http://elec3004.com

Digital Filter Analysis

ELEC 3004: **Systems**: Signals & Controls
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Lecture 10

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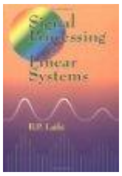
Lecture Schedule:

Week	Date	Lecture Title
1	27-Feb	Introduction
	1-Mar	Systems Overview
2	6-Mar	Systems as Maps & Signals as Vectors
	8-Mar	Systems: Linear Differential Systems
3	13-Mar	Sampling Theory & Data Acquisition
	15-Mar	Aliasing & Antialiasing
4	20-Mar	Discrete Time Analysis & Z-Transform
	22-Mar	Second Order LTID (& Convolution Review)
5	27-Mar	Frequency Response
	29-Mar	**Filter Analysis**
6	3-Apr	Digital Filters (IIR) & Filter Analysis
	5-Apr	PS 1: Q & A
7	10-Apr	Digital Filter (FIR) & Digital Windows
	12-Apr	FFT
8	17-Apr	Active Filters & Estimation & Holiday
	19-Apr	Holiday
	24-Apr	
	26-Apr	
9	1-May	Introduction to Feedback Control
	3-May	Servoregulation/PID
10	8-May	PID & State-Space
	10-May	State-Space Control
11	15-May	Digital Control Design
	17-May	Stability
12	22-May	State Space Control System Design
	24-May	Shaping the Dynamic Response
13	29-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
[TK5102.9.L38 1998]

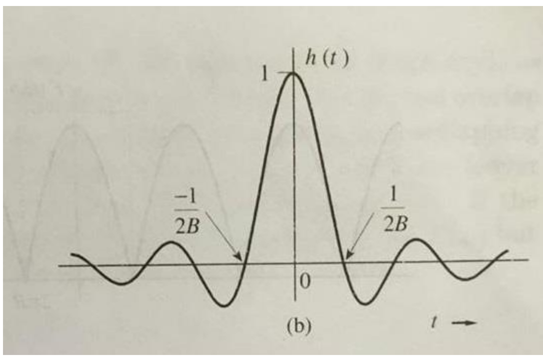
Today

- Chapter 10
  (Discrete-Time System Analysis Using the *z*-Transform)
  – § 10.3 Properties of DTFT
  – § 10.5 Discrete-Time Linear System analysis by DTFT
  – § 10.7 Generalization of DTFT to the $\mathcal{Z}$–Transform

- Chapter 12
  (**Frequency Response and Digital Filters**)
- § 12.1 Frequency Response of Discrete-Time Systems
- § 12.3 Digital Filters
- § 12.4 Filter Design Criteria
- § 12.7 Nonrecursive Filters

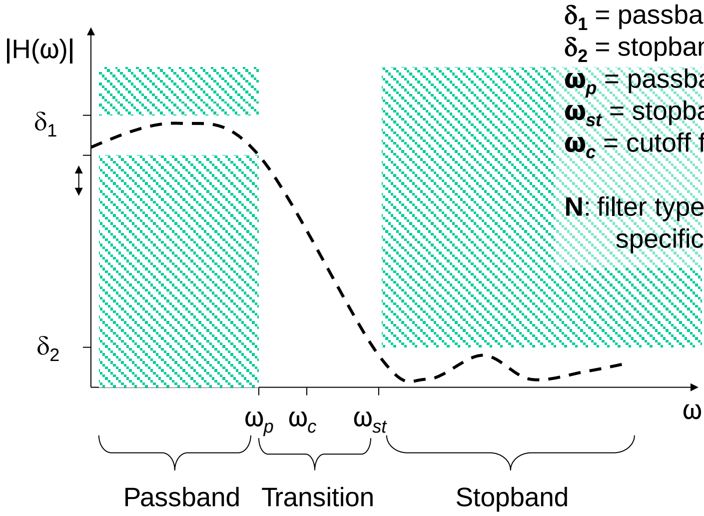
Next Time

Announcement!



- Please don't link external images/content please
  – It might expire and worse might disallow us from grading your solution ∵ it could be used to change the answer *a posteriori*
- Please don't link from Facebook as this [reveals source] ☺
  (12527949_1066290980057720_1984531858_n.jpg) (➔ https://www.facebook.com/1066290980057720)

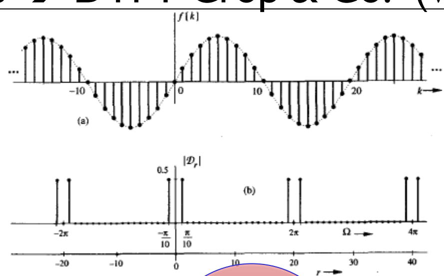
Filter Specification in the Frequency Domain



Where:
δ_1 = passband ripple (dB)
δ_2 = stopband attenuation (dB)
$\boldsymbol{\omega_p}$ = passband edge (Hz)
$\boldsymbol{\omega_{st}}$ = stopband edge (Hz)
$\boldsymbol{\omega_c}$ = cutoff frequency (@ 3dB)

N: filter type/order to meet specification

Digital Filters → DTFT Crop & Go! (Well, **No!**)



Lathi, p. 621

- First Thought:

FFT → Crop → Un-FFT → Go!

- How to get DTFT? FFT?
- Slightly Naïve ∵
  - For finite time span (or compact support), H(ω) cannot be exactly zero over any *band* of frequencies **(Paley-Wiener Theorem)**

Recall: DTFT is a Convolution

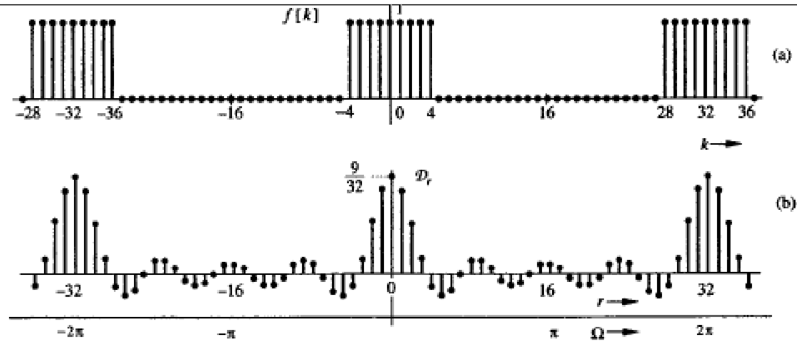


Fig. 10.2 Periodic sampled gate pulse and its Fourier spectrum. Lathi, p. 623

- The frequency response is limited to 2π
- DTFT is a convolution responses in time domain…

