Digital Filters:
Active Filters & Estimation

ELEC 3004: **Systems**: Signals & Controls
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Lecture 15

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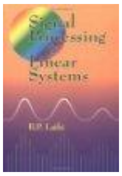
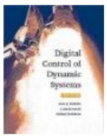
Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	Systems: Linear Differential Systems
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	Aliasing & Antialiasing
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	Filter Analysis
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	PS 1: Q & A
7	10-Apr	Digital Windows
	12-Apr	Digital Filter (FIR)
8	**17-Apr**	**Active Filters & Estimation**
	19-Apr	Holiday
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation & PID Control
10	8-May	Guest Lecture: FFT
	10-May	State-Space Control
11	15-May	Digital Control Design
	17-May	Stability
12	22-May	State Space Control System Design
	24-May	Shaping the Dynamic Response
13	29-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998



G. Franklin, J. Powell, M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990
[Available as UQ Ebook]

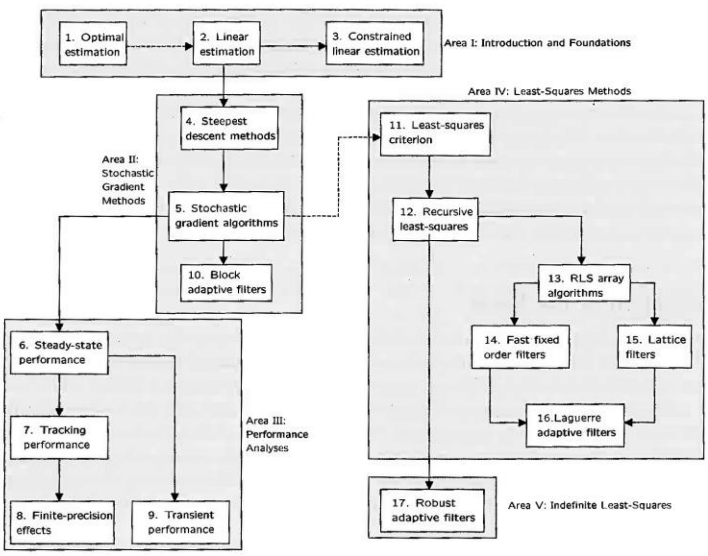
Today

- Chapter 12
  Frequency Response & Digital Filters

- Lathi: Chapter 13
  - **State-Space Analysis**
- FPW: Chapter 2
  - Chapter 2: Linear, Discrete, Dynamic-Systems Analysis

Next Time

Adaptive Filters



Random or Stochastic Variables

- A random variable is one described by its **Expectation (E)**

$$\bar{x} \triangleq \mathsf{E}x, \qquad \sigma_x^2 \triangleq \mathsf{E}(x-\bar{x})^2 = \mathsf{E}x^2 - \bar{x}^2$$

- The Variance is the Expectation of the difference between the variable and its mean

- When x has zero mean, its variance is simply given by

$$\sigma_x = Ex^2$$

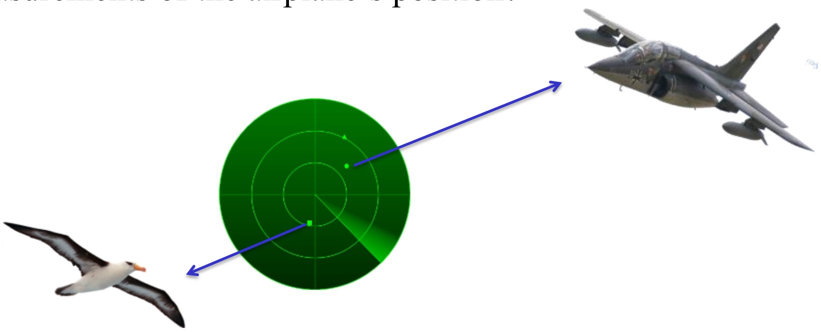
Expectation → Estimation: "Bayesian Perspective"

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Based on Material from Jur van den Berg, Introduction to Robtoics

Kalman Filtering

- (Optimal) estimation of the (hidden) state of a linear dynamic process of which we obtain noisy (partial) measurements
- Example: radar tracking of an airplane. What is the state of an airplane given noisy radar measurements of the airplane's position?



Model

- Discrete time steps, continuous state-space
- (Hidden) state: $\mathbf{x}_t$, measurement: $\mathbf{y}_t$
- Airplane example:
- Position, speed and acceleration

$$\mathbf{x}_t = \begin{pmatrix} x_t \\ \dot{x}_t \\ \ddot{x}_t \end{pmatrix}, \quad \mathbf{y}_t = \left(\tilde{x}_t\right)$$

Dynamics and Observation model

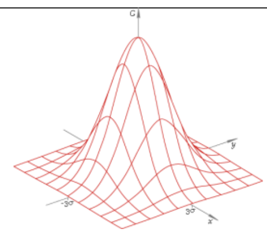
- Linear dynamics **model** describes relation between the state and the next state, and the observation:
- Airplane example (if process has time-step δ):

$$
\begin{aligned}
\mathbf{x}_{t+1} &= A\mathbf{x}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim W_t = N(\mathbf{0}, Q) \\
\mathbf{y}_t &= C\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim V_t = N(\mathbf{0}, R)
\end{aligned}
$$

$$
A = \begin{pmatrix} 1 & \delta & \frac{1}{2}\delta^2 \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}
$$

