http://elec3004.com

Systems Overview

ELEC 3004: **Systems**: Signals & Controls

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Lecture 2

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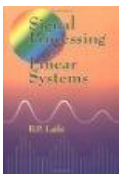
Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	Systems: Linear Differential Systems
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	Aliasing & Antialiasing
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	Digital Filter (FIR)
7	10-Apr	Digital Windows
	12-Apr	FFT
8	17-Apr	Active Filters & Estimation & Holiday
	19-Apr	Holiday
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation/PID
10	8-May	PID & State-Space
	10-May	State-Space Control
11	15-May	Digital Control Design
	17-May	Stability
12	22-May	State Space Control System Design
	24-May	Shaping the Dynamic Response
13	29-May	System Identification & Information Theory
	31-May	Summary and Course Review

Signals & Systems: A Primer!

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 1 (**Introduction to Signals and Systems**)
  - § 1.2: Classification of Signals
  - § 1.2: Some Useful Signal Operations
  - § 1.6 Systems
- Chapter B (Background)
  - B.5 Partial fraction expansion
  - B.6 Vectors and Matrices

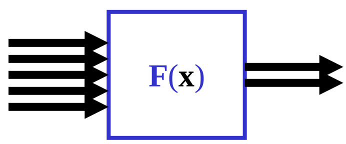
Modelling Ties Back with ELEC 2004

- Linear Circuit Theorems
- Operational Amplifiers
- Capacitors and Inductors, RL and RC Circuits
- AC Steady State Analysis
- AC Power, Frequency Response
- Laplace Transform
- Reduction of Multiple Sub-Systems
- Fourier Series and Transform
- Filter Circuits

➔ Linear Algebra is a Modelling Tools!
(**Modelling** means forecasting)

An Overview of Systems

- Today we are going to look at **F**(x)!



- F(x): System Model
  - The rules of operation that describe it's behaviour of a "system"
  - Predictive power of the responses
  - Analytic forms > Empirical ones
    - Analytic formula offer various levels of detail
    - Not everything can be experimented on *ad infinitum*
    - Also offer Design Intuition (let us devise new "systems")
    - Let's us do **analysis**! (determine the outputs for an input)
  - Various Analytic Forms
    - Constant, Polynomial, **Linear**, Nonlinear, Integral, **ODE**, PDE, Bayesian…

System Ter·mi·nol·o·gy
\ tər-mə-ˈnä-lə-jē \

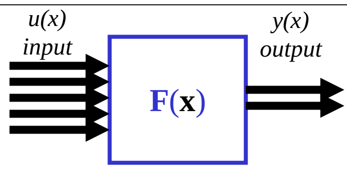
System Classifications/Attributes

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Linear Systems

- Model describes the relationship between the input **u**(x) and the output **y**(x)
- If it is a Linear System (wk 3):

$$y(t) = \int_0^t F(t-\tau)\,u(\tau)\,d\tau$$



- If it is also a (Linear and) **lumped**, it can be expressed **algebraically** as:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

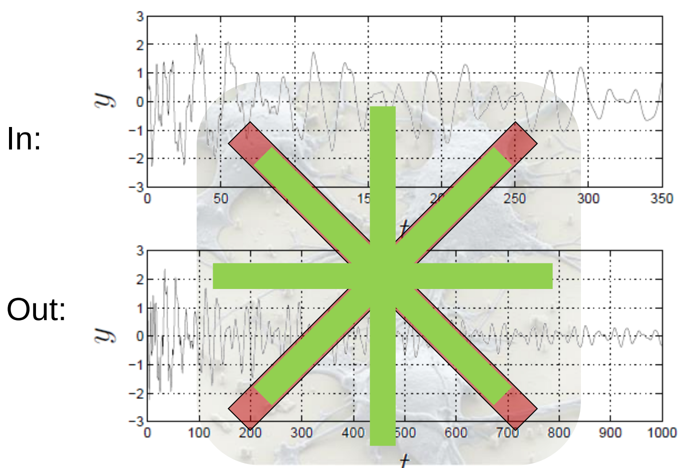
- If it is also (Linear and) **time invariant** the matrices can be reduced to:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