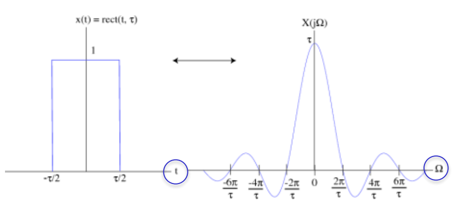
$$\underbrace{\mathcal{F}\{x * h\}}_{Y(\omega)} = \underbrace{\mathcal{F}\{x\}}_{X(\omega)} \cdot \underbrace{\mathcal{F}\{h\}}_{H(\omega)}$$

$$y[n] = x[n] * h[n] = \mathcal{F}^{-1}\big\{X(\omega) \cdot H(\omega)\big\},$$

Recall: Fourier Series & Rectangular Functions

$\mathfrak{F}$: Fourier Tranform

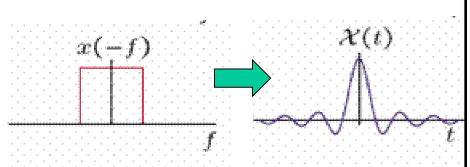
$$\mathfrak{F}\{rect(t)\} = sinc\left(\frac{\omega}{2}\right)$$



Ref: http://cnx.org/content/m32899/1.8/
http://www.thefouriertransform.com/pairs/box.php

$\mathfrak{F}^{-1}$: Inverse Fourier Tranform

$$\mathfrak{F}^{-1}\left\{rect\left(\frac{\omega}{2}\right)\right\} = \frac{sinc(t)}{\pi}$$



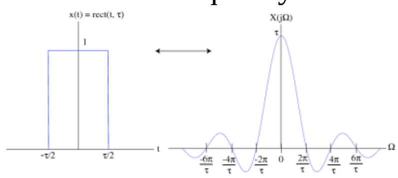
Ref: http://cnx.org/content/m26719/1.1/
http://www.wolframalpha.com/input/?i=IFFT%28sinc%28f%29%29

See Also:

- Table 7.1 (p. 702) Entry 17 & Table 9.1 (p. 852) Entry 7

Recall: Fourier Series & Rectangular Functions [2]

- The sinc function might look familiar
  - This is the frequency content of a square wave (box)



Ref: http://www.wolframalpha.com/input/?i=FFT%28rect%28t%29%29
http://cnx.org/content/m32899/1.8/

- This also applies to **signal reconstruction!**

➔ Whittaker–Shannon interpolation formula
  - This says that the "better way" to go from Discrete to Continuous (i.e. D to A) is not ZOH, but rather via the sinc!

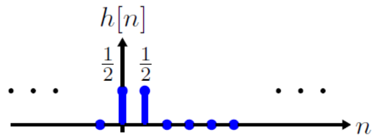
$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

Before we get to Filters…

(digital) **Signal Types**
& Corresponding Digital Filter Types!

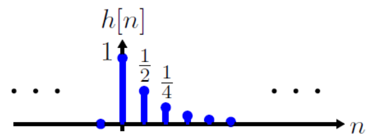
Two Types of Impulse Response

$$y[n] = \frac{1}{2}u[n-1] + \frac{1}{2}u[n]$$



'Finite impulse response" (FIR)

$$y[n] = \frac{1}{2}y[n-1] + u[n]$$



"Infinite impulse response" (IIR)

→ Digital Filters Types

FIR

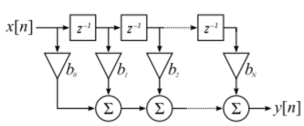
From H(z):

→ $$H(\omega) = h_0 + h_1 e^{-\mathrm{i}\omega} + \cdots + h_{n-1} e^{-\mathrm{i}(n-1)\omega} = \sum_{t=0}^{n-1} h_t \cos t\omega - \mathrm{i} \sum_{t=0}^{n-1} h_t \sin t\omega$$

→ Filter becomes a "multiply, accumulate, and delay" system:

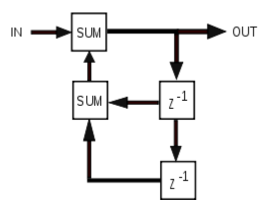
$$y(t) = \sum_{\tau=0}^{n-1} h_\tau u(t-\tau)$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N]$$



IIR

- Impulse response function that is non-zero over an infinite length of time.



FIR Properties

- Require no feedback.
- Are inherently stable.
- They can easily be designed to be linear phase by making the coefficient sequence symmetric
- Flexibility in shaping their magnitude response
- Very Fast Implementation (based around FFTs)

- The main disadvantage of FIR filters is that considerably more computation power in a general purpose processor is required compared to an IIR filter with similar sharpness or selectivity, especially when low frequency (relative to the sample rate) cutoffs are needed.

FIR as a class of LTI Filters

- Transfer function of the filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

- Finite Impulse Response (FIR) Filters: (N = 0, no feedback)

➔From H(z):

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-\mathrm{i}\omega} + \cdots + h_{n-1} e^{-\mathrm{i}(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega - \mathrm{i} \sum_{t=0}^{n-1} h_t \sin t\omega \end{aligned}$$

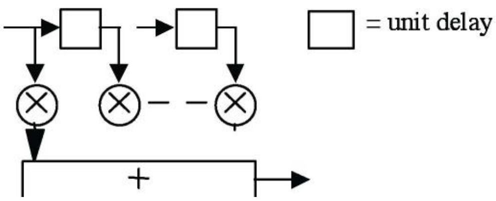
∵ H(ω) is periodic and conjugate

∴ Consider ω ∈ [0, π]

FIR Filters

- Let us consider an FIR filter of length M
- Order N=M-1 **(watch out!)**
- Order → number of delays

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$



FIR Impulse Response

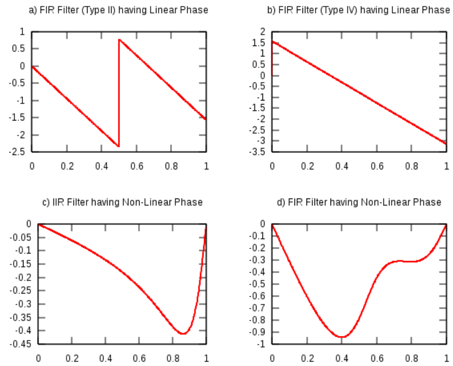
Obtain the impulse response immediately with $x(n)$= $\delta(n)$:

$$h(n) = y(n) = \sum_{k=0}^{M-1} b_k \delta(n-k) = b_n$$

- The impulse response is of finite length M (good!)
- FIR filters have only zeros (no poles) (as they must, N=0 !!)
  - Hence known also as **all-zero** filters
- FIR filters also known as **feedforward** or **non-recursive,** or **transversal** filters

FIR & Linear Phase

- The phase response of the filter is a linear function of frequency



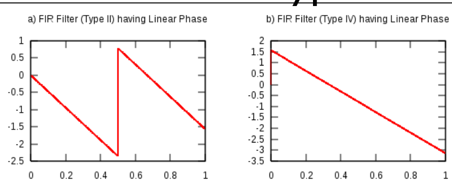
Ref: Wikipedia (Linear Phase)

- Linear phase has constant group delay, all frequency components have equal delay times. ∴ No distortion due to different time delays of different frequencies

- FIR Filters with:

$$\sum_{n=-\infty}^{\infty} h[n] \cdot \sin(\omega \cdot (n-\alpha)+\beta) = 0$$

FIR & Linear Phase → Four Types



Ref: Wikipedia (Linear Phase)

