcal{C} = [\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \ldots]$$

$$\alpha_c(\mathbf{\Phi}) = \mathbf{\Phi}^n + \alpha_1\mathbf{\Phi}^{n-1} + \alpha_2\mathbf{\Phi}^{n-2} + \cdots + \alpha_n\mathbf{I},$$

  - α_i: coefficients of the desired characteristic equation

$$\alpha_c(z) = |\, z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K} \,| = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n.$$

Ackermann's Formula Example (FPW p.246)

Example 6.2: Applying Ackermann's formula to the satellite attitude-control system of Example 6.1, we find from (6.9) that

$$\alpha_1 = -1.6, \qquad \alpha_2 = +0.70,$$

and therefore

$$\alpha_c(\mathbf{\Phi}) = \begin{bmatrix} 1 & 2T \\ 0 & 1 \end{bmatrix} - 1.6 \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + 0.70 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix}.$$

Furthermore, we find that

$$[\mathbf{\Gamma} \;\; \mathbf{\Phi\Gamma}] = \begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix}$$

and

$$[\mathbf{\Gamma} \;\; \mathbf{\Phi\Gamma}]^{-1} = 1/T^2 \begin{bmatrix} -1 & +3T/2 \\ 1 & -T/2 \end{bmatrix},$$

and finally

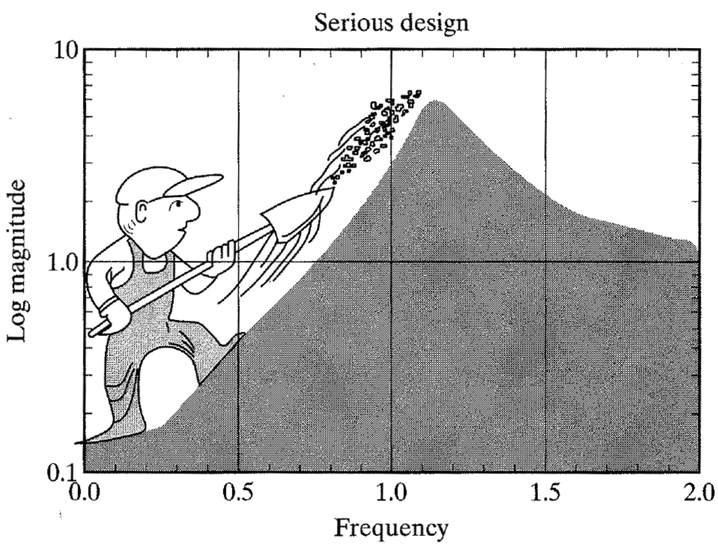
$$\mathbf{K} = [K_1 \;\; K_2] = (1/T^2)[0 \;\; 1] \begin{bmatrix} -1 & 3T/2 \\ 1 & -T/2 \end{bmatrix} \begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix};$$

therefore

$$[K_1 \;\; K_2] = \frac{1}{T^2}[0.1 \;\; 0.35T]$$
$$= [10 \quad 3.5],$$

which is the same result as that obtained earlier.

Meaning…



SDR: Introduction [Extended Reading…]

6.1 INTRODUCTION

At last we have arrived at the point of using state-space methods for control system design. In this chapter we will develop a simple method of designing a control system for a process in which all the state variables are accessible for measurement—the method known as *pole-placement*. We will find that in a controllable system, with all the state variables accessible for measurement, it is possible to place the closed-loop poles anywhere we wish in the complex s plane. This means that we can, in principle, completely specify the closed-loop dynamic performance of the system. In principle, we can start with a sluggish open-loop system and force it to behave with alacrity; in principle, we can start with a system that has very little open-loop damping and provide any amount of damping desired. Unfortunately, however, what can be attained in principle may not be attainable in practice. Speeding the response of a sluggish system requires the use of large control signals which the actuator (or power supply) may not be capable of delivering. The consequence is generally that the actuator saturates at the largest signal that it can supply. In some instances the system behavior may be acceptable in spite of the saturation. But in other cases the effect of saturation is to make the closed-loop system unstable. It is usually not possible to alter open-loop dynamic behavior very drastically without creating practical difficulties.