$\mathscr{L}$aplacian: $$y(s) = F(s)\,u(s)$$

WHY? This can help simplify matters…

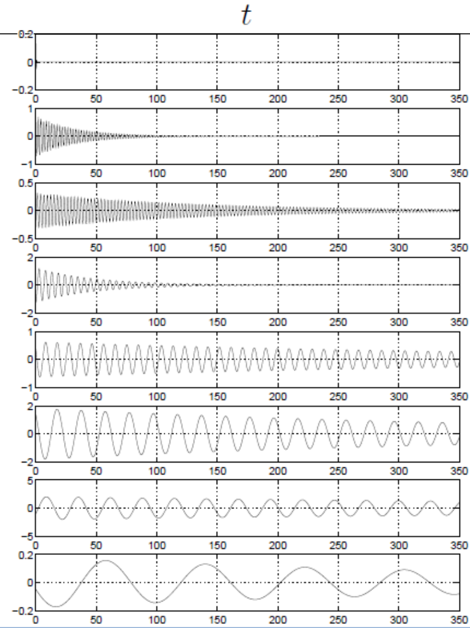
For Example: Consider the following system:



- How to model and predict (and later control) the output?

Source: EE263 (s.1-13)

This can help simplify matters…

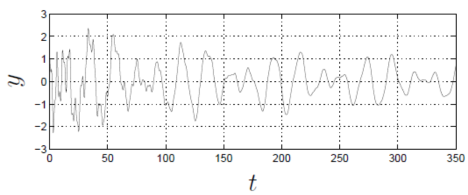


Source: EE263 (s.1-13)

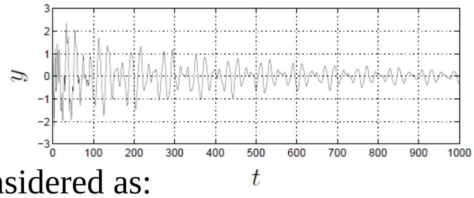
This can help simplify matters…

- That is:

In:



Out:



- May be considered as:

$$\dot{x} = Ax, \qquad y = Cx$$

- $x(t) \in \mathbb{R}^8$, $y(t) \in \mathbb{R}^1$ → 8-state, single-output system
- **No Control** ☹ ∵ **It's Autonomous** ☺ (∵ No input yet! ($u(t) = 0$))

Source: EE263 (s.1-13)

Linear Systems

Linearity:

- **A most desirable property for many systems to possess**
- Ex: Circuit theory, where it allows the powerful technique or voltage or current superposition to be employed.

Two requirements must be met for a system to be *linear:*

- *Additivity*
- *Homogeneity or Scaling*

Additivity **∪** ***Scaling*** **➔ Superposition**

Linear Systems: Additivity

- **Given** input $x_1(t)$ produces output $y_1(t)$ and input $x_2(t)$ produces output $y_2(t)$
- **Then** the input $x_1(t) + x_2(t)$ must produce the output $y_1(t) + y_2(t)$ for arbitrary $x_1(t)$ and $x_2(t)$
- Ex:
  - Resistor
  - Capacitor
- **Not** Ex:
  - $y(t) = \sin[x(t)]$

Linear Systems: Homogeneity or Scaling

- **Given** that $x(t)$ produces $y(t)$
- **Then** the scaled input $a \cdot x(t)$
  must produce the scaled output $a \cdot y(t)$
  for an arbitrary $x(t)$ and a
- Ex:
  - $y(t) = 2x(t)$
- **Not** Ex:
  - $y(t) = x^2(t)$
  - $y(t) = 2x(t) + 1$

Linear Systems: Superposition

- **Given** input $x_1(t)$ produces output $y_1(t)$
  and input $x_2(t)$ produces output $y_2(t)$
- **Then**: The linearly combined input

$$x(t) = ax_1(t) + bx_2(t)$$

  must produce the linearly combined output

$$y(t) = ay_1(t) + by_2(t)$$

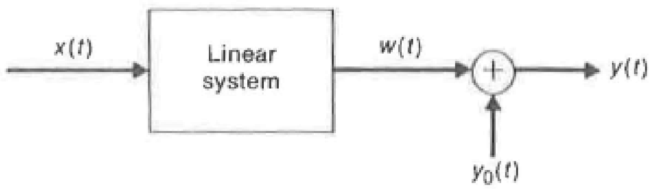
  for arbitrary a and b
- **Generalizing:**
  - Input: $x(t) = \sum_k a_k x_k(t)$
  - Output: $y(t) = \sum_k a_k y_k(t)$

Linear Systems: Superposition [2]

Consequences:

- Zero input for all time yields a zero output.
  - This follows readily by setting $a = 0$, then $0 \cdot x(t) = 0$

- DC output/Bias ➔ **Incrementally linear**
- Ex: $y(t) = [2x(t)] + [1]$
- Set offset to be added offset [Ex: $y_0(t)$=1]



"Dynamical" Systems… (→ Differential Equations)

- A system with a memory
  - Where past history (or derivative states) are **relevant** in determining the response
- Ex:
  - RC circuit: Dynamical
    - Clearly a function of the "capacitor's past" (initial state) and
    - Time! (charge / discharge)
  - R circuit: is memoryless ∵ the output of the system (recall V=IR) at some time **t** only depends on the input at time **t**

- Lumped/Distributed
  - Lumped: Parameter is constant through the process & can be treated as a "point" in space
- Distributed: System dimensions ≠ small over signal
  - Ex: waveguides, antennas, microwave tubes, etc.