Impulse response	# coefs	$H(\omega)$	Type
$h(n) = h(M-1-n)$	Odd	$e^{-j\omega(M-1)/2}\left(h\left(\frac{M-1}{2}\right) + 2\sum_{k=1}^{(M-3)/2} h\left(\frac{M-1}{2}-k\right)\cos(\omega k)\right)$	1
$h(n) = h(M-1-n)$	Even	$e^{-j\omega(M-1)/2} 2\sum_{k=1}^{(M-3)/2} h\left(\frac{M}{2}-k\right)\cos\left(\omega\left(k-\frac{1}{2}\right)\right)$	2
$h(n) = -h(M-1-n)$	Odd	$e^{-j[\omega(M-1)/2-\pi/2]}\left(2\sum_{k=1}^{(M-1)/2} h\left(\frac{M-1}{2}-k\right)\sin(\omega k)\right)$	3
$h(n) = -h(M-1-n)$	Even	$e^{-j[\omega(M-1)/2-\pi/2]} 2\sum_{k=1}^{(M-1)/2} h\left(\frac{M}{2}-k\right)\sin\left(\omega\left(k-\frac{1}{2}\right)\right)$	4

- Type 1: most versatile
- Type 2: frequency response is always 0 at $\omega=\pi$
  (not suitable as a high-pass)
- Type 3 and 4: introduce a $\pi/2$ phase shift, 0 at $\omega=0$
  (not suitable as a high-pass)

Discrete Time Fourier Transform

2D DFT

$$
\begin{aligned}
\mathcal{F}(u,v) &= \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux+vy)/N} \\
f(x,y) &= \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \mathcal{F}(u,v)e^{j2\pi(ux+vy)/N}
\end{aligned}
$$

2D DFT

- Each DFT coefficient is a complex value
  - There is a single DFT coefficient for each spatial sample
  - A complex value is expressed by two real values in either Cartesian or polar coordinate space.
    - Cartesian: R(u,v) is the *real* and I(u, v) the *imaginary* component
    - Polar: |F(u,v)| is the *magnitude* and phi(u,v) the *phase*

$$\mathcal{F}(u,v) = R(u,v) + jI(u,v)$$

$$\mathcal{F}(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

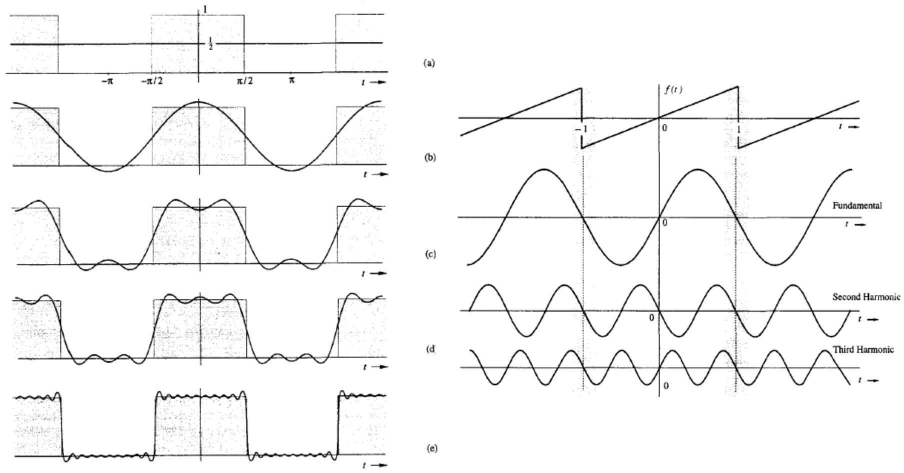
2D DFT

- Representing the DFT coefficients as magnitude and phase is a more useful for processing and reasoning.
  - The magnitude is a measure of strength or length
  - The phase is a direction and lies in [-pi, +pi]
- The magnitude and phase are easily obtained from the real and imaginary values

$$\begin{aligned} |\mathcal{F}(u,v)| &= \sqrt{R^2(u,v) + I^2(u,v)} \\ \phi(u,v) &= \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]. \end{aligned}$$

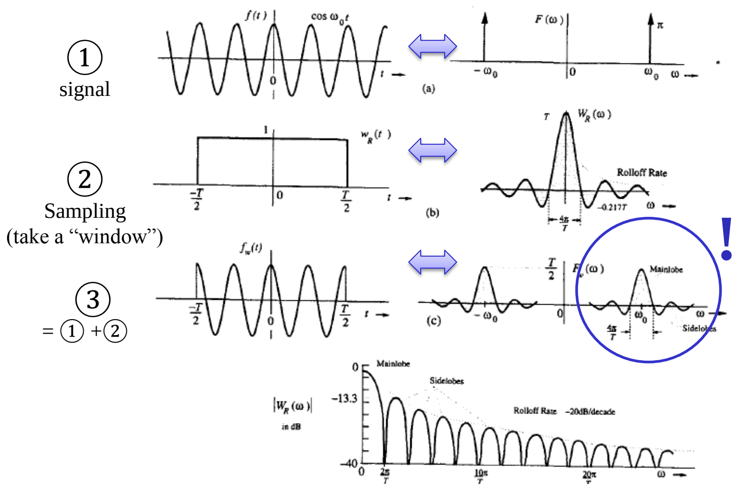
Harmonics

- Synthesis of a square pulse: periodic signal by successive addition of its harmonics (Lathi, p. 202-3)



☆Windowing for the DFT

$$f_w(t) = f(t)w(t) \quad \text{and} \quad F_w(\omega) = \frac{1}{2\pi}F(\omega) * W(\omega)$$



Source: Lathi, p.303

Digitial Filters
the DT Fourier Transform
And the Z-Transform!

DTFT ➔ z-Transform

The above results motivate the definitions of the z transform, the discrete-time Fourier transform (DTFT), and the discrete Fourier series (DFS) to be presented in this chapter and the next. In particular, if the basis functions for the input can be enumerated as

$$\phi_k[n] = z_k^n,$$

that is, if $x(t)$ can be expressed in the form of Eq. (6.1.1) as

$$x[n] = \sum_k a_k z_k^n, \qquad (6.1.10)$$

then the corresponding output is simply, from Eqs. (6.1.2) and (6.1.8),

$$y[n] = \sum_k a_k H(z_k) z_k^n. \qquad (6.1.11)$$

The discrete Fourier series for periodic signals is of this form, with $z_k = e^{j2\pi k/N}$. If, on the other hand, the required basis functions cannot be enumerated, we must utilize the continuum of functions $\phi[n] = z^n$ to represent $x[n]$ and $y[n]$ in the form of integrals. When z is restricted to have unit magnitude (that is, $z = e^{j\Omega}$), the resulting representation is called the *discrete-time Fourier transform*, while if z is an arbitrary complex variable, the full *z-transform* representation results.