Adding a great deal of damping to a system having poles near the imaginary axis is also problematic, not only because of the magnitude of the control signals needed, but also because the control system gains are very sensitive to the location of the open-loop poles. Slight changes in the open-loop pole

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SDR: Introduction [Extended Reading…] [2]

location may cause the closed-loop system behavior to be very different from that for which it is designed.

We will first address the design of a regulator. Here the problem is to determine the gain matrix G in a linear feedback law

$$u = -Gx \tag{6.1}$$

which shapes the dynamic response of the process in the absence of disturbances and reference inputs. Afterward we shall consider the more general problem of determining the matrices G and G_0 in the linear control law

$$u = -Gx - G_0 x_0 \tag{6.2}$$

where x_0 is the vector of exogenous variables. The reason it is necessary to separate the exogenous variables from the process state x, rather than deal directly with the metastate

$$\mathbf{x} = \begin{bmatrix} x \\ \hline x_0 \end{bmatrix} \tag{6.3}$$

introduced in Chap. 5, is that in developing the theory for the design of the gain matrix, we must assume that the underlying process is *controllable*. Since the exogenous variables are not true state variables, but additional inputs that cannot be affected by the control action, they cannot be included in the state vector when using a design method that requires controllability.

SDR: Introduction [Extended Reading…] [3]

The assumption that all the state variables are accessible to measurement in the regulator means that the gain matrix G in (6.1) is permitted to be any function of the state x that the design method requires. In most practical instances, however, the state variables are not all accessible for measurement. The feedback control system design for such a process must be designed to use only the measurable output of the process

$$y = Cx$$

where y is a vector of lower dimension than x. In some cases it may be possible to determine the gain matrix G_y for a control law of the form

$$u = -G_y y \tag{6.4}$$

which produces acceptable performance. But more often it is not possible to do so. It is then necessary to use a more general feedback law, of the form

$$u = -G\hat{x} \tag{6.5}$$

where $\hat{x}$ is the state of an appropriate dynamic system known as an "observer." The design of observers is the subject of Chap. 7. And in Chap. 8, we shall show that when a feedback law of the form of (6.5) is used with a properly designed observer, the dynamic properties of the overall system can be specified at will, subject to practical limitations on control magnitude and accuracy of implementation.

Design of regulators for single-input, single-output systems

6.2 DESIGN OF REGULATORS FOR SINGLE-INPUT, SINGLE-OUTPUT SYSTEMS

The present section is concerned with the design of a gain matrix

$$G = g' = [g_1, g_2, \ldots, g_k] \tag{6.6}$$

for the single-input, single-output system

$$\dot{x} = Ax + Bu \tag{6.7}$$

where

$$B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \tag{6.8}$$

With the control law $u = -Gx = -g'x$ (6.7) becomes

$$\dot{x} = (A - bg')x$$

Our objective is to find the matrix $G = g'$ which places the poles of the closed-loop dynamics matrix

$$A_c = A - bg' \tag{6.9}$$

Design of regulators for single-input, single-output systems

at the locations desired. We note that there are k gains $g_1, g_2, \ldots, g_k$ and k poles for a kth order system, so there are precisely as many gains as needed to specify each of the closed-loop poles.

One way of determining the gains would be to set up the characteristic polynomial for A_c:

$$|sI - A_c| = |sI - A + bg'| = s^k + \bar{a}_1 s^{k-1} + \cdots + \bar{a}_k \tag{6.10}$$

The coefficients $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$ of the powers of s in the characteristic polynomial will be functions of the k unknown gains. Equating these functions to the numerical values desired for $\bar{a}_1, \ldots, \bar{a}_k$ will result in k simultaneous equations the solution of which will yield the desired gains $g_1, \ldots, g_k$.

This is a perfectly valid method of determining the gain matrix g', but it entails a substantial amount of calculation when the order k of the system is higher than 3 or 4. For this reason, we would like to develop a direct formula for g in terms of the coefficients of the open-loop and closed-loop characteristic equations.