Linear + Dynamical: A Type of Linear Systems:

- **LDS (Linear Dynamical System):**:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

Where:

- Continuous-time linear dynamical system (CT LDS):

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)$$

- $t \in \mathbb{R}$ denotes time
- $x(t) \in \mathbb{R}^n$ is the state (vector)
- $u(t) \in \mathbb{R}^m$ is the input or control
- $y(t) \in \mathbb{R}^p$ is the output

A Type of Linear Systems

- **LDS (Linear Dynamical System)**:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- $A(t) \in \mathbb{R}^{n \times n}$ is the dynamics matrix
- $B(t) \in \mathbb{R}^{n \times m}$ is the input matrix
- $C(t) \in \mathbb{R}^{p \times n}$ is the output or sensor matrix
- $D(t) \in \mathbb{R}^{p \times m}$ is the feedthrough matrix

➔ state equations, or "*m*-input, *n*-state, *p*-output' LDS

A Type of Linear Systems

- **LDS (Linear Dynamical System):**

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- **Time-invariant:** where A(t), B(t), C(t) and D(t) are **constant**
- **Autonomous:** there is no input *u* (B,D are irrelevant)
- **No Feedthrough**: D = 0

- SISO: *u(t)* and *y(t)* are scalars
- MIMO: *u(t)* and *y(t)*: They're vectors: Big Deal ?

To Recap: LDS & LTI-LTS

- **LDS (Linear Dynamical System):**

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- **LTI – LDS (Linear Time Invariant – LDS):**

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Discrete-time Linear Dynamical System

- **Discrete-time Linear Dynamical System (DT LDS)** has the form:

$$x(t+1) = A(t)x(t) + B(t)u(t), \qquad y(t) = C(t)x(t) + D(t)u(t)$$

- $t \in \mathbb{Z}$ denotes time index : $\mathbb{Z}=\{0, \pm 1, \ldots, \pm \mathbf{n}\}$
- *x(t), u(t), y(t)* ∈ are sequences
- Differentiation handled as difference equation:
  ➔ first-order vector recursion

Discrete Variations & Stability

$$y(s) = F(s)\,u(s)$$

- Is in continuous time …
- To move to discrete time it is more than just "sampling" at: 2 × (biggest Frequency)
- **Discrete-Time Exponential**

$$F(t) \to F[kT]$$

$$e^{\frac{k}{T}} = \gamma^k$$

$$\frac{1}{T} = \ln \gamma$$

- SISO to MIMO
  - Single Input, Single Output
  - Multiple Input, Multiple Output
- BIBO:
  - Bounded Input, Bounded Output
- Lyapunov:
  - Conditions for Stability
  - ➔ Are the results of the system asymptotic or exponential

Causality:
Causal (physical or nonanticipative) systems

- **Is one for which the output at any instant t_0 depends only on the value of the input x(t) for $t \leq t_0$.** Ex:

$u(t) = x(t-2) \Rightarrow \text{causal}$ $\quad$ $u(t) = x(t-2) + x(t+2) \Rightarrow \text{noncausal}$

- A "*real-time*" system must be causals
  - How can it respond to future inputs?
- A prophetic system: knows future inputs and acts on it (now)
  - The output would begin before t_0
- In some cases Noncausal maybe modelled as causal with delay
- Noncausal systems provide **an upper bound** on the performance of causal systems

Causality:
Looking at this from the output's perspective…

- **Causal** = The output *before* some time *t* does not depend on the input *after* time *t.*

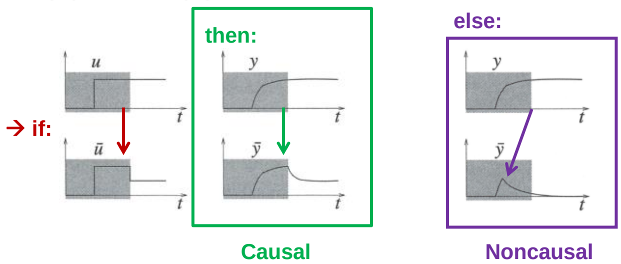
Given: $y(t) = F(u(t))$

For:

$\hat{u}(t) = u(t), \forall 0 \leq t < T \; or \; [0, T)$

Then for a T>0:

$\rightarrow \hat{y}(t) = y(t), \; \forall 0 \leq t < T$



Systems with Memory

- A system is said t have *memory* if the output at an arbitrary time $t = t_*$ depends on input values other than, or in addition to, $x(t_*)$

- Ex: Ohm's Law

$$V(t_o) = Ri(t_o)$$

- **Not** Ex: Capacitor

$$V(t_0) = \frac{1}{C}\int_{-\infty}^{t} i(t)\, dt$$

Time-Invariant Systems

- **Given** a shift (delay or advance) in the input signal
- **Then/Causes** simply a like shift in the output signal

- If $x(t)$ produces output $y(t)$
- Then $x(t - t_0)$ produces output $y(t - t_0)$

- Ex: Capacitor
- $V(t_0) = \frac{1}{C}\int_{-\infty}^{t} i(\tau - t_0)\, d\tau$

  $= \frac{1}{C}\int_{-\infty}^{t-t_0} i(\tau)\, d\tau$

  $=V(t - t_0)$

Time-Invariant Systems

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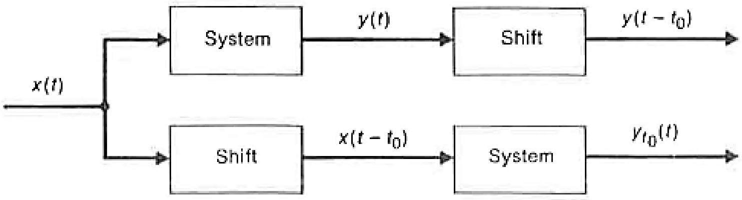
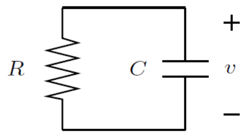
- Then $x(t - t_0)$ produces output $y(t - t_0)$



Linear Systems Examples! : First Order Systems

Examples

simple RC circuit:



circuit equation: $RCv' + v = 0$

solution: $v(t) = v(0)e^{-t/(RC)}$

population dynamics:

- $y(t)$ is population of some bacteria at time t
- growth (or decay if negative) rate is $y' = by - dy$ where b is birth rate, d is death rate
- $y(t) = y(0)e^{(b-d)t}$ (grows if $b > d$; decays if $b < d$)

First Order Systems

First order systems

$$ay' + by = 0 \quad \text{(with } a \neq 0)$$

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

$$Ty' + y = 0 \quad \text{or} \quad y' + ry = 0$$

where

- $T = a/b$ is a *time* (units: seconds)
- $r = b/a = 1/T$ is a *rate* (units: $1/\text{sec}$)

First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT + 1} = \frac{y(0)}{s + 1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, y *decays* exponentially

- T gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half $(0.693 = \log 2)$
- $4.6T$ gives time to decay by 0.01 $(4.6 = \log 100)$

if $T < 0$, y *grows* exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100

Linear Systems: Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a(\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}(y'')}) + b(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

Second Order Systems

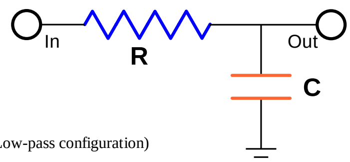
so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

EXAMPLE: First Order RC Filter

- Passive, First-Order **R**esistor-**C**apacitor Design:



(Low-pass configuration)

$$T(s) = \frac{a_1 s + a_0}{s + \omega_0}$$

- 3dB (½ Signal Power):

$$\omega = 2\pi f$$

$$f_c = \frac{1}{2\pi \mathbf{RC}}$$

- Magnitude:

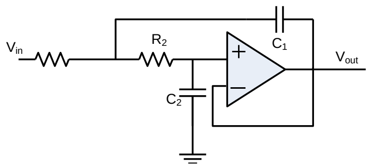
$$|V_{\text{out}}| = \sqrt{\frac{1}{(\omega RC)^2}}\,|V_{\text{in}}|$$

- Phase:

$$\phi = \tan^{-1}(-\omega RC)$$

2^{nd} Order Active RC Filter (Sallen–Key)

- 2^{nd} Order System Sallen–Key Low-Pass Topology:



Build this for Real in **ELEC 4403**

- KCL: $\frac{v_{\text{in}}-v_x}{R_1} = C_1 s\,(v_x - v_{\text{out}}) + \frac{v_x - v_{\text{out}}}{R_2}$