Normal distributions

- Let X_0 be a normal distribution of the initial state $\mathbf{x}_0$
- Then, every X_t is a normal distribution of hidden state $\mathbf{x}_t$. Recursive definition:



- And every Y_t is a normal distribution of observation $\mathbf{y}_t$. Definition:

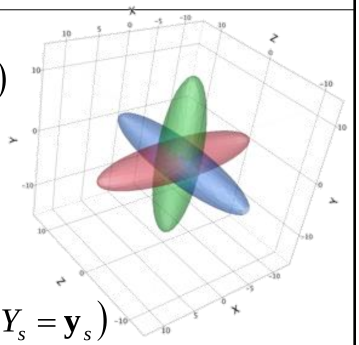
$$X_{t+1} = AX_t + W_t$$

- **Goal of filtering**: compute conditional distribution

$$Y_t = CX_t + V_t$$

$$\left(X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_t = \mathbf{y}_t\right)$$

Normal distribution



- Because X_t's and Y_t's are normal distributions, $(X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_t = \mathbf{y}_t)$ is also a normal distribution
- Normal distribution is fully specified by mean and covariance
- We denote:

$$
\begin{aligned}
X_{t|s} &= (X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_s = \mathbf{y}_s) \\
&= N\left(\mathrm{E}(X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_s = \mathbf{y}_s), \mathrm{Var}(X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_s = \mathbf{y}_s)\right) \\
&= N\left(\hat{\mathbf{x}}_{t|s}, P_{t|s}\right)
\end{aligned}
$$

Problem reduces to computing $\underline{\mathbf{x}}_{t|t}$ and $P_{t|t}$

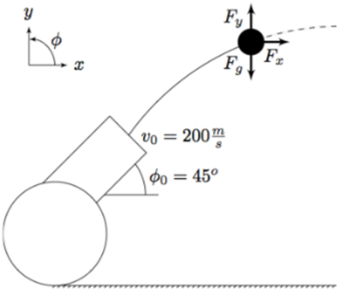
Kalman Filter:

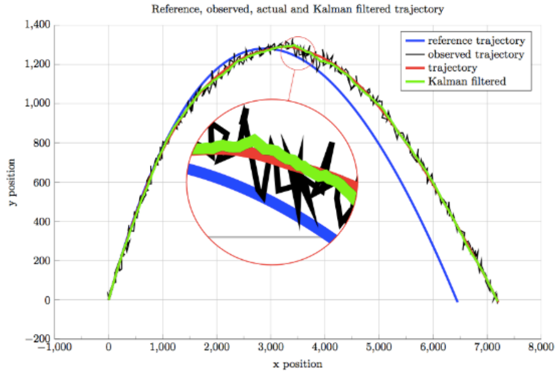
Estimating (μ) with Confidence (σ)

Along multiple dimensions

The Kalman Filter

- Ex: Projectile Motion





The Kalman Filter

- Question: What does it do?
- Answer: It estimates $x(t)$ based on $y(t)$ from:

$$x(t+1) = A\,x(t) + u(t)$$
$$y(t) \qquad = C\,x(t) + w(t)$$

State Space

- We collect our set of uncertain variables into a vector …
  $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T$
- The set of values that **x** might take on is termed the *state space*
- There is a *single* true value for **x**,
  but it is unknown

State Space Dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$H(s) = C(sI - A)^{-1}B$$

Measured versus True

- Measurement errors are inevitable
- So, add Noise to State...
  – State Dynamics becomes:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + w$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + v$$

- Can represent this as a "Normal" Distribution

$$\mathcal{N}(x; \mu, \sigma) = \frac{1}{(\sqrt{2\pi})\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Recovering The Truth

- Numerous methods
- Termed “Estimation” because we are trying to estimate the truth from the signal
- A strategy discovered by Gauss
- Least Squares in Matrix Representation

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} n & \sum_1^n t_i \\ \sum_1^n t_i & \sum_1^n t_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_1^n z_i \\ \sum_1^n t_i z_i \end{bmatrix}$$

