http://elec3004.com

# Digital Filters:
# FIR & Adaptive Windows

ELEC 3004: **Systems**: Signals & Controls
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Lecture 14

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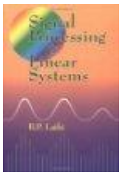
## Lecture Schedule:

| Week | Date | Lecture Title |
|---|---|---|
| 1 | 27-Feb | Introduction |
| | 1-Mar | Systems Overview |
| 2 | 6-Mar | Systems as Maps & Signals as Vectors |
| | 8-Mar | Systems: Linear Differential Systems |
| 3 | 13-Mar | Sampling Theory & Data Acquisition |
| | 15-Mar | Aliasing & Antialiasing |
| 4 | 20-Mar | Discrete Time Analysis & Z-Transform |
| | 22-Mar | Second Order LTID (& Convolution Review) |
| 5 | 27-Mar | Frequency Response |
| | 29-Mar | Filter Analysis |
| 6 | 3-Apr | Digital Filters (IIR) & Filter Analysis |
| | 5-Apr | PS 1: Q & A |
| 7 | 10-Apr | Digital Filter (FIR) & Digital Windows |
| | **12-Apr** | **Active Filters & Estimation** |
| 8 | 17-Apr | Introduction to Feedback Control |
| | 19-Apr | **Holiday** |
| | 24-Apr | |
| | 26-Apr | |
| 9 | 1-May | Servoregulation |
| | 3-May | PID Control |
| 10 | 8-May | Guest Lecture: FFT |
| | 10-May | State-Space Control |
| 11 | 15-May | Digital Control Design |
| | 17-May | Stability |
| 12 | 22-May | State Space Control System Design |
| | 24-May | Shaping the Dynamic Response |
| 13 | 29-May | System Identification & Information Theory |
| | 31-May | Summary and Course Review |

## Follow Along Reading:



**B. P. Lathi**
***Signal processing and linear systems***
**1998**
**TK5102.9.L38 1998**

**G. Franklin,**
**J. Powell,**
**M. Workman**
*Digital Control of Dynamic Systems*
1990

**TJ216.F72 1990**
[Available as UQ Ebook]

**Today**

- Chapter 12
  **Frequency Response & Digital Filters**

- Lathi: Chapter 13
  – **State-Space Analysis**

- FPW: Chapter 2
  – Chapter 2: Linear, Discrete, Dynamic-Systems Analysis

***Next Time***

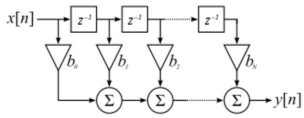


# FIR:
# Filter In RECT++ Window ☺

## ** FIR Filter Design **

- How to get all these coefficients?

$$H(\omega) = \boxed{h_0 + h_1 e^{-i\omega} + \cdots + h_{n-1} e^{-i(n-1)\omega}}$$



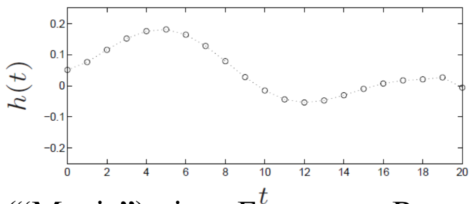

**FIR Design Methods:**

1. Impulse Response Truncation
   + Simplest
   – Undesirable frequency domain-characteristics, not very useful
2. Windowing Design Method
   + Simple
   – Not optimal (not minimum order for a given performance level)
3. Optimal filter design methods
   + "More optimal" (treat the whole thing as a system to solve ☺)
   – Less simple…

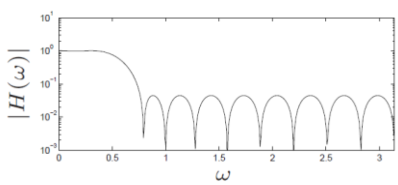


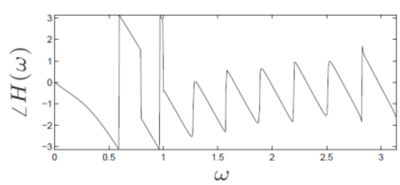
## FIR Filter Design & Operation
## Ex: Lowpass FIR filter