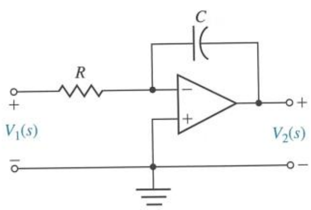
- Combined with Op-Amp Law:

$$\frac{v_{\text{in}}-v_{\text{out}}(C_2 s R_2+1)}{R_1} = C_1 s v_{\text{out}}\,(C_2 s R_2 + 1) - v_{\text{out}} + \frac{v_{\text{out}}(C_2 s R_2+1)-v_{\text{out}}}{R_2}$$

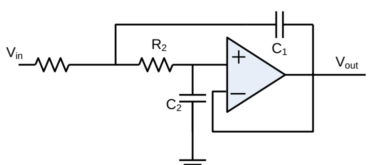
- Solving for Gives a 2^{nd} order System:

$$\frac{v_{out}}{v_{in}} = \frac{1}{C_1 C_2 R_1 R_2 s^2 + C_2 (R_1 + R_2) s + 1}$$

Example$_1$: 1^{st} or 2^{nd} Order Circuit Elements (



$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$



$$\frac{v_{out}}{v_{in}} = \frac{1}{C_1 C_2 R_1 R_2 s^2 + C_2 (R_1 + R_2) s + 1}$$

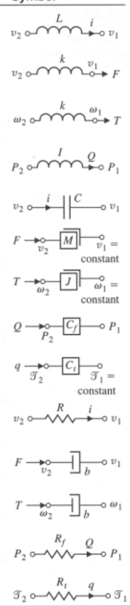
Linear Systems: **Equivalence Across Domains**

Table 2.1 Summary of Through- and Across-Variables for Physical Systems

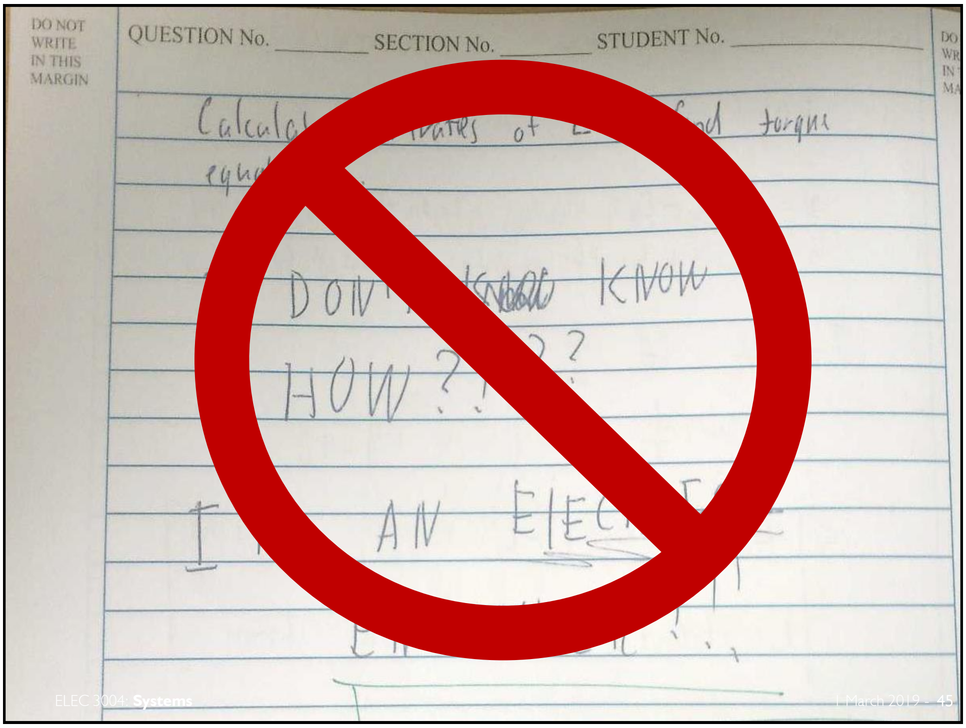
System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Source: Dorf & Bishop, *Modern Control Systems*, 12th Ed., p. 73