If the original system is in the companion form given in (3.90), the task is particularly easy, because

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6.11}$$

Design of regulators for single-input, single-output systems

$$bg' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [g_1, g_2, \ldots, g_k] = \begin{bmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence

$$A_c = A - bg' = \begin{bmatrix} -a_1 - g_1 & -a_2 - g_2 & \cdots & -a_k - g_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The gains $g_1, \ldots, g_k$ are simply added to the coefficients of the open-loop A matrix to give the closed-loop matrix A_c. This is also evident from the block-diagram representation of the closed-loop system as shown in Fig. 6.1. Thus for a system in the companion form of Fig. 6.1, the gain matrix elements are given by

$$a_i + g_i = \hat{a}_i \qquad i = 1, 2, \ldots, k$$

or

$$g = \hat{a} - a \tag{6.12}$$

where

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \qquad \hat{a} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_k \end{bmatrix} \tag{6.13}$$

Design of regulators for single-input, single-output systems

are vectors formed from the coefficients of the open-loop and closed-loop characteristic equations, respectively.

The dynamics of a typical system are usually not in companion form. It is necessary to transform such a system into companion form before (6.12) can be used. Suppose that the state of the transformed system is $\bar{x}$, achieved through the transformation

$$\bar{x} = Tx \tag{6.14}$$

Then, as shown in Chap. 3,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \tag{6.15}$$

where

$$\bar{A} = TAT^{-1} \quad \text{and} \quad \bar{b} = Tb$$

For the transformed system the gain matrix is

$$\bar{g} = \hat{a} - \bar{a} = \hat{a} - a \tag{6.16}$$

since $\bar{a} = a$ (the characteristic equation being invariant under a change of state variables). The desired control law in the original system is

$$u = -g'x = -g'T^{-1}\bar{x} = -\bar{g}'\bar{x} \tag{6.17}$$

From (6.17) we see that

$$\bar{g}' = g'T^{-1}$$

Thus the gain in the original system is

$$g = T'\bar{g} = T'(\hat{a} - a) \tag{6.18}$$

Design of regulators for single-input, single-output systems

In words, the desired gain matrix for a general system is the difference between the coefficient vectors of the desired and actual characteristic equation, premultiplied by the inverse of the transpose of the matrix T that transforms the general system into the companion form of (3.90), the A matrix of which has the form (6.11).

The desired matrix T is obtained as the product of two matrices U and V:

$$T = VU \tag{6.19}$$

The first of these matrices transforms the original system into an intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} \tag{6.20}$$

in the second companion form (3.107) and the second transformation U transforms the intermediate system into the first companion form.

Consider the intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u \tag{6.21}$$

with $\tilde{A}$ and $\tilde{b}$ in the form of (3.107). Then we must have

$$\bar{A} = U\tilde{A}U^{-1} \quad \text{and} \quad \bar{b} = U\tilde{b} \tag{6.22}$$

Design of regulators for single-input, single-output systems

The desired matrix U is precisely the inverse of the controllability test matrix Q of Sec. 5.4. To prove this fact, we must show that

$$U^{-1}\tilde{A} = AU^{-1} \tag{6.23}$$

or

$$Q\tilde{A} = AQ \tag{6.24}$$

Now, for a single-input system

$$Q = [b, Ab, \ldots, A^{k-1}b]$$

Thus, with $\tilde{A}$ given by (3.107), the left-hand side of (6.23) is

$$Q\tilde{A} = [b, Ab, \ldots, A^{k-1}b]\begin{bmatrix} 0 & 0 & \cdots & -a_k \\ 1 & 0 & \cdots & -a_{k-1} \\ 0 & 1 & \cdots & -a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_1 \end{bmatrix}$$

$$= [Ab, A^2b, \ldots, A^{k-1}b, -a_kb - a_{k-1}Ab - \cdots - a_kA^{k-1}b] \tag{6.25}$$

The last term in (6.25) is

$$(-a_kI - a_{k-1}A - \cdots - a_kA^{k-1})b \tag{6.26}$$

Now, by the Cayley-Hamilton theorem, (see Appendix):

$$A^k = -a_1A^{k-1} - a_2A^{k-2} - \cdots - a_kI$$

so (6.26) is A^kb. Thus the left-hand side of (6.24) as given by (6.25) is

$$Q\tilde{A} = [Ab, A^2b, \ldots, A^kb] = A[b, Ab, \ldots, A^{k-1}b] = AQ$$

which is the desired result.