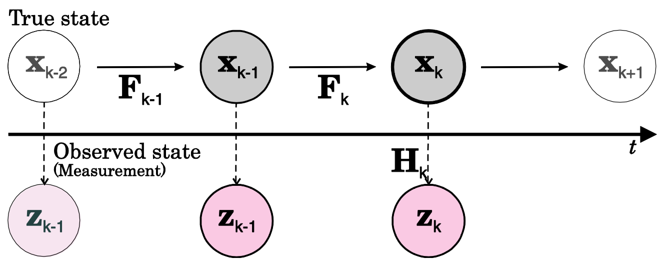
Recovering the Truth: Terminology

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} + \mathbf{w}$$

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

$\mathbf{x}$: the state vector
$\mathbf{x}_{A|B}$: the state of $\mathbf{x}$ at time A based on data taken up to time B
$\hat{\mathbf{x}}$: estimate of the true state vector
$\mathbf{F}$: system dynamics matrix in continuous time (equivalent to $\mathbf{A}$ in Eq. 1)
$\mathbf{G}$: system control matrix relating deterministic input, $\mathbf{u}$, to the state (equivalent to $\mathbf{B}$ in Eq. 1)
$\mathbf{H}$: measurement matrix in continuous time (equivalent to $\mathbf{C}$ in Eq. 2)
$\mathbf{F}_i$: system model in **discrete** time at $t = t_i$
$\mathbf{H}_i$: measurement model in **discrete** time at $t = t_i$
$\mathbf{P}_i$: estimate covariance in **discrete** time at $t = t_i$
$\mathbf{w}$: process uncertainty (noise) vector (of type $\mathcal{N}(0, s)$)
$\mathbf{Q}$: process noise matrix, $\mathbf{Q} = E\left[ww^{\mathsf{T}}\right]$
$\mathbf{Q}_i$: $\mathbf{Q}$ in discrete time at $t = t_i$
$\mathbf{v}$: measurement noise vectors (of type $\mathcal{N}(0, \sigma)$)
$\mathbf{R}_i$: the measurement variance matrix, $\mathbf{R} = E\left[vv^{\mathsf{T}}\right]$, in discrete time at $t = t_i$

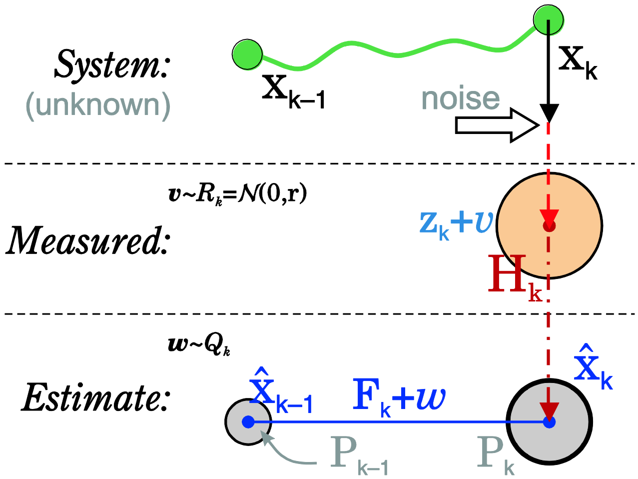
General Problem…



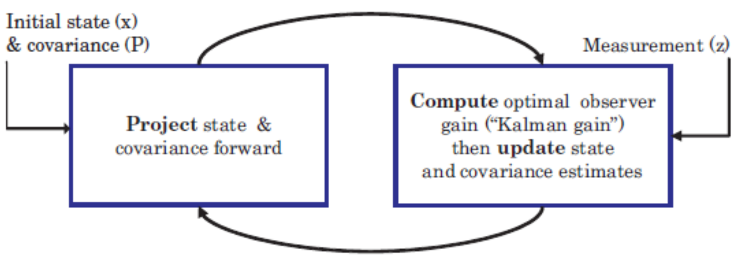
Duals and Dual Terminology

	Estimation		**Control**
Model:	$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$ (discrete: $\mathbf{x} = \mathbf{F}_k\mathbf{x}$)	$\leftrightarrow$	$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, $\mathbf{A} = \mathbf{F}^T$
Regulates:	P (covariance)	$\leftrightarrow$	M (performance matrix)
Minimized function:	Q (or GQG^T)	$\leftrightarrow$	V
Optimal Gain:	K	$\leftrightarrow$	G
Completeness law:	Observability	$\leftrightarrow$	Controllability

Estimation Process in Pictures



Kalman Filter Process



KF Process in Equations

$$
\begin{aligned}
\text{Prediction: } \hat{\mathbf{x}}_{k|k-1} &= \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1}, && \text{(state prediction)}\\
\mathbf{P}_{k|k-1} &= \mathbf{Q}_{k-1} + \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\mathsf{T}}{}_{k-1}, && \text{(covariance prediction)}\\
\text{Kalman Gain: } \mathbf{K}_k &= \mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}}[\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_k]^{-1}, &&\\
\text{Update: } \mathbf{P}_{k|k} &= [\mathbf{I} - \mathbf{K}_k\mathbf{H}]\mathbf{P}_{k|k-1}, && \text{(covariance update)}\\
\hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1}) && \text{(state update)}
\end{aligned}
$$

KF Considerations

$$
\begin{aligned}
\underbrace{\hat{\mathbf{x}}_{k|k-1}}_{n\times 1} &= \underbrace{\mathbf{F}_{k-1}}_{n\times n}\hat{\mathbf{x}}_{k-1|k-1} + \underbrace{\mathbf{G}_{k-1}}_{n\times j}\underbrace{\mathbf{u}_{k-1}}_{j\times 1}\\
\underbrace{\mathbf{P}_{k|k-1}}_{n\times n} &= \underbrace{\mathbf{Q}_{k-1}}_{n\times n} + \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\mathsf{T}}{}_{k-1}\\
\underbrace{\mathbf{K}_k}_{n\times m} &= \mathbf{P}_{k|k-1}\underbrace{\mathbf{H}^{\mathsf{T}}}_{n\times m}\underbrace{[\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_k]^{-1}}_{m\times m}\\
\mathbf{P}_{k|k} &= [\mathbf{I} - \mathbf{K}_k\mathbf{H}]\mathbf{P}_{k|k-1}\\
\hat{\mathbf{x}}_{k|k} &= \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\underbrace{\mathbf{z}_k}_{m\times 1} - \underbrace{\mathbf{H}}_{m\times n}\hat{\mathbf{x}}_{k|k-1} - \mathbf{H}\mathbf{G}_k\mathbf{u}_{k-1})
\end{aligned}
$$