The Discrete-Time Fourier Transform

- Synthesis:

The function $X(e^{j\Omega})$ defined by

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n} \qquad (7.1.1)$$

(if it converges) is called the *discrete-time Fourier transform* (*DTFT*) of the signal $x[n]$. In particular, if the region of convergence for the z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

includes the unit circle, then the DTFT equals $X(z)$ evaluated on the unit circle, that is,

$$X(e^{j\Omega}) = X(z)\big|_{z=e^{j\Omega}}. \qquad (7.1.2)$$

The Discrete-Time Fourier Transform

- Analysis/Inverse:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\Omega})e^{j\Omega n}\, d\Omega.$$

- *x[n]* is the (limiting) sum of sinusoidal components of the form $\left[\frac{1}{2\pi}X\left(e^{j\Omega}\right)d\Omega\right] e^{j\Omega n}$
- Together: Forms the DTFT Pair

The Discrete-Time Fourier Transform

- Ex:

$$x[n] = a^n u[n]$$

has the z transform

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|,$$

and thus $X(e^{j\Omega})$ exists for $|a| < 1$ because the ROC then contains the unit circle. Specifically,

$$X(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}}, \qquad |a| < 1. \qquad (7.1.8)$$

The corresponding *magnitude spectrum* $|X(e^{j\Omega})|$ and *phase spectrum* $\angle X(e^{j\Omega})$ are shown in Fig. 6.8. Clearly, from the defining sum in Eq. (7.1.1), the DTFT of $x[n]$ does not converge for $|a| > 1$, and we defer until later the case of $|a| = 1$.

On the other hand, the anticausal exponential

$$w[n] = -a^n u[-n-1]$$

has the z transform

$$W(z) = \frac{1}{1 - az^{-1}}, \qquad |z| < |a|,$$

and thus $W(e^{j\Omega})$ exists for $|a| > 1$, but not for $|a| < 1$. That is,

$$W(e^{j\Omega}) = \frac{1}{1 - ae^{-j\Omega}}, \qquad |a| > 1. \qquad (7.1.9)$$

Again the case of $|a| = 1$ is deferred until later.

The Discrete-Time Fourier Transform

- Observe:
  "Kinship Of Difference Equations To Differential Equations"

$$\frac{dy}{dt} + cy(t) = x(t) \qquad (3.15a)$$

Consider uniform samples of $x(t)$ at intervals of T seconds. As usual, we use the notation $x[n]$ to denote $x(nT)$, the nth sample of $x(t)$. Similarly, $y[n]$ denotes $y[nT]$, the nth sample of $y(t)$. From the basic definition of a derivative, we can express Eq. (3.15a) at $t = nT$ as

$$\lim_{T \to 0} \frac{y[n] - y[n-1]}{T} + cy[n] = x[n]$$

Clearing the fractions and rearranging the terms yields (assuming nonzero, but very small T)

$$y[n] + \alpha y[n-1] = \beta x[n] \qquad (3.15b)$$

where

$$\alpha = \frac{-1}{1 + cT} \quad \text{and} \quad \beta = \frac{T}{1 + cT}$$

We can also express Eq. (3.15b) in advance operator form as

$$y[n+1] + \alpha y[n] = \beta x[n+1] \qquad (3.15c)$$

The Discrete-Time Fourier Transform

- Ex(2): The DTFT of the real sinusoid

$$x[n] = \sin \Omega_0 n = \frac{1}{2j}(e^{j\Omega_0 n} - e^{-j\Omega_0 n})$$

is simply

$$X(e^{j\Omega}) = 2\pi\left(\frac{1}{2j}\right)[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$$

$$= -j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$$

for $|\Omega|, |\Omega_0| \leq \pi$, while that of the cosine signal

$$y[n] = \cos \Omega_0 n = \tfrac{1}{2}(e^{j\Omega_0 n} + e^{-j\Omega_0 n})$$

is likewise

$$Y(e^{j\Omega}) = 2\pi(\tfrac{1}{2})[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$$

$$= \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)].$$

In addition, the DTFT pair for the dc signal $x[n] = 1$ is simply

$$1 \leftrightarrow 2\pi\,\delta(\Omega), \qquad |\Omega| \leq \pi,$$

as opposed to the dual relationship

$$\delta[n] \leftrightarrow 1, \qquad \text{all } \Omega.$$

BREAK

Now: (digital) **Filters!**

Filter Design

- Previously we have analysed
  - difference equations (y[n])
  - transfer functions (H(z))
- To obtain time/frequency domain response
  - Impulse (h[n]) or frequency (H(w)) response
- Now we have a specification
  - frequency response (filters)
  - time response (control)
- Goal to design a filter that meets specification
  - i.e., determine transfer function
  - and therefore difference equation (implementation)

Transfer Function → Difference Equation

- Example, consider

$$H(z)=\frac{z^2-0.2z-0.08}{z^2+0.5}$$

Make H(z) causal × by $\frac{z^{-2}}{z^{-2}}$

- Normalise to negative powers of z (causal)
  - re-arrange and take inverse z transform

$$H(z)=\frac{1-0.2z^{-1}-0.08z^{-2}}{1+0.5z^{-2}}=\frac{Y(z)}{X(z)}$$

$$Y(z)\left(1+0.5z^{-2}\right)=X(z)\left(1-0.2z^{-1}-0.08z^{-2}\right)$$

$$y[n]+0.5y[n-2]=x[n]-0.2x[n-1]-0.08x[n-2]$$

$$y[n]=x[n]-0.2x[n-1]-0.08x[n-2]-0.5y[n-2]$$

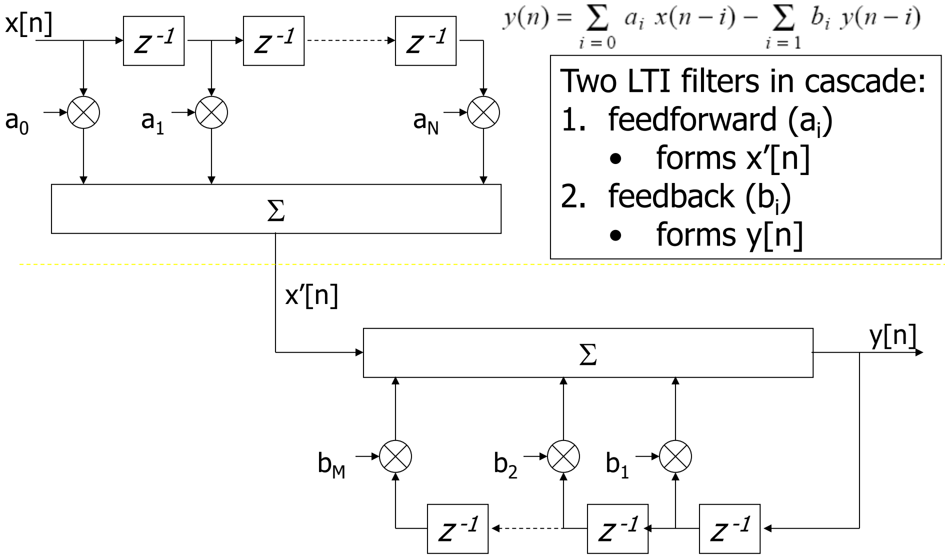
Direct Form I: Direct realisation of digital filter