Ex: Servo Motor Control

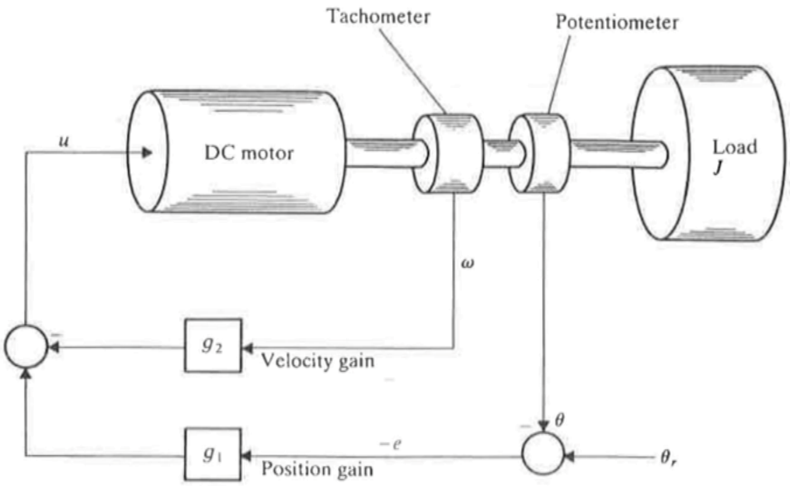


Figure 6.2 Implementation of an instrument servo.

Ex: Servo Motor Control [2]

Example 6A Instrument servo A dc motor driving an inertial load constitutes a simple instrument servo for keeping the load at a fixed position.

As shown in Chap. 2 (Example 2B), the state-space equations for the motor-driven inertia are

$$\dot{\theta} = \omega \tag{6A.1}$$

$$\dot{\omega} = -\alpha\omega + \beta u \tag{6A.2}$$

where θ is the angular position of the load, ω is the angular velocity, u is the applied voltage, and α and β are constants that depend on the physical parameters of the motor and load:

$$\alpha = -K^2/JR \qquad \beta = K/JR$$

If the desired position θ_r is a constant then we can define the servo error

$$e = \theta - \theta_r$$

Then

$$\dot{e} = \dot{\theta} - \dot{\theta}_r = \omega \qquad (\theta_r = \text{const}) \tag{6A.3}$$

and (6A.3) replaces (6A.1) to give

$$\begin{bmatrix} \dot{e} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u \tag{6A.4}$$

The angular position measurement can be instrumented by a potentiometer on the motor shaft and the angular velocity by a tachometer. Thus, the closed-loop system would have the configuration illustrated in Fig. 6.2. Note that the position gain is shown multiplying the negative of the system error which in turn is added to the control signal. This is consistent with the convention normally used for servos, wherein the position gain multiplies the difference $\theta_r - \theta$ between the reference and the actual positions. The quantity e defined above (6A.3) is the negative of the system error as normally defined in elementary texts.

The characteristic polynomial of the system is

$$|sI - A| = \begin{vmatrix} s & -1 \\ 0 & s+\alpha \end{vmatrix} = s^2 + \alpha s$$

Thus

$$a = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

The controllability test matrix Q and the matrix W are given respectively by

$$Q = [b, Ab] = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \qquad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

Ex: Servo Motor Control [3]

Thus

$$QW = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} = (QW)'$$

and

$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Thus the desired gain matrix, by the Bass-Gura formula (6.34), is

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 - \alpha \\ \bar{a}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_2/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix} \tag{6A.5}$$

where $\bar{a}_1$ and $\bar{a}_2$ are the coefficients of the desired characteristic polynomial.

While the above calculation illustrates the general procedure, the gains could have been more easily computed directly. For a control law of the form

$$u = -g_1 e - g_2 \omega$$

(6A.4) becomes

$$\dot{e} = \omega$$

$$\dot{\omega} = -g_1\beta e - (\alpha + \beta g_2)\omega$$

which has the closed-loop matrix

$$A_c = \begin{bmatrix} 0 & 1 \\ -g_1\beta & -(\alpha + g_2\beta) \end{bmatrix}$$

with the characteristic equation

$$|sI - A_c| = s^2 + (\alpha + g_2\beta)s + g_1\beta$$

Thus

$$\bar{a}_1 = \alpha + g_2\beta \qquad \bar{a}_2 = g_1\beta$$

Ex: Servo Motor Control [4]

or

$$g_1 = \bar{a}_2/\beta \qquad g_2 = (\bar{a}_1 - \alpha)/\beta$$

which is the same as (6A.5).