Ex: Kinematic KF: Tracking

- Consider a System with Constant Acceleration

$$\ddot{y} = -g$$
$$\dot{y} = gt + p_1$$
$$y = p_0 + p_1 t + \frac{gt^2}{2}$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{F_k} = \begin{bmatrix} 0 & t_s & \frac{t_s{}^2}{2} \\ 0 & 0 & t_s \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathbf{x}}_k = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1})$$

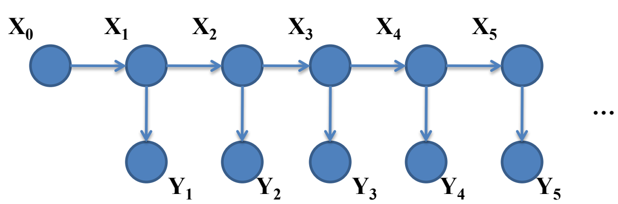
In Summary

- KF:
  - The true state (*x*) is separate from the measured (*z*)
  - Lets you **combine** prior controls knowledge with measurements to filter signals and find the <u>truth</u>
  - It **regulates** the covariance (P)
    - As P is the scatter between *z* and *x*
    - So, if P → 0, then *z* → *x* (measurements → truth)
- EKF:
  - Takes a Taylor series approximation to get a local “F” (and “G” and “H”)

(Bayesian) Kalman Filter: A Gaussian way to beat the noise

Recursive update of state

- Kalman filtering algorithm: repeat…
  - Time update:
    from $X_{t|t}$, compute **a priori** distribution $X_{t+1|t}$
  - Measurement update:
    from $X_{t+1|t}$ (and given $\mathbf{y}_{t+1}$), compute
    a posteriori distribution $X_{t+1|t+1}$



Time update

- From $X_{t|t}$, compute **a priori** distribution $X_{t+1|t}$:

$$\begin{aligned} X_{t+1|t} &= AX_{t|t} + W_t \\ &= N\left(\mathrm{E}\left(AX_{t|t} + W_t\right), \mathrm{Var}\left(AX_{t|t} + W_t\right)\right) \\ &= N\left(A\,\mathrm{E}\left(X_{t|t}\right) + \mathrm{E}\left(W_t\right), A\,\mathrm{Var}\left(X_{t|t}\right)A^T + \mathrm{Var}\left(W_t\right)\right) \\ &= N\left(A\hat{\mathbf{x}}_{t|t}, AP_{t|t}A^T + Q\right) \end{aligned}$$

- So:

$$\hat{\mathbf{x}}_{t+1|t} = A\hat{\mathbf{x}}_{t|t}$$

$$P_{t+1|t} = AP_{t|t}A^T + Q$$

Measurement update

From $X_{t+1|t}$ (and given $\mathbf{y}_{t+1}$), compute $X_{t+1|t+1}$.

1. Compute **a priori** distribution of the observation $Y_{t+1|t}$ from $X_{t+1|t}$:

$$\begin{aligned} Y_{t+1|t} &= CX_{t+1|t} + V_{t+1} \\ &= N\left(\mathrm{E}\left(CX_{t+1|t} + V_{t+1}\right), \mathrm{Var}\left(CX_{t+1|t} + V_{t+1}\right)\right) \\ &= N\left(C\,\mathrm{E}\left(X_{t+1|t}\right) + \mathrm{E}\left(V_{t+1}\right), C\,\mathrm{Var}\left(X_{t+1|t}\right)C^T + \mathrm{Var}\left(V_{t+1}\right)\right) \\ &= N\left(C\hat{\mathbf{x}}_{t+1|t}, CP_{t+1|t}C^T + R\right) \end{aligned}$$

Measurement update (cont'd)

2. Look at joint distribution of $X_{t+1|t}$ and $Y_{t+1|t}$:

$$\begin{aligned}(X_{t+1|t}, Y_{t+1|t}) &= N\left(\begin{pmatrix} \mathrm{E}(X_{t+1|t}) \\ \mathrm{E}(Y_{t+1|t}) \end{pmatrix}, \begin{pmatrix} \mathrm{Var}(X_{t+1|t}) & \mathrm{Cov}(X_{t+1|t}, Y_{t+1|t}) \\ \mathrm{Cov}(Y_{t+1|t}, X_{t+1|t}) & \mathrm{Var}(Y_{t+1|t}) \end{pmatrix}\right) \\ &= N\left(\begin{pmatrix} \hat{\mathbf{x}}_{t+1|t} \\ C\hat{\mathbf{x}}_{t+1|t} \end{pmatrix}, \begin{pmatrix} P_{t+1|t} & P_{t+1|t}C^T \\ CP_{t+1|t} & CP_{t+1|t}C^T + R \end{pmatrix}\right)\end{aligned}$$

where

$$\begin{aligned}\mathrm{Cov}(Y_{t+1}, X_{t+1|t}) &= \mathrm{Cov}(CX_{t+1|t} + V_{t+1}, X_{t+1|t}) \\ &= C\,\mathrm{Cov}(X_{t+1|t}, X_{t+1|t}) + \mathrm{Cov}(V_{t+1}, X_{t+1|t}) \\ &= C\,\mathrm{Var}(X_{t+1|t}) \\ &= CP_{t+1|t}\end{aligned}$$