Table 2.2 Summary of Governing Differential Equations for Ideal Elements

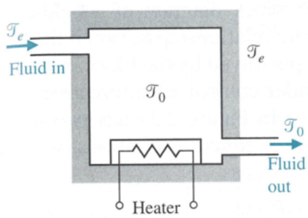
Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\frac{dv_{21}}{dt}$	$E = \frac{1}{2}Cv_{21}{}^2$	
	Translational mass	$F = M\frac{dv_2}{dt}$	$E = \frac{1}{2}Mv_2{}^2$	
	Rotational mass	$T = J\frac{d\omega_2}{dt}$	$E = \frac{1}{2}J\omega_2{}^2$	
	Fluid capacitance	$Q = C_f\frac{dP_{21}}{dt}$	$E = \frac{1}{2}C_fP_{21}{}^2$	
	Thermal capacitance	$q = C_t\frac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \frac{1}{R}v_{21}$	$\mathcal{P} = \frac{1}{R}v_{21}{}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}{}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}{}^2$	
	Fluid resistance	$Q = \frac{1}{R_f}P_{21}$	$\mathcal{P} = \frac{1}{R_f}P_{21}{}^2$	
	Thermal resistance	$q = \frac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t}\mathcal{T}_{21}$	

Source: Dorf & Bishop, *Modern Control Systems*, 12th Ed., p. 74



Example$_2$: Thermal Systems

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$ = temperature difference due to thermal process

C_t = thermal capacitance

Q = fluid flow rate = constant

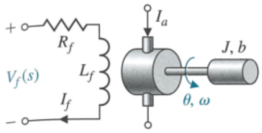
S = specific heat of water

R_t = thermal resistance of insulation

$q(s)$ = transform of rate of heat flow of heating element

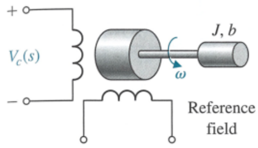
Example$_3$: Motors

5. DC motor, field-controlled, rotational actuator



$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$$

7. AC motor, two-phase control field, rotational actuator



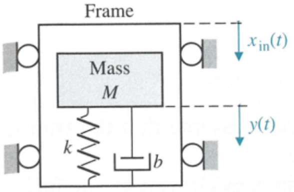
$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

Example$_4$: Mechanical Systems

15. Accelerometer, acceleration sensor



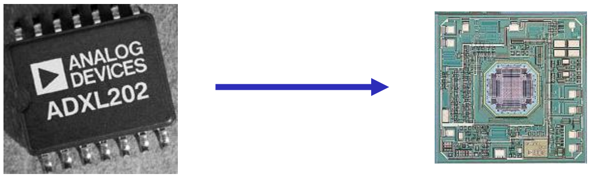
$$x_o(t) = y(t) - x_{in}(t),$$

$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$

For low-frequency oscillations, where $\omega < \omega_n$,

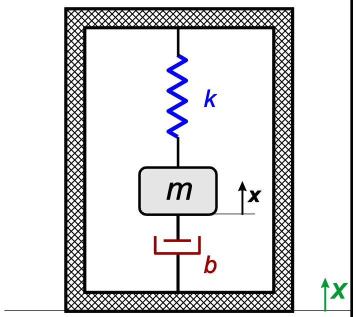
$$\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

Another 2nd Order System: Accelerometer or Mass Spring Damper (MSD)



- General accelerometer:
  - Linear spring (k) (0^{th} order w/r/t o)
  - Viscous damper (b) (1^{st} order)
  - Proof mass (m) (2^{nd} order)

➔ Electrical system analogy:
  - resistor (R) : damper (b)
  - inductance (L) : spring (k)
  - capacitance (C) : mass (m)



Measuring Acceleration: Sense *a* by measuring spring motion *Z*

- Start with Newton's 2nd Law:

$$ma = F$$

- Substitute:

$$m\frac{d^2x}{dt^2} = k(X-x) + b\frac{d(X-x)}{dt}$$

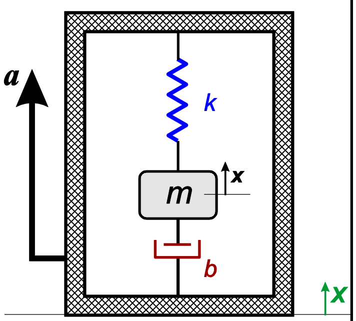
The "displacement" measured by the unit (the motion of *m* relative the accelerometer frame)

$$\mathbf{Z} \equiv (X-x) \rightarrow x = X - \mathbf{Z}$$

$$\Rightarrow m\frac{d^2X}{dt^2} = m\frac{d^2\mathbf{Z}}{dt^2} + k\mathbf{Z} + b\frac{d\mathbf{Z}}{dt}$$

- Solve ODE:

$$X(t) = X_0 e^{i\omega t} \qquad \mathbf{Z}(t) = Z_0 e^{i\omega t}$$



Measuring Acceleration [2]