$$y(n)=\sum_{i=0}^{N} a_i\, x(n-i) - \sum_{i=1}^{M} b_i\, y(n-i)$$

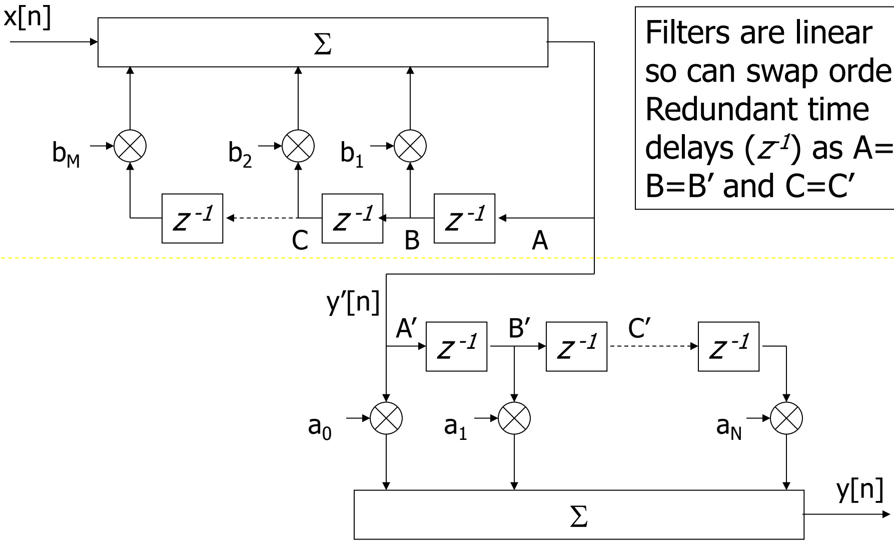
+

Two LTI filters in cascade:

1. feedforward (a_i)
   - forms x′[n]
2. feedback (b_i)
   - forms y[n]



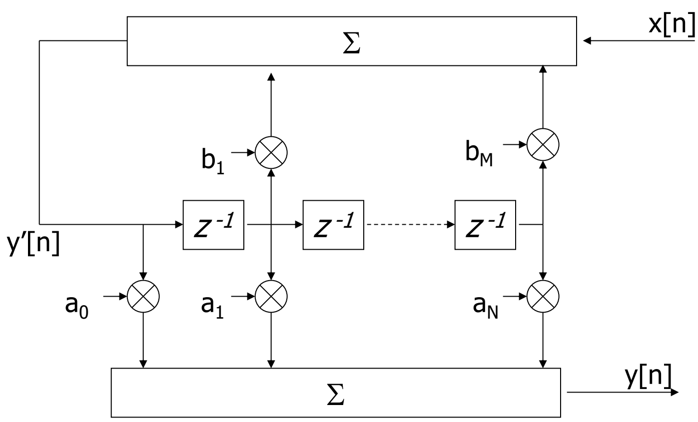
Reordered form of realisation



Filters are linear so can swap order. Redundant time delays (z^{-1}) as A=A′ B=B′ and C=C′

Note: y′[n] ≠ x′[n] of previous slide BUT y[n] = y[n] ☺ so, same filter

Direct form II: Canonical form of realisation (minimum memory)



redundant time delays removed

Derivation of Canonical Form

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{i=0}^{N} a_i z^{-i}}{(1 - \sum_{i=1}^{M} b_i z^{-i})}$$ General form of transfer function

$$Y(z) = H(z)\, X(z)$$ Re-arranging in terms of output

$$Y(z) = \sum_{i=0}^{N} a_i z^{-i}\, Y'(z)$$ where $$Y'(z) = \frac{X(z)}{(1 - \sum_{i=1}^{M} b_i z^{-i})}$$

Which as a difference equation is

Direct II $$y(n) = \sum_{i=0}^{N} a_i\, y'(n-i)$$ where $$y'(n) = x(n) + \sum_{i=1}^{M} b_i\, y'(n-i),$$

Remember

Direct I $$y(n) = \sum_{i=0}^{N} a_i\, x(n-i) + \sum_{i=1}^{M} b_i\, y(n-i)$$

Canonical terms
A′ B′ C′

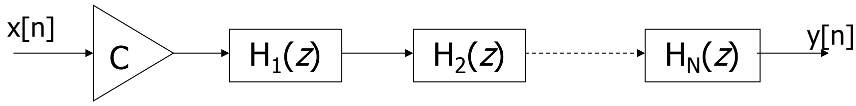
Canonical Realisation

- Direct Form I
  - Conceptually simplest realisation
  - Often less susceptible to noise
- Canonical/Direct Form II
  - Minimimum memory (storage)
- Filter design
  - Determine value of filter coefficients (all ai & bi)
  - Poles controlled by bi coefficients
    - if any bi ≠ 0 then filter IIR (recursive)
    - if all bi = 0 then filter FIR (non-recursive)
  - Zeros controlled by ai coefficients

Cascade Form

- Transfer function factorised to
  - Product of second order terms Hn(z)
  - C is a constant (gain)

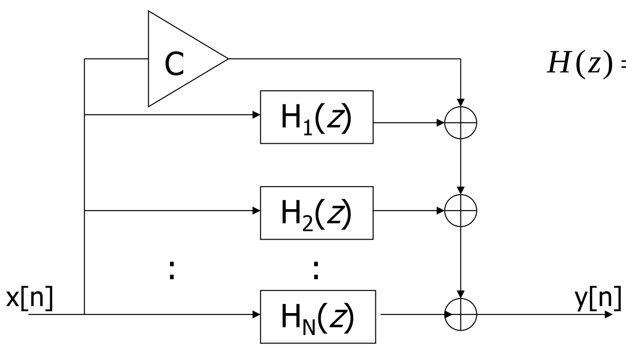
$$H(z) = C\prod_{n=1}^{N} H_n(z)$$



Most common realisation
Often assumed by many filter design packages
many 2nd order sections have integer coefficients

Parallel Form

- Transfer function expressed as
  - partial fraction expansion of second order terms

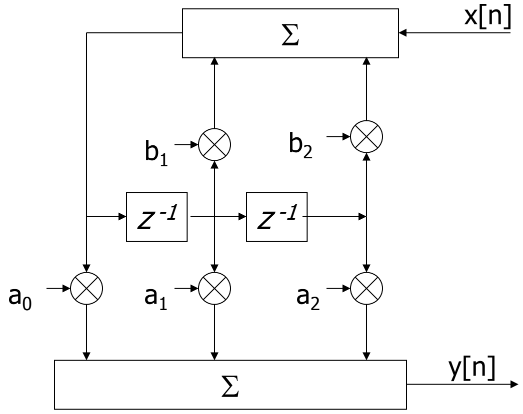


$$H(z) = C + \sum_{n=1}^{N} H_n(z)$$

Least sensitive to coefficient errors, i.e., when limited No. bits to represent real ($\Re$) coefficient