Measurement update (cont'd)

- Recall that if

$$(Z_1, Z_2) = N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$$

then

$$(Z_1 \mid Z_2 = \mathbf{z}_2) = N\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{z}_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

3. Compute $X_{t+1|t+1} = (X_{t+1|t} | Y_{t+1|t} = \mathbf{y}_{t+1})$:

$$\begin{aligned}X_{t+1|t+1} &= (X_{t+1|t} \mid Y_{t+1|t} = \mathbf{y}_{t+1}) \\ &= N\Big(\hat{\mathbf{x}}_{t+1|t} + P_{t+1|t}C^T(CP_{t+1|t}C^T + R)^{-1}(\mathbf{y}_{t+1} - C\hat{\mathbf{x}}_{t+1|t}), \\ &\qquad P_{t+1|t} - P_{t+1|t}C^T(CP_{t+1|t}C^T + R)^{-1}CP_{t+1|t}\Big)\end{aligned}$$

Measurement update (cont'd):

This can also (often) be written in terms of the **Kalman gain** matrix:

$$
\begin{aligned}
K_{t+1} &= P_{t+1|t} C^T \left(C P_{t+1|t} C^T + R \right)^{-1} \\
\hat{\mathbf{x}}_{t+1|t+1} &= \hat{\mathbf{x}}_{t+1|t} + K_{t+1} \left(\mathbf{y}_{t+1} - C \hat{\mathbf{x}}_{t+1|t} \right) \\
P_{t+1|t+1} &= P_{t+1|t} - K_{t+1} C P_{t+1|t}
\end{aligned}
$$

Initialization

- Choose distribution of initial state by picking $\underline{\mathbf{x}}_0$ and P_0
- Start with measurement update given measurement $\mathbf{y}_0$
- Choice for Q and R (identity)
  - small Q: dynamics "trusted" more
  - small R: measurements "trusted" more

(Bayesian) Kalman Filter Summary

I. Model:

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim W_t = N(\mathbf{0}, Q) \\ \mathbf{y}_t &= C\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim V_t = N(\mathbf{0}, R) \end{aligned}$$

II. Algorithm: Repeat…

– Time update:

$$\hat{\mathbf{x}}_{t+1|t} = A\hat{\mathbf{x}}_{t|t}$$

$$P_{t+1|t} = AP_{t|t}A^T + Q$$

– Measurement update:

$$\begin{aligned} K_{t+1} &= P_{t+1|t}C^T\left(CP_{t+1|t}C^T + R\right)^{-1} \\ \hat{\mathbf{x}}_{t+1|t+1} &= \hat{\mathbf{x}}_{t+1|t} + K_{t+1}\left(\mathbf{y}_{t+1} - C\hat{\mathbf{x}}_{t+1|t}\right) \\ P_{t+1|t+1} &= P_{t+1|t} - K_{t+1}CP_{t+1|t} \end{aligned}$$

(Bayesian) Kalman Filter Summary [II]

Take Aways:

- Kalman filter can be used in real time
- Use $\underline{\mathbf{x}}_{t|t}$'s as optimal estimate of state at time t, and use $P_{t|t}$ as a measure of uncertainty.

Extensions:

- Dynamic process with known **control input**
- **Non-linear** dynamic process
- **Kalman smoothing**: compute optimal estimate of state $\mathbf{x}_t$ given all data $\mathbf{y}_1, \ldots, \mathbf{y}_T$, with T > t (not real-time).
- Automatic parameter (Q and R) fitting using EM-algorithm