- Set Impulse response (order n = 21)
- "Determine" h(t)
  – h(t) is a 20 element vector that we'll use to as a weighted sum



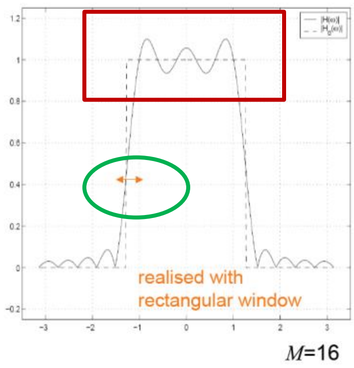

- FFT ("Magic") gives Frequency Response & Phase



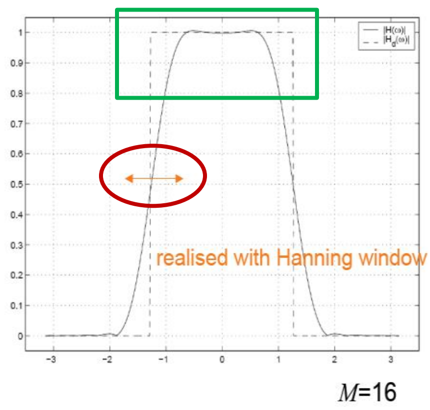



# FIR: Rectangular & Hanning Windows

- Rectangular




- Hanning




➔ Hanning: Less ripples, but wider transition band

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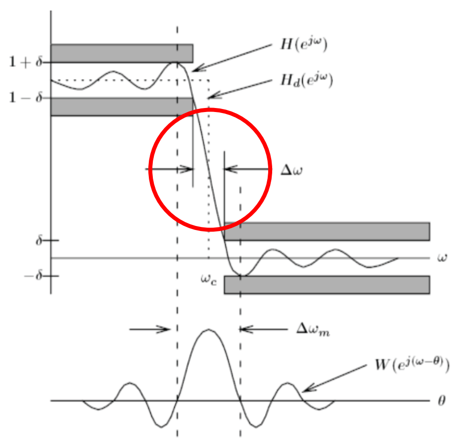
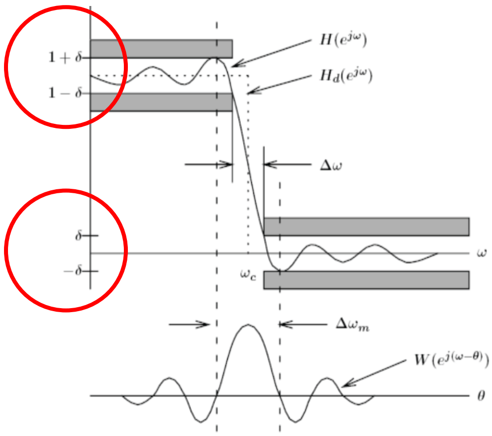
# Windowed FIR Property 1: Equal transition bandwidth




Punskaya, Slide 96

- Equal transition bandwidth on both sides of the ideal cutoff frequency

# Windowed FIR Property 2: Peak Errors same in Passband & Stopband

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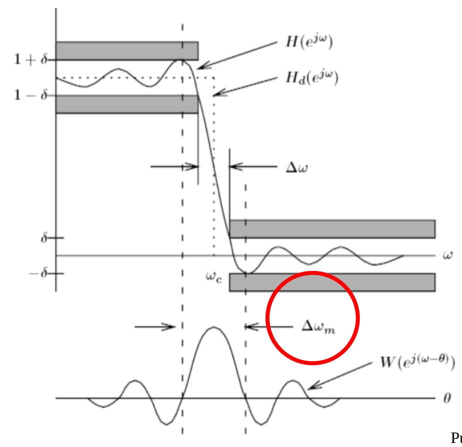
- Peak approximation error in the passband (1+δ → 1-δ) is equal to that in the stopband (δ → -δ)