Bi-quadratic Digital Filter

- Canonic form of Second order system
- 2nd order, system 'building block'



Difference equation:

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + b_1 y[n-1] + b_2 y[n-2]$$

IIR Filter Design Methods

- Normally based on analogue prototypes
  - Butterworth, Chebyshev, Elliptic etc
- Then transform H(s) → H(z)
- Three popular methods:
- Impulse invariant
  - produces H(z) whose impulse response is a sampled version of h(t) (also step invariant)
- Matched z – transform
  - poles/zeros H(s) directly mapped to poles/zeros H(z)
- Bilinear z – transform
  - left hand s – plane mapped to unit circle in z - plane

Impulse Invariant

- Simplest approach, proceeds as follows,
- Select prototype analogue filter
- Determine H(s) for desired wc and ws
- Inverse Laplace,
  - i.e., calculate impulse response, h(t)
- Sample impulse response $h(t)|t=n\Delta td$
  - $h[n] = \Delta td\ h(n\Delta td)$
- Take z - transform of $h[n] \Rightarrow H(z)$
  - poles, p1 map to $\exp(p1\Delta td)$ (maintains stability)
  - zeros have no simple mapping

Impulse Invariant

- Useful approach when
  - Impulse (or step) invariance is required, or
    - e.g., control applications
  - Designing Lowpass or Bandpass filters
- Has problems when
  - H(w) does not $\rightarrow 0$ as $w \rightarrow \infty$
  - i.e., if H(w) is not bandlimited, aliasing occurs
  - e.g., highpass or bandstop filters

Matched z - transform

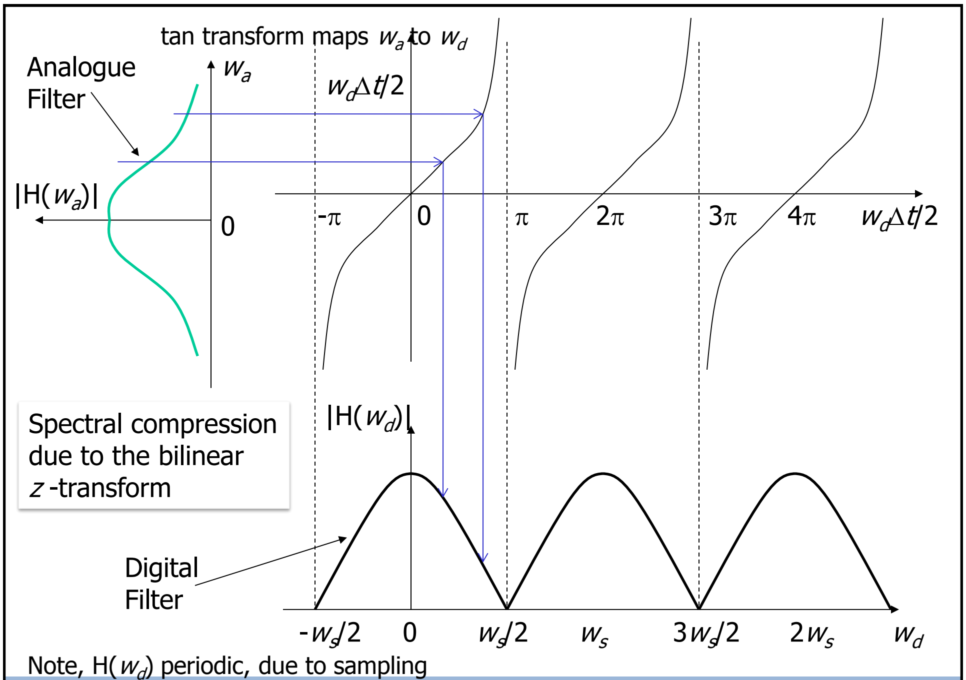
- Maps poles/zeros in s – plane directly
  - to poles/zeros in z – plane
- No great virtues/problems
- Fairly old method
  - not commonly used
  - so we won't consider it further

Bilinear z - transform

- Maps complete imaginary s –plane (±∞)
  - to unit circle in z -plane
- i.e., maps analogue frequency wa to
  - discrete frequency wd
- uses continuous transform,

$$w_a = \frac{2}{\Delta t}\tan\left(\frac{w_d \Delta t}{2}\right)$$

This compresses (warps) w_a to have finite extent $\pm w_s/2$
i.e., this removes possibility of any aliasing ☺



Bilinear Transform

$$\omega_a = \frac{2}{\Delta t}\tan\left(\frac{\omega_d \Delta t}{2}\right)$$

The bilinear transform

$$s = \frac{2}{\Delta t}\,\frac{j\sin\left(\frac{\omega_d \Delta t}{2}\right)}{\cos\left(\frac{\omega_d \Delta t}{2}\right)}$$

Transforming to s-domain
Remember: $s = j\omega_a$
and $\tan\theta = \sin\theta/\cos\theta$
Where $\theta = \omega_d \Delta t/2$

$$s = \frac{2}{\Delta t}\,\frac{\frac{1}{2}\left(\exp\left(\frac{j\omega_d \Delta t}{2}\right) - \exp\left(\frac{-j\omega_d \Delta t}{2}\right)\right)}{\frac{1}{2}\left(\exp\left(\frac{j\omega_d \Delta t}{2}\right) + \exp\left(\frac{-j\omega_d \Delta t}{2}\right)\right)}$$

Using Euler's relation
This becomes…
(note: j terms cancel)

$$s = \frac{2}{\Delta t}\,\frac{(1-\exp(-j\omega_d \Delta t))}{(1+\exp(-j\omega_d \Delta t))}$$

Multiply by $\exp(-j\theta)/\exp(-j\theta)$

$$s = \frac{2(1-z^{-1})}{\Delta t(1+z^{-1})}$$

As $z = \exp(s_d \Delta t) = \exp(j\omega_d \Delta t)$

Bilinear Transform

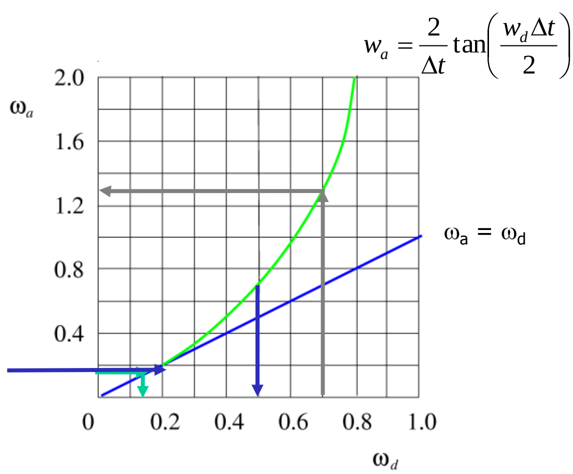
- Convert H(s) ⇒ H(z) by substituting,

$$s = \frac{2\left(1 - z^{-1}\right)}{\Delta t\left(1 + z^{-1}\right)}$$

Note: this comes directly from $e^{st} \cdot (\frac{½}{½})$ or the tan transform

- However, this transformation compresses the analogue frequency response, which means
  - digital cut off frequency will be lower than the analogue prototype
- Therefore, analogue filter must be "pre-warped" prior to transforming H(s) ⇒ H(z)