BREAK

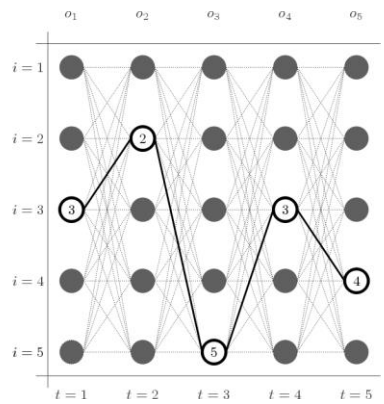
Viterbi Algorithm

Based on Material from S Salzberg CMSC 828N

The Viterbi Algorithm

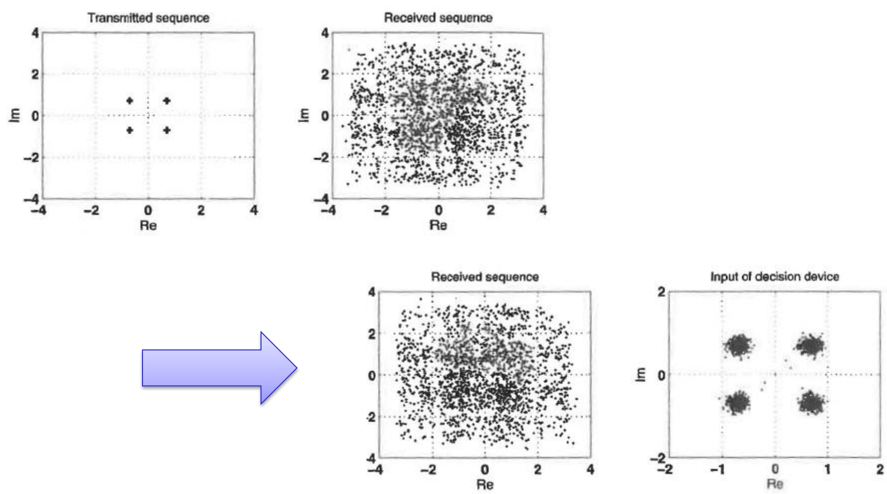
Question: What does it do?

Answer: It **finds** the **most likely** sequence of hidden states:



What Sequence?

- Remember QPSK Constellations?

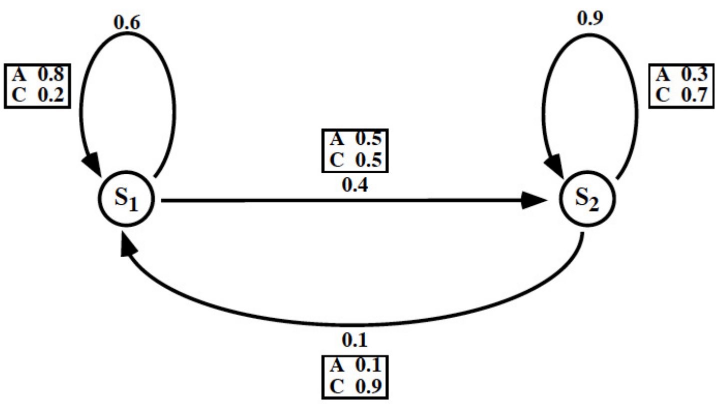


Viterbi algorithm

$$V_i(t) = \begin{cases} 0 & : \quad t = 0 \wedge i \neq S_I \\ 1 & : \quad t = 0 \wedge i = S_I \\ \max V_j(t-1) a_{ji} b_{ji}(y) & : \quad t > 0 \end{cases}$$

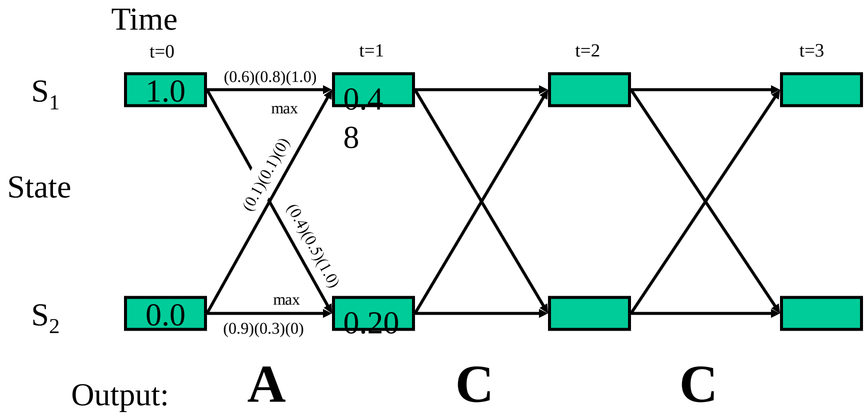
Where $V_i(t)$ is the probability that the HMM is in state *i* after generating the sequence $y_1, y_2, \ldots, y_t$, following the *most probable path* in the HMM