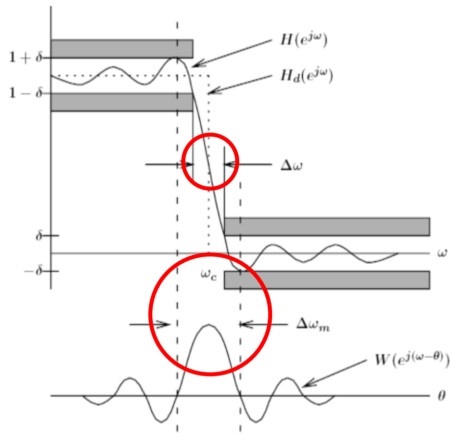
# Windowed FIR Property 3: Mainlobe Width




Punskaya, Slide 99

- The distance between approximation error peaks is approximately equal to the width of the mainlobe $\Delta w_m$

## Windowed FIR Property 4: Mainlobe Width [2]

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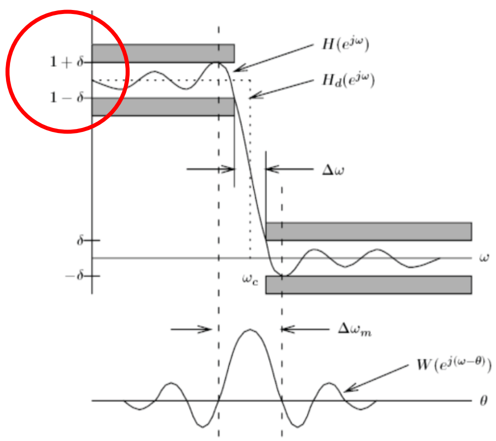
- The width of the mainlobe is wider than the transition bandwidth



## Windowed FIR Property 5: Peak Δδ is determined by the window shape

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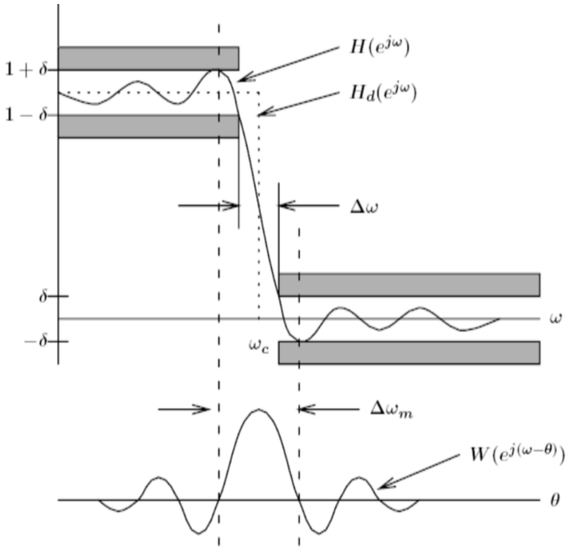
- peak approximation error is determined by the window shape, <u>independent</u> of the filter order

## Window Design Method Design Terminology

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Where:

- $\omega_c$: cutoff frequency
- $\delta$: maximum passband ripple
- $\Delta\omega$: transition bandwidth
- $\Delta\omega_m$: width of the window mainlobe



## Passband / stopband ripples

$\omega_s$ and $\omega_p$: Corner Frequencies

Passband / stopband ripples are often expressed in dB:

- passband ripple = 20 $log_{10}$ $(1+\delta_p)$ dB
- peak-to-peak passband ripple ≅ 20 $log_{10}$ $(1+2\delta_p)$ dB
- minimum stopband attenuation = -20 $log_{10}$ $(\delta_s)$ dB

## Passband / stopband ripples

$\omega_s$ and $\omega_p$: Corner Frequencies

Passband / stopband ripples are often expressed in dB:

- passband ripple = $\cancel{20\ log_{10}\ (1+\delta_p\ )\ \text{dB}} \cong 20\ log_{10}\ (\delta_p\ )$ dB
- peak-to-peak passband ripple $\cong \cancel{20\ log_{10}\ (1+2\delta_p)\ \text{dB}}$
  $\cong 20\ log_{10}\ (2\delta_p)$ dB
- minimum stopband attenuation = $\cancel{-20\ log_{10}\ (\delta_s)\ \text{dB}}$
  $\cong 20\ log_{10}\ (\delta_s)$ dB



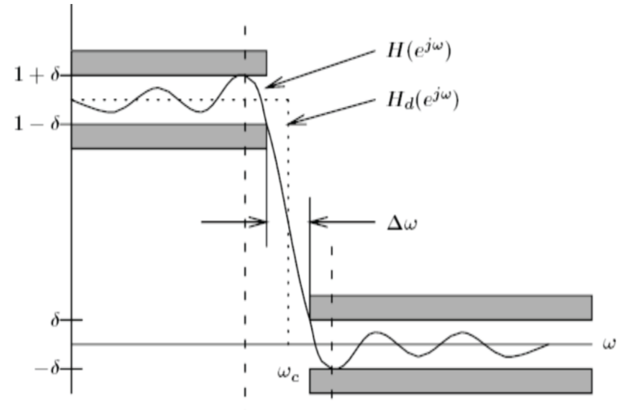
## Summary of Design Procedure

1. Select a suitable window function
2. Specify an ideal response $H_d(\omega)$
3. Compute the coefficients of the ideal filter $h_d(n)$
4. Multiply the ideal coefficients by the window function to give the filter coefficients
5. Evaluate the frequency response of the resulting filter and iterate if necessary (e.g. by increasing $M$ if the specified constraints have not been satisfied).

Punskaya, Slide 105

## Windowed Filter Design Example

- Design a type I low-pass filter with:
  - $\omega_p$ **=0.2π**
  - $\omega_s$ **=0.3π**
  - δ **=0.01**






## Windowed Filter Design Example: Step 1: Select a suitable Window Function

| No. | Window $w(t)$ | Mainlobe Width | Rolloff Rate (dB/oct) | Peak Sidelobe level (dB) | Peak $20\log_{10}\delta$ |
|---|---|---|---|---|---|
| 1 | Rectangular: $\text{rect}\left(\frac{t}{T}\right)$ | $\frac{4\pi}{T}$ | −6 | −13.3 | -21dB |
| 2 | Bartlett: $\Delta\left(\frac{t}{2T}\right)$ | $\frac{8\pi}{T}$ | −12 | −26.5 | |
| 3 | Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$ | $\frac{8\pi}{T}$ | −18 | −31.5 | -44dB |
| 4 | Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$ | $\frac{8\pi}{T}$ | −6 | −42.7 | -53dB |
| 5 | Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$ | $\frac{12\pi}{T}$ | −18 | −58.1 | -74dB |
| 6 | Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{t}{T}\right)^2}\right]}{I_0(\alpha)}$ $0 \le \alpha \le 10$ | $\frac{11.2\pi}{T}$ | −6 | −59.9 ($\alpha = 8.168$) | |

- LP with: $\omega_p$ **=0.2π**, $\omega_s$ **=0.3π**, δ **=0.01**
- δ **=0.01:** The required peak error spec: =20log10 (δ) = –40 dB

Hanning Window

- Main-lobe width:

$\omega_s$-$\omega_p$=0.3π-0.2π =0.1π → 0.1π = 8π / *M*

→ Filter length M ≥ 80 & Filter order N ≥ 79

- BUT, Type-I filters have even order so N = 80

## Windowed Filter Design Example: Step 2: Specify the Ideal Response

- From Property 1 (Midpoint rule)

$\omega_c = (\omega_s + \omega_p)/2 = (0.2\pi+0.3\pi)/2 = 0.25\pi$

∴ An ideal response will be:

$$H_d(\omega) = \begin{cases} 1 & \text{if} \quad |\omega| \le 0.25\pi \\ 0 & \text{if} \quad 0.25\pi < |\omega| < \pi \end{cases}$$