Note that the position and velocity gains g_1 and g_2, respectively, are proportional to the amounts we wish to move the coefficients from their open-loop positions. The position gain g_1 is necessary to produce a stable system: $\bar{a}_2 > 0$. But if the designer is willing to settle for $a_1 = \alpha$, i.e., to accept the open-loop damping, then the gain g_2 can be zero. This of course eliminates the need for a tachometer and reduces the hardware cost of the system. It is also possible to alter the system damping without the use of a tachometer, by using an estimate $\hat{\omega}$ of the angular velocity ω. This estimate is obtained by means of an observer as discussed in Chap. 7.

Break ☺

Control System Design: Obtaining a Time Response

From SS to Time Response — Impulse Functions

- Given: $\dot{x} = Ax + Bu$
- Solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Substituting $t_0 = 0$ into this:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Write the impulse as: $\mathbf{u}(t) = \delta(t)\mathbf{w}$
  - where **w** is a vector whose components are the magnitudes of **r** impulse functions applied at t=0

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)\mathbf{w}\,d\tau$$

$$= e^{\mathbf{A}t}\mathbf{x}(0-) + e^{\mathbf{A}t}\mathbf{B}\mathbf{w}$$

From SS to Time Response — Step Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = \boldsymbol{k}$
  Where **k** is a vector whose components are the magnitudes of r step functions applied at t=0.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{k}\,d\tau$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[\int_0^t \left(\mathbf{I} - \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} - \cdots\right) d\tau\right]\mathbf{B}\mathbf{k}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{\mathbf{A}^2t^3}{3!} - \cdots\right)\mathbf{B}\mathbf{k}$$

  – Assume A is non-singular

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[-(\mathbf{A}^{-1})(e^{-\mathbf{A}t} - \mathbf{I})\right]\mathbf{B}\mathbf{k}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}\mathbf{k}$$

From SS to Time Response — Ramp Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = t\boldsymbol{v}$
  Where **v** is a vector whose components are magnitudes of ramp functions applied at $t = 0$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\tau\mathbf{v}\,d\tau$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\tau\,d\tau\,\mathbf{B}\mathbf{v}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\frac{\mathbf{I}}{2}t^2 - \frac{2\mathbf{A}}{3!}t^3 + \frac{3\mathbf{A}^2}{4!}t^4 - \frac{4\mathbf{A}^3}{5!}t^5 + \cdots\right)\mathbf{B}\mathbf{v}$$

  – Assume A is non-singular

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + (\mathbf{A}^{-2})(e^{\mathbf{A}t} - \mathbf{I} - \mathbf{A}t)\mathbf{B}\mathbf{v}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + \left[\mathbf{A}^{-2}(e^{\mathbf{A}t} - \mathbf{I}) - \mathbf{A}^{-1}t\right]\mathbf{B}\mathbf{v}$$

Example: Obtain the Step Response

- Given:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u(t) = 1(t)$$
- Solution:
$$\mathbf{A} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$
$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+1 & 0.5 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2 + s + 0.5} \begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{s + 0.5 - 0.5}{(s+0.5)^2 + 0.5^2} & \dfrac{-0.5}{(s+0.5)^2 + 0.5^2} \\ \dfrac{1}{(s+0.5)^2 + 0.5^2} & \dfrac{s + 0.5 + 0.5}{(s+0.5)^2 + 0.5^2} \end{bmatrix}$$
$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t) & -e^{-0.5t}\sin 0.5t \\ 2e^{-0.5t}\sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t) \end{bmatrix}$$
  - Set k=1, x(0)=0:
$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}k$$
$$= \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}$$
$$= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t}\sin 0.5t \end{bmatrix}$$
$$= \begin{bmatrix} e^{-0.5t}\sin 0.5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1 \end{bmatrix}$$
$$y(t) = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 = e^{-0.5t}\sin 0.5t$$

Example II: Obtain the Step Response

- Given:
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad u(t) = 1(t)$$
- Solution:
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$
$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$
$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$
  - Assume x(0)=0:
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Solving State Space Method II: Tustin's Method (Analog Emulation)