Our sample HMM

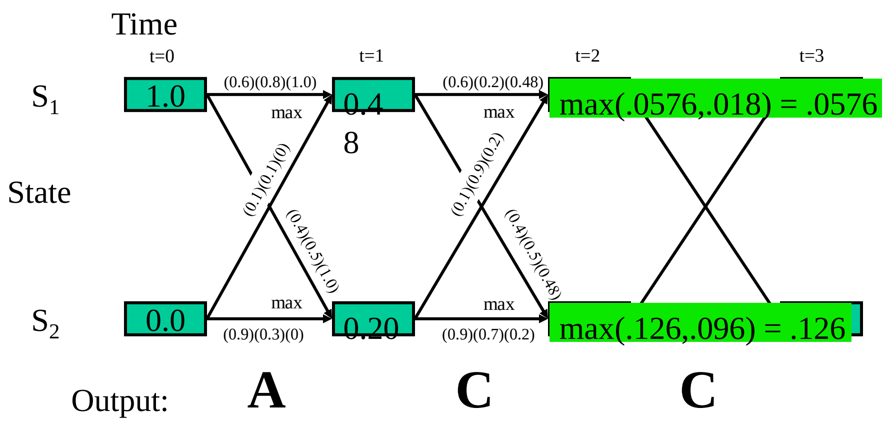


Let S_1 be initial state, S_2 be final state

A trellis for the Viterbi Algorithm



A trellis for the Viterbi Algorithm

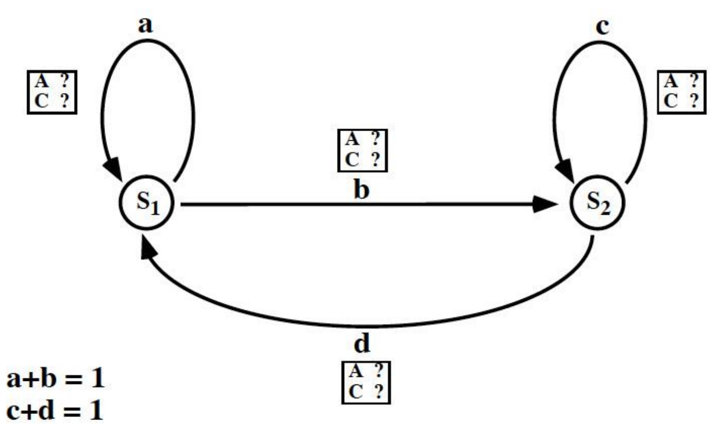


Learning in HMMs: the E-M algorithm

- In order to learn the parameters in an "empty" HMM, we need:
  - The topology of the HMM
  - Data - the more the better

- The learning algorithm is called "Estimate-Maximize" or E-M
  - Also called the Forward-Backward algorithm
  - Also called the Baum-Welch algorithm

An untrained HMM



Some HMM training data

- CACAACAAAACCCCCACAA
- ACAACACACACACACACCAAAC
- CAACACACAAACCCC
- CAACCACCACACACACACCCCA
- CCCAAAACCCCAAAAACCC
- ACACAAAAAACCCAACACACAACA
- ACACAACCCCAAAACCACCAAAAA

Step 1: Guess all the probabilities

- We can start with random probabilities, the learning algorithm will adjust them
- If we can make good guesses, the results will generally be better

Step 2: the Forward algorithm

- Reminder: each box in the trellis contains a value $\alpha i(t)$
- $\alpha i(t)$ is the probability that our HMM has generated the sequence y1, y2, …, yt and has ended up in state i.

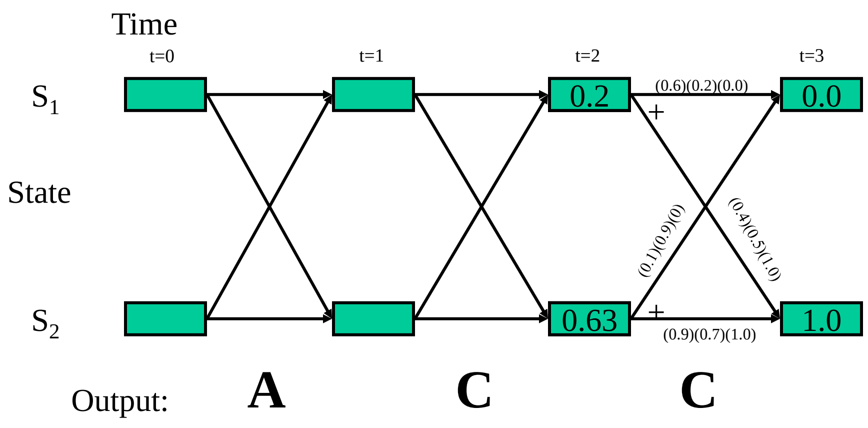
Reminder: notations

- sequence of length T: y_1^T
- all sequences of length T: Y_1^T
- Path of length T+1 generates Y: x_1^{T+1}
- All paths: X_1^{T+1}

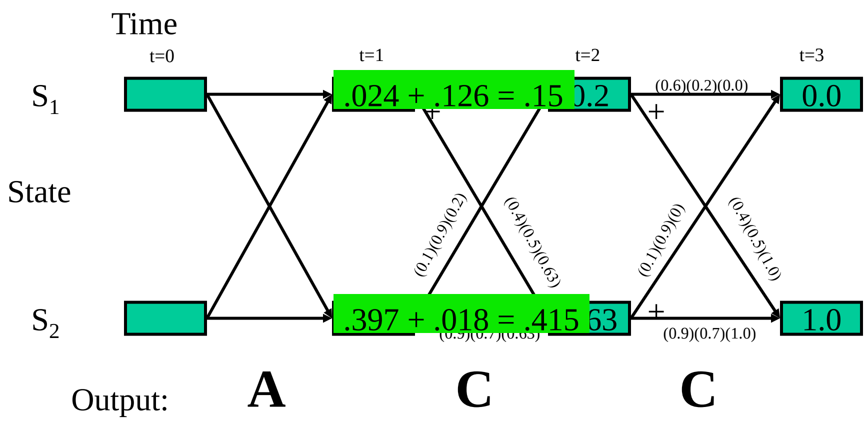
Step 3: the Backward algorithm

- Next we need to compute βi(t) using a Backward computation
- βi(t) is the probability that our HMM will generate the rest of the sequence yt+1,yt+2, …, yT beginning in state i