## Windowed Filter Design Example: Step 3: Compute the coefficients of the ideal filter

- The ideal filter coefficients $h_d$ are given by the Inverse **Discrete time** Fourier transform of $H_d(\omega)$

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n}. \end{aligned}$$

\+ Delayed impulse response (to make it causal)

$$\widetilde{h}(n) = \widehat{h}\left(n - \frac{N-1}{2}\right)$$

- Coefficients of the ideal filter (via equation or IFFT):

$$h(n) = \frac{\sin(0.25\pi(n-40))}{\pi(n-40)}$$

## Windowed Filter Design Example: Step 4: Multiply to obtain the filter coefficients

$$h(n) = \frac{\sin(0.25\pi(n-40))}{\pi(n-40)}$$

- Multiply by a Hamming window function for the passband:

$$w(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{M}\right)$$



## Windowed Filter Design Example: Step 5: Evaluate the Frequency Response and Iterate

- The frequency response is computed as the DFT of the filter coefficient vector
- **If** the resulting filter does not meet the specifications, **then**:
  - Adjust the ideal filter frequency response (for example, move the band edge) and repeat (step 2)
  - Adjust the filter length and repeat (step 4)
  - change the window (& filter length) (step 4)
- And/Or consult with **`Matlab`**:
  - **`FIR1`** and **`FIR2`**
  - **`B=FIR1(N,Wn):`** Designs a N^th order FIR Window-Based FIR filter with passband given by
  - **`B=FIR2(N,F,M):`** Designs a N^th order FIR digital filter with arbitrary frequency response specified by vectors ***F*** and ***M*** .

➔All elements of `Wn` must be [0 1):
➔where **1 corresponds to the Nyquist frequency**: 0 < Wn < 1. The Nyquist frequency is half the sample rate or **π rad/sample**.

## Windowed Filter Design Example: Consulting Matlab:

- **`FIR1`** and **`FIR2`**
  - **`B=FIR2(N,F,M):`** Designs a Nth order FIR digital filter
  - **`F`** and **`M`** specify frequency and magnitude breakpoints for the filter such that **plot(N,F,M)** shows a plot of desired frequency
  - Frequencies **`F`** must be in increasing order between 0 and $\frac{F_s}{2}$, with $F_s$ corresponding to the sample rate.
  - **B** is the vector of length N+1, it is real, has linear phase and symmetric coefficients
  - Default window is Hamming – others can be specified



## FIR Properties

- Require no feedback.
- Are inherently stable.
- They can easily be designed to be linear phase by making the coefficient sequence symmetric
- Flexibility in shaping their magnitude response
- Very Fast Implementation (based around FFTs)

- The main disadvantage of FIR filters is that considerably more computation power in a general purpose processor is required compared to an IIR filter with similar sharpness or selectivity, especially when low frequency (relative to the sample rate) cutoffs are needed.

## FIR as a class of LTI Filters

- Transfer function of the filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

- Finite Impulse Response (FIR) Filters: (N = 0, no feedback)

➔From H(z):

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-\mathrm{i}\omega} + \cdots + h_{n-1} e^{-\mathrm{i}(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega - \mathrm{i} \sum_{t=0}^{n-1} h_t \sin t\omega \end{aligned}$$

∵ H(ω) is periodic and conjugate

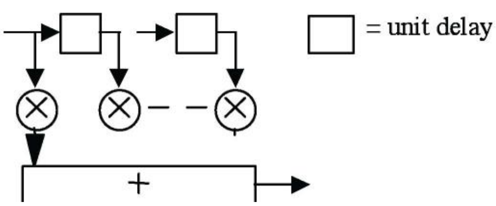
∴ Consider ω ∈ [0, π]



## FIR Filters

- Let us consider an FIR filter of length *M*
- Order *N*=*M*-1 **(watch out!)**
- Order → number of delays