Bilinear Pre-warping



Bilinear Transform: Example

- Design digital Butterworth lowpass filter
  - order, n = 2, cut off frequency wd = 628 rad/s
  - sampling frequency ws = 5024 rad/s (800Hz)
- pre-warp to find wa that gives desired wd

$$w_a = \left(\frac{2}{1/800}\right)\tan\left(\frac{628}{2\times 800}\right) = 663\,\text{rad/s}$$

Note: $w_d < w_a$ due to compression

- Butterworth prototype (unity cut off) is,

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Bilinear Transform: Example

- De-normalised analogue prototype (s' = s/ ω_c)
  - $\omega_c = 663\ rad/s$ (required ω_a to give desired)

$$H(s_d) = \frac{1}{\left(\frac{s}{663}\right)^2 + \frac{\sqrt{2}s}{663} + 1}$$

  - Convert H(s) ⇒ H(z) by substituting

$$s = \frac{2(1 - z^{-1})}{\Delta t(1 + z^{-1})}$$

$$H(z) = \frac{1}{\left(\frac{2\times 800(1-z^{-1})}{663(1+z^{-1})}\right)^2 + \sqrt{2}\left(\frac{2\times 800(1-z^{-1})}{663(1+z^{-1})}\right) + 1}$$

$$H(z) = \frac{0.098z^2 + 0.195z + 0.098}{z^2 - 0.942z + 0.333}$$

Note: H(z) has both poles and zeros
H(s) was all-pole

Bilinear Transform: Example

$$H(z)=\frac{Y(z)}{X(z)}=\frac{0.098z^2+0.195z+0.098}{z^2-0.942z+0.333}$$

- Multiply out and make causal:

$$Y(z)(z^2-0.942z+0.333)=X(z)(0.098z^2+0.195z+0.098)$$

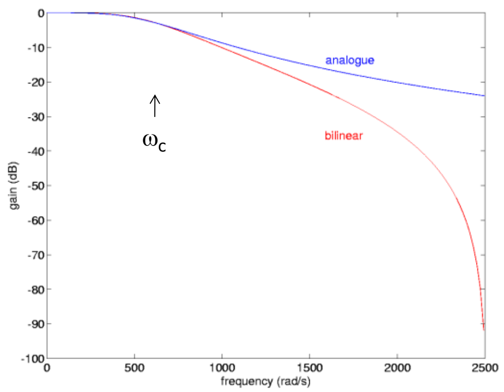
$$Y(z)(1-0.942z^{-1}+0.333z^{-2})=X(z)(0.098+0.195z^{-1}+0.098z^{-2})$$

Finally, apply inverse z-transform to yield the difference equation:

$$y[n]=0.098x[n]+0.195x[n-1]+0.098x[n-2]$$
$$+0.942y[n-1]-0.333y[n-2]$$

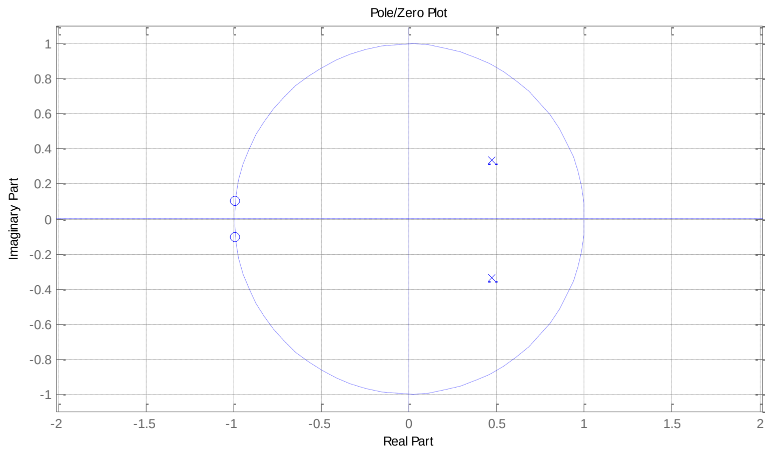
Bilinear Transform: Example

Magnitude response



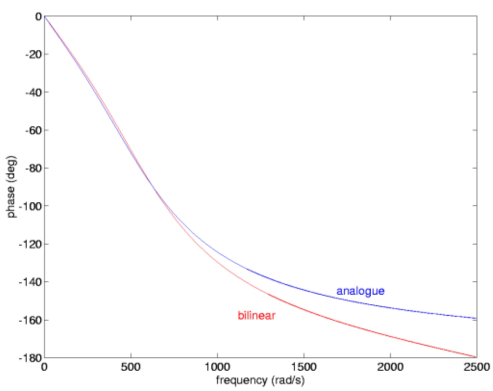
1. same cut off frequency,
2. increased roll off and attenuation in stopband
3. ∞ attenuation at $w_s/2$

Bilinear Transform: Example



Bilinear Transform: Example

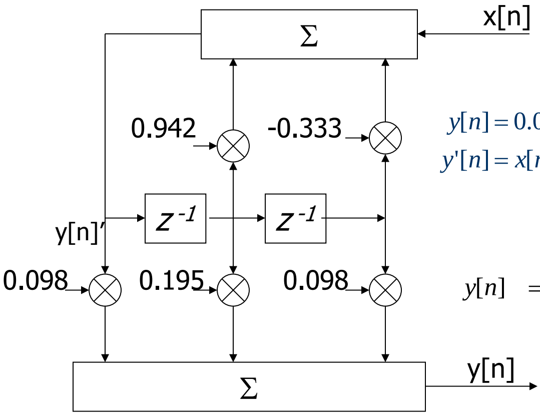
Phase response



Increased phase delay

Bilinear transform has effectively increased digital filter order (by adding zeros)

Bilinear Transform: Example



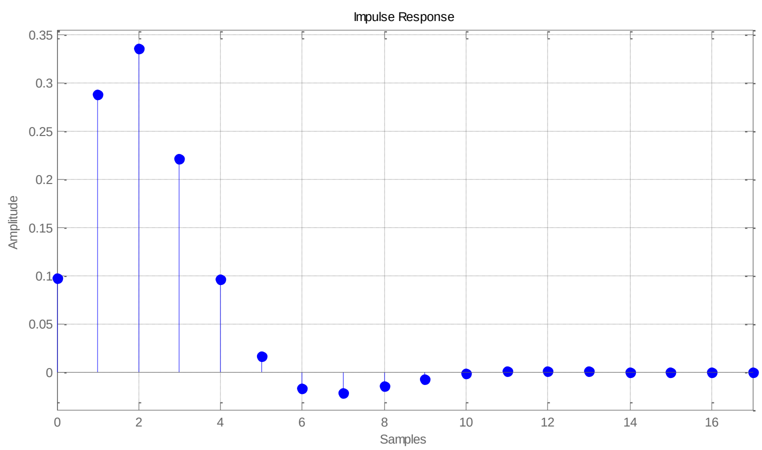
Canonical Implementation

$$y[n] = 0.098y'[n] + 0.195y'[n-1] + 0.098y'[n-2]$$
$$y'[n] = x[n] + 0.942y'[n-1] - 0.333y'[n-2]$$

of the difference equation

$$y[n] = 0.098x[n] + 0.195x[n-1] + 0.098x[n-2] + 0.942y[n-1] - 0.333y[n-2]$$

Bilinear Transform: Example



Bilinear Design Summary

- Calculate pre-warping analogue cutoff frequency
- De-normalise filter transfer function using pre-warping cut-off
- Apply bilinear transform and simplify
- Use inverse z-transform to obtain difference equation