Tustin's method

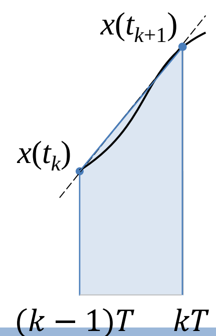
- Tustin uses a trapezoidal integration approximation (compare Euler's rectangles)
- Integral between two samples treated as a straight line:

$$u(kT) = \frac{T}{2}[x(k-1) + x(k)]$$

Taking the derivative, then z-transform yields:

$$s = \frac{2}{T}\frac{z-1}{z+1}$$

which can be substituted into continuous models



Matched pole-zero

- If $z = e^{sT}$, why can't we just make a direct substitution and go home?

$$\frac{Y(s)}{X(s)} = \frac{s+a}{s+b} \Rightarrow \frac{Y(z)}{X(z)} = \frac{z-e^{-aT}}{z-e^{-bT}}$$

- Kind of!
  - Still an approximation
  - Produces quasi-causal system (hard to compute)
  - Fortunately, also very easy to calculate.

Matched pole-zero

The process:

1. Replace continuous poles and zeros with discrete equivalents:

$$(s+a) \Rightarrow (z-e^{-aT})$$

2. Scale the discrete system DC gain to match the continuous system DC gain

3. If the order of the denominator is higher than the enumerator, multiply the numerator by $(z+1)$ until they are of equal order*

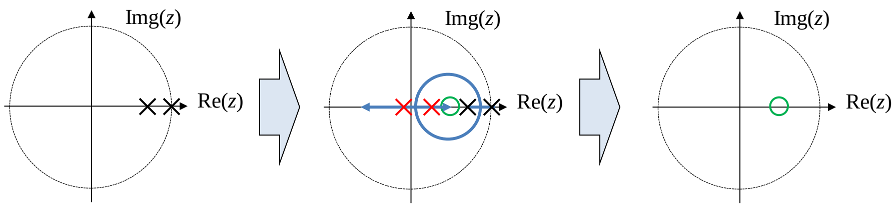
   * This introduces an averaging effect like Tustin's method

Modified matched pole-zero

- We're prefer it if we didn't require instant calculations to produce timely outputs
- Modify step 2 to leave the dynamic order of the numerator one less than the denominator
  - Can work with slower sample times, and at higher frequencies

Discrete design process

1. Derive the dynamic system model ODE
2. Convert it to a discrete transfer function
3. Design a digital compensator
4. Implement difference equations in software
5. Platypus Is Divine!



Discrete design process

- Handy rules of thumb:
  - Sample rates can be as low as twice the system bandwidth
    - but 5 to 10× for "stability"
    - 20 to 30 × for better performance
  - A zero at $z = -1$ makes the discrete root locus pole behaviour more closely match the s-plane
  - Beware "dirty derivatives"
    - dy/dt terms derived from sequential digital values are called 'dirty derivatives' – these are especially sensitive to noise!
    - Employ actual velocity measurements when possible

Solving State Space Method III:

The Direct Method of Digital Controls –

NOT to be confused with
Controller Emulation
(e.g., Tustin's Method)

Direct Design Method Of Ragazzini (See also: FPW 5.7 pp.216-222)

Start with 3 Discrete Transfer Functions:

- **G(z)**: TF[1] of a **plant + a hold** (e.g., from a ZOH)
- **D(z)**: A **controller** TF to do the job (what we want here)
- **H(z)**: The final desired TF between **R** (reference) and **Y** (output)
- Thus[2]:

$$\boxed{\begin{aligned} H(z) &= \frac{DG}{1+DG} \\ \rightarrow D(z) &= \frac{1}{G}\frac{H}{1-H} \end{aligned}}$$

- This calls for a **D(z)** that will **cancel the plant effects** and that will **add whatever is necessary to give the desired result**. The problem is to discover and implement constraints on **H(z)** so that we do not ask for the impossible.
  - This implies that we **need some constraints** on both **H(z)** and **D(z)**

1: Transfer Function

2: Mental Quiz: What does **1+DG** say about the sign of the feedback (positive or negative)? That is, what is the characteristic equation for a system with positive feedback?

Direct Design Method Of Ragazzini [2]: Design Constraints: I. Causality

- Remember/Recall an Interesting Point:
  - From z-transform theory we know that **if D(z) is causal**, **then** as $z \rightarrow \infty$ its transfer function is well behaved & it **does not** have a pole at infinity.