- Substitute candidate solutions:

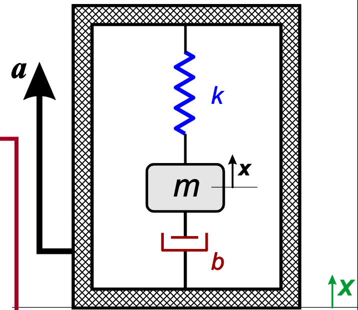
$$m\frac{d^2\left(X_0e^{i\omega t}\right)}{dt^2} = m\frac{d^2\left(Z_0e^{i\omega t}\right)}{dt^2} + k\left(Z_0e^{i\omega t}\right) + b\frac{d\left(Z_0e^{i\omega t}\right)}{dt}$$

$$-m\omega^2X_0e^{i\omega t} = -m\omega^2Z_0e^{i\omega t} + kZ_0e^{i\omega t} + (i\omega)\,bZ_0e^{i\omega t}$$

- Define Natural Frequency (ω_0)
  & Simplify for Z_0
  (the spring displacement "magnitude"):

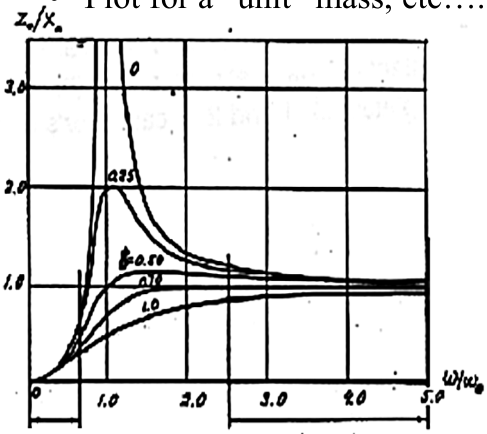
$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

$$Z_0 = \frac{m\omega^2X_0}{m\omega^2 - k - i\omega b} = \frac{X_0}{\sqrt{1 - \frac{{\omega_0}^2}{\omega^2} - \frac{b^2}{m^2\omega^2}}}$$



Acceleration: 2nd Order System

- Plot for a "unit" mass, etc….



Accelerometer Seismometer

- For $\omega \ll \omega_0$:

$$Z_0 \approx \frac{\omega^2 X_0}{\omega_0^2} = \frac{a}{\omega_0^2}$$

$$\rightarrow a = Z_0\omega_0^2$$

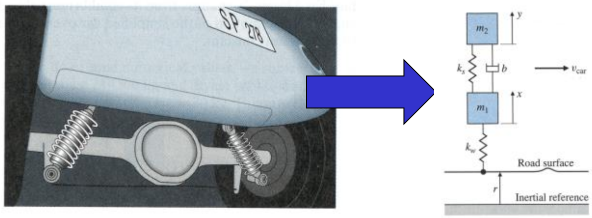
➔ it's an Accelerometer

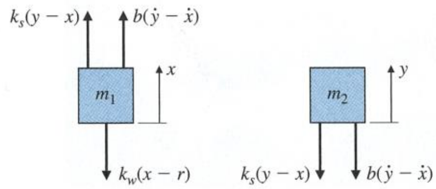
- For $\omega \sim \omega_0$
  - As: b→ 0, Z → ∞
  - Sensitivity ↑
- For $\omega \gg \omega_0$:

$$Z_0 \approx X_0$$

→ it's a Seismometer

Cascades of Linear Systems: Ex $_6$: Quarter-Car Model





REF: FPE, Feedback Control of Dynamic Systems, 6th Ed, p.25

Example: Quarter-Car Model (2)

$$\ddot{x} + \frac{b}{m_1}(\dot{x}-\dot{y}) + \frac{k_s}{m_1}(x-y) + \frac{k_w}{m_1}x = \frac{k_w}{m_1}r,$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y}-\dot{x}) + \frac{k_s}{m_2}(y-x) = 0.$$