Direct Synthesis

- Not based on analogue prototype
  - But direct placement of poles/zeros
- Useful for
  - First order lowpass or highpass
    - simple smoothers
  - Resonators and equalisers
    - single frequency amplification/removal
  - Comb and notch filters
    - Multiple frequency amplification/removal

First Order Filter: Example

- General first order transfer function
  - Gain, G, zero at –b, pole at a (a, b both < 1)

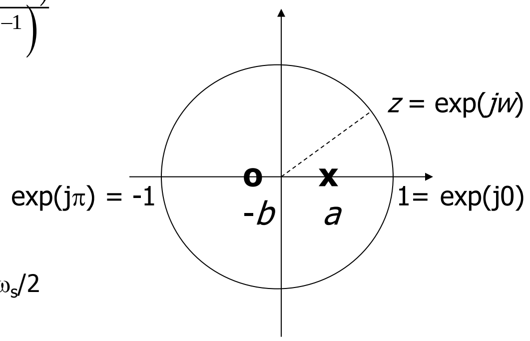
Remember: $H(w) = H(z)|_{z = \exp(jw\Delta t)}$

$$H(z) = \frac{G(1+bz^{-1})}{(1-az^{-1})}$$

with *a* +ve & *b* –ve
this is a lowpass filter
i.e.,

$$H(0) = \frac{G(1+b)}{(1-a)}$$

$$H(\pi) = \frac{G(1-b)}{(1+a)} \qquad \omega_s/2$$



First Order Filter: Example

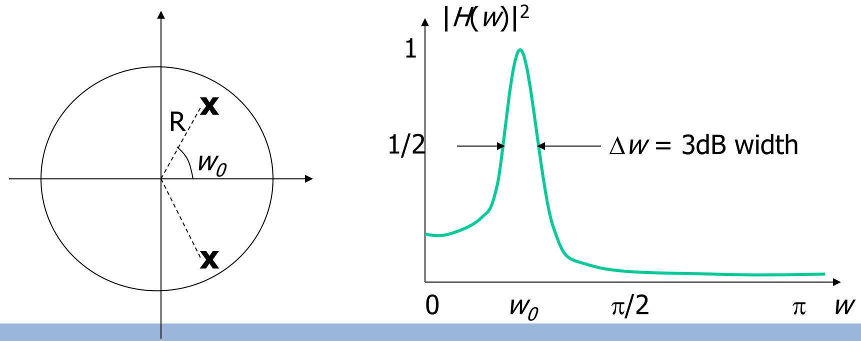
- Possible design criteria
  - cut-off frequency, wc
    - 3dB = 20 log(|H (wc)|)
    - e.g., at wc = π/2, (1+b)/(1+a) = √2
  - stopband attenuation
    - assume wstop = π (Nyquist frequency)
    - e.g., δ2 = H (π)/H (0) = 1/21 i.e.,

$$\frac{H(\pi)}{H(0)} = \frac{(1-b)(1-a)}{(1+b)(1+a)} = \frac{1}{21}$$

two unknowns (*a*,*b*)
two (simultaneous)
design equations.

Digital Resonator

- Second order ‘resonator’
  - single narrow peak frequency response
  - i.e., peak at resonant frequency, w0



Quality factor (Q-factor)

- Dimensionless parameter that compares
  - Time constant for oscillator decay/bandwidth ($\Delta\omega$) to
  - Oscillation (resonant) period/frequency (ω0)
    - High Q = less energy dissipated per cycle

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{f_0}{\Delta f}$$

- Alternative to damping factor (ζ) as

$$Q = \frac{1}{2\zeta} \qquad H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

- Note: Q < ½ overdamped (not an oscillator)

Digital Resonator Design

- To make a peak at w0 place pole
  - Inside unit circle (for stability)
  - At angle w0 distance R from origin
    - i.e., at location p = R exp(jw0)
      - R controls Δw
        - » Closer to unit circle → sharper peak
    - plus complex conj pole at p* = R exp(-jw0)

$$\begin{aligned} H(z) &= \frac{1}{\left(1 - R \cdot \exp(jw_0) z^{-1}\right)\left(1 - R \cdot \exp(-jw_0) z^{-1}\right)} \\ &= \frac{1}{1 - R\left(\exp(jw_0) + \exp(-jw_0)\right) z^{-1} + R^2 z^{-2}} \\ &= \frac{G}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned}$$

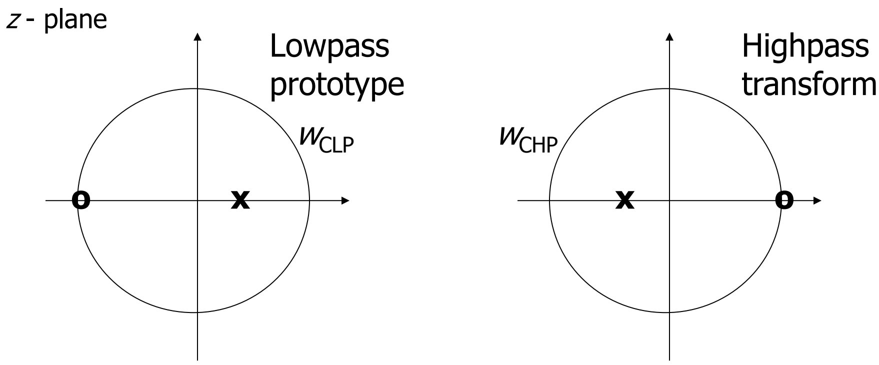
Where (via Euler's relation)

$$a_1 = -2R\cos(w_0) \text{ and } a_2 = R^2$$

Discrete Filter Transformations

- By convention, design Lowpass filters
  - transform to HP, BP, BS, etc
- Simplest transformation
  - Lowpass H(z') → highpass H(z)
  - HHP(z) = HLP(z)|z' → -z
    - reflection about imaginary axis (ws/4)
    - changing signs of poles and zeros
- LP cutoff frequency, wCLP becomes
    - HP cut-in frequency, wCHP = ½ - wCLP

Lowpass → highpass (z' = -z)



$p_L = ¼$, $z_L