- $D(z) = \frac{1}{G}\frac{H}{1-H}$ implies that **if** G**(z)** = 0 (at ∞), **then** **D(z)** would have a pole (at ∞) **unless H(z)** cancels it.

∴

H(z) must have a zero (at ∞) of the same order as G**(z)**'s 0s (at ∞)

→ Which means: **If** there is a lag in the plant (**G(z) starts with** z^{-l}) **then** causality requires that the delay of **H(z)** is that the closed-loop system must be at least as long a delay of the plant.

(Whoa! It might sound deep, but it's rather intuitive ☺)

Direct Design Method Of Ragazzini [3]: Design Constraints: II. Stability

- The characteristic equation and the closed loop roots:

$$1 + D(z)G(z) = 0$$

- Define[3] $D = \frac{c}{d}$ and G= $\frac{b}{a}$ ➔ $ad + bc = 0$
- Define $z - \alpha$ as a pole of **G(z)** and a <u>common factor</u> in DG that represents **D(z)** cancelling a pole/zero of **G(z)**.
- **Then** this common factor *remains a factor of the characteristic polynomial.*
- **If** this factor is outside the unit circle, **then** the system is unstable!

∴

1-H(z) must contain as zeros
all the poles of **G(z)** that are outside the unit circle &
H(z) must contain as zeros
all the zeros of G(z) that are outside the unit circle

3: Note the switching of the "alphabetical-ness" of these two fractions

Direct Design Method Of Ragazzini [4]: Design Constraints: III. Steady State Accuracy

- The error from **H(z)** is given by:

$$\mathrm{E(z)} = \mathrm{R(z)}\big(1 - \mathrm{H(z)}\big)$$

- **If** the system is "Type 1" (with a constant velocity/first derivative (K_v)
  - **Then**[4] $E_{ss}^{Step} = 0$ and $E_{ss}^{Ramp} = {}^{1}/_{K_v}$

∴

$$\mathrm{H(z)} = 1$$

&

$$-\mathrm{T_s}\left.\frac{dH(z)}{d(z)}\right|_{z=1} = \frac{1}{K_v}\mathrm{H(z)} = 1$$

4: E_{ss}: steady-state error

Direct Design Method Of Ragazzini [5]: An Example

- Consider the plant: $s^2 + s + 1 = 0$

With T_s=1 → z-Transform: $z^2 + 0.786z + 0.368$=0

- Let's design this **system** such that
  - $K_v = 1$
  - Poles at the roots of the plant equation & additional poles as needed

$$\rightarrow \mathrm{H(z)} = \frac{b_0+b_1z^{-1}+b_2z^{-2}+b_3z^{-3}+\cdots}{1-0.786z^{-1}+0.368\,z^{-2}}$$

I. Causality: $\mathrm{H(z)}|_{z=\infty} = 0 \rightarrow b_0 = 0$

II. Stability: All poles/zeros of G(z) are in the unit circle
  - except for b_0, which is taken care of by $b_0 = [Const] = 0$

III. Tracking:
  - $\mathrm{H}(1) = b_1 + b_2 + b_3 + \cdots = 1 \cdot (1 - 0.786 + 0.368)$ &
  - $-\{1\} \left.\frac{dH(z)}{d(z^{-1})}\right|_{z=1} = \frac{1}{\{1\}} \rightarrow \frac{b_1+2b_2+3b_3+\cdots\ -[-.05014]}{(1-0.786+0.368)}$ (note the z^{-1})
  - Truncate the number of unknowns to 2 "zeros" … thus solve for b_1 and b_2 (& set b_3,b_4,…=0)

$$\therefore \mathrm{H(z)} = \frac{b_1z+b_2}{z^2-0.786z+0.368} \rightarrow$$

$$D(z) = \frac{(z-1)(z-0.9048)(0.6321)}{(0.04837)(z+0.9672)}\,\frac{(z-0.07932)}{(z-1)(z-0.4180)}$$

$$= 13.07\,\frac{(z-0.9048)}{(z+0.9672)}\,\frac{(z-0.07932)}{(z-0.4180)}.$$

Next Time…

- **The "-Ity Lecture"**
  - **Observability | Stability**
- **Examples of Digital Feedback Control**

- Review:
  - Chapter 5 of FPW

- More Pondering??