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$



## FIR Impulse Response

Obtain the impulse response immediately with $x(n) = \delta(n)$:

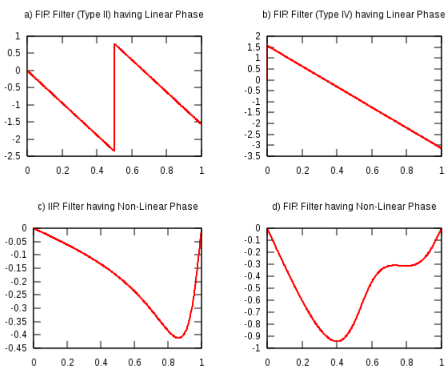
$$h(n) = y(n) = \sum_{k=0}^{M-1} b_k \delta(n-k) = b_n$$

- The impulse response is of finite length $M$ (good!)
- FIR filters have only zeros (no poles) (as they must, N=0 !!)
  - Hence known also as **all-zero** filters
- FIR filters also known as **feedforward** or **non-recursive,** or **transversal** filters



## FIR & Linear Phase

- The phase response of the filter is a linear function of frequency
- Linear phase has constant group delay, all frequency components have equal delay times. ∴ No distortion due to different time delays of different frequencies

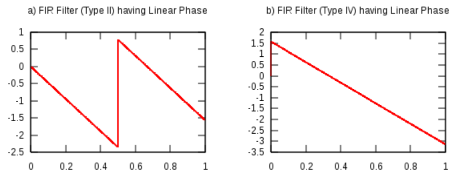



Ref: Wikipedia (Linear Phase)

- FIR Filters with:

$$\sum_{n=-\infty}^{\infty} h[n] \cdot \sin(\omega \cdot (n-\alpha) + \beta) = 0$$

## FIR & Linear Phase → Four Types




Ref: Wikipedia (Linear Phase)

| Impulse response | # coefs | $H(\omega)$ | Type |
|---|---|---|---|
| $h(n) = h(M-1-n)$ | Odd | $e^{-j\omega(M-1)/2}\left(h\left(\frac{M-1}{2}\right) + 2\sum_{k=1}^{(M-3)/2} h\left(\frac{M-1}{2} - k\right)\cos(\omega k)\right)$ | 1 |
| $h(n) = h(M-1-n)$ | Even | $e^{-j\omega(M-1)/2} 2\sum_{k=1}^{(M-3)/2} h\left(\frac{M}{2} - k\right)\cos\left(\omega\left(k - \frac{1}{2}\right)\right)$ | 2 |
| $h(n) = -h(M-1-n)$ | Odd | $e^{-j[\omega(M-1)/2-\pi/2]}\left(2\sum_{k=1}^{(M-1)/2} h\left(\frac{M-1}{2} - k\right)\sin(\omega k)\right)$ | 3 |
| $h(n) = -h(M-1-n)$ | Even | $e^{-j[\omega(M-1)/2-\pi/2]} 2\sum_{k=1}^{(M-1)/2} h\left(\frac{M}{2} - k\right)\sin\left(\omega\left(k - \frac{1}{2}\right)\right)$ | 4 |

- Type 1: most versatile
- Type 2: frequency response is always 0 at $\omega=\pi$
  **(not suitable as a high-pass)**
- Type 3 and 4: introduce a $\pi/2$ phase shift, 0 at $\omega=0$
  **(not suitable as a high-pass)**



## In Summary

- FIR Filters are digital (can not be implemented in analog) and exploit the difference and delay operators
- A window based design builds on the notion of a truncation of the "ideal" box-car or rectangular low-pass filter in the Frequency domain (which is a sinc function in the time domain)
- Other Design Methods exist:
  - Least-Square Design
  - Equiripple Design
  - Remez method
  - The Parks-McClellan Remez algorithm
  - Optimisation routines …

BREAK



## Next Time…

- **Adaptive Filters**
  - Who says the "taps" (filter coefficients) are constant ☺