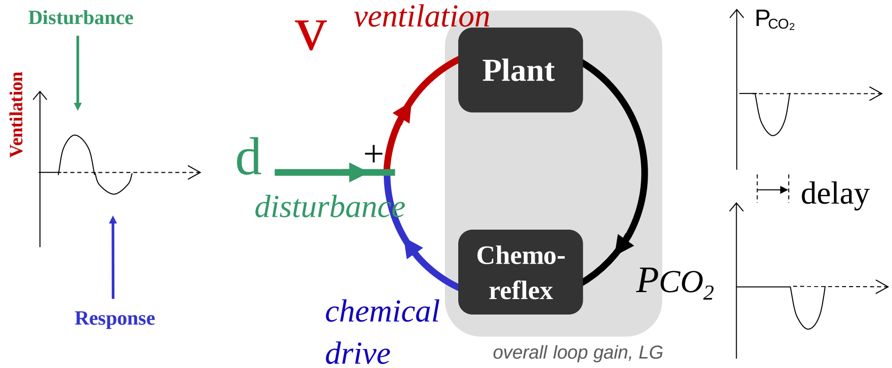
$$s^2X(s) + s\frac{b}{m_1}(X(s)-Y(s)) + \frac{k_s}{m_1}(X(s)-Y(s)) + \frac{k_w}{m_1}X(s) = \frac{k_w}{m_1}R(s),$$

$$s^2Y(s) + s\frac{b}{m_2}(Y(s)-X(s)) + \frac{k_s}{m_2}(Y(s)-X(s)) = 0,$$

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s+\frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1}+\frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1}+\frac{k_s}{m_2}+\frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}.$$

Ex $_7$: Using Response to Guide Order

- Loop Gain" to Quantify Ventilatory Stability:

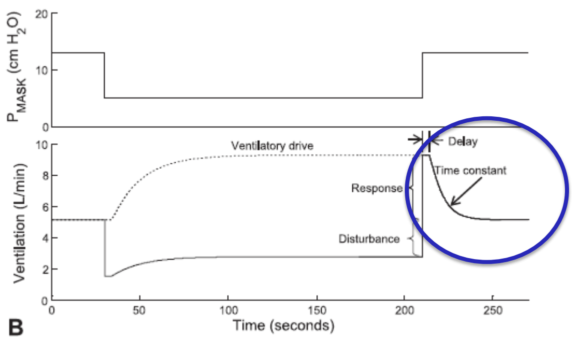


- $\text{Loop Gain} = \frac{Response}{Disturbance}$
- Loop Gain>1 implies an unstable control system
- Loop Gain<1 implies a stable control system
- Like EEG, disturbance can be characterised by frequency

REF: P. Terrill and U. Abeyratne

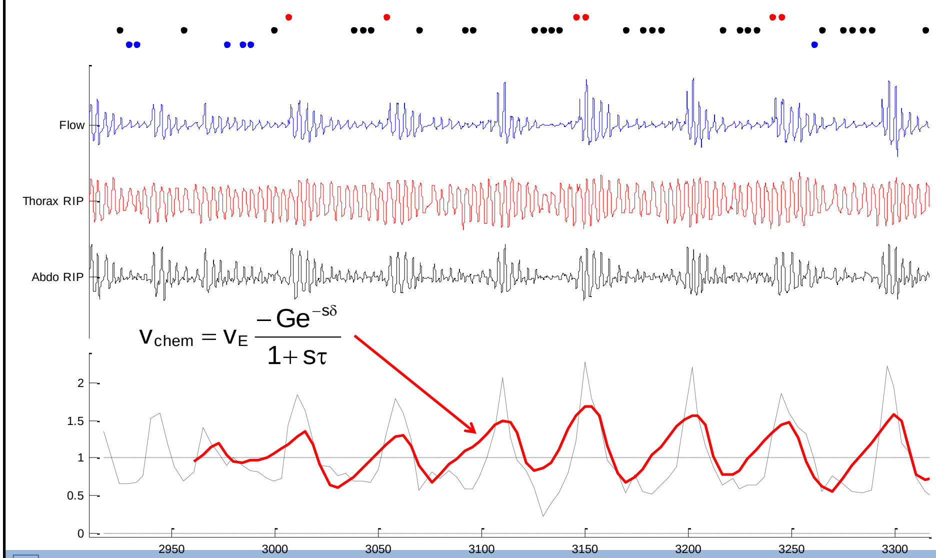
Measuring LG – 3min CPAP Drop:

- This is an invasive procedure
- As such, unsuitable for clinical sleep lab:
  - Large scale clinical studies Difficult
  - Clinical practice…..



[Wellman, JAP, 2011]

Estimating LG from Clinical PSG:



LGn=0.64 LG60=0.9 LG30=0.46

Next Time…

- We'll talk about Other System Properties ☺
- We will introduce this via the lens of:
  "Systems as Maps. Signals as Vectors"
- Review:
  - Phasers, complex numbers, polar to rectangular, and general functional forms.
  - Chapter B and Chapter 1 of Lathi
    (particularly the first sections on signals & classification thereof)
- Register on Platypus
- Try the practise assignment