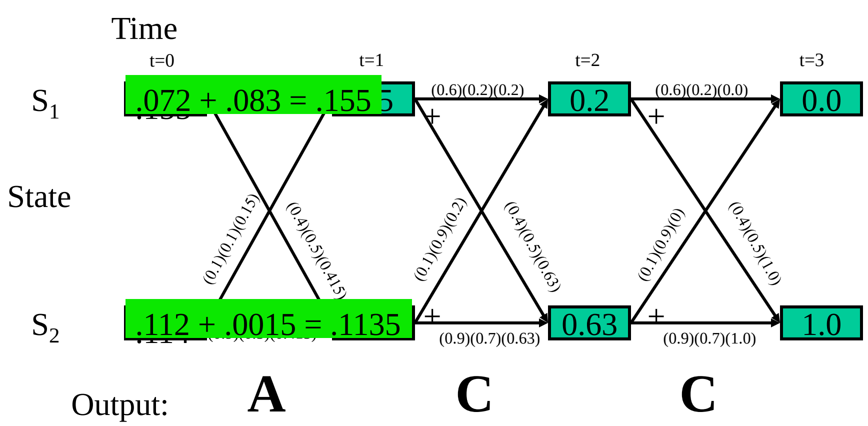
A trellis for the Backward Algorithm



A trellis for the Backward Algorithm (2)



A trellis for the Backward Algorithm (3)



Step 4: Re-estimate the probabilities

- After running the Forward and Backward algorithms once, we can re-estimate all the probabilities in the HMM
- α_{SF} is the prob. that the HMM generated the entire sequence
- Nice property of E-M: the value of α_{SF} never decreases; it converges to a local maximum
- We can read off α and β values from Forward and Backward trellises

Compute new transition probabilities

- γ is the probability of making transition i-j at time t, given the observed output
  - γ is dependent on data, plus it only applies for one time step; otherwise it is just like $a_{ij}(t)$

$$\gamma_{ij}(t) = P(X_t = i, X_{t+1} = j \mid y_1^T)$$

$$\gamma_{ij}(t) = \frac{\alpha_i(t-1) a_{ij} b_{ij}(y_t) \beta_j(t)}{\alpha_{S_F}}$$

What is gamma?

- Sum γ over all time steps, then we get the expected number of times that the transition i-j was made while generating the sequence Y:

$$C_1 = \sum_{t=1}^{T} \gamma_{ij}(t)$$

How many times did we leave i?

- Sum γ over all time steps and all states that can follow i, then we get the expected number of times that the transition i-x as made for any state x:

$$C_2 = \sum_{t=1}^{T} \sum_{k} \gamma_{ik}(t)$$

Recompute transition probability

$$a_{ij} = \frac{C_1}{C_2}$$

In other words, probability of going from state i to j is estimated by counting how often we took it for our data (C1), and dividing that by how often we went from i to other states (C2)

Recompute output probabilities

- Originally these were $b_{ij}(k)$ values
- We need:
  - expected number of times that we made the transition i-j and emitted the symbol k
  - The expected number of times that we made the transition i-j

New estimate of $b_{ij}(k)$

$$b_{ij}(k) = \frac{\sum_{t:y_t=k} \gamma_{ij}(t)}{\sum_{t=1}^{T} \gamma_{ij}(t)}$$

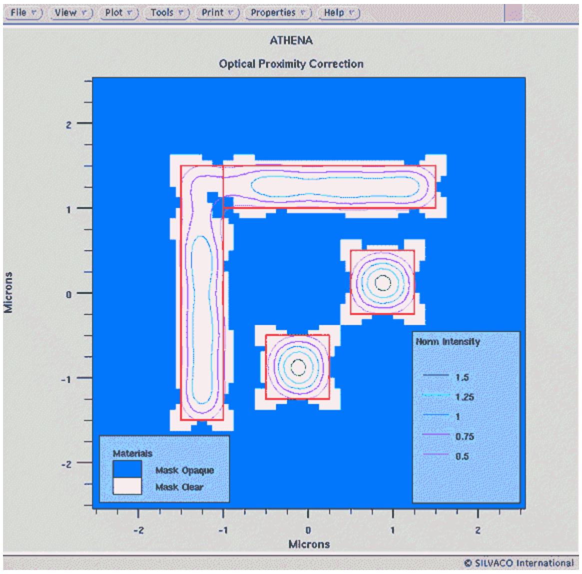
Step 5: Go to step 2

- Step 2 is Forward Algorithm
- Repeat entire process until the probabilities converge
  - Usually this is rapid, 10-15 iterations
- "Estimate-Maximize" because the algorithm first estimates probabilities, then maximizes them based on the data
- "Forward-Backward" refers to the two computationally intensive steps in the algorithm

Computing requirements

- Trellis has N nodes per column, where N is the number of states
- Trellis has S columns, where S is the length of the sequence
- Between each pair of columns, we create E edges, one for each transition in the HMM
- Total trellis size is approximately S(N+E)

Advanced Application: Optical Proximity Correction



Next Time…

- **State-Space & Digital Control!**
  - State-space filters add feedback and becomes a "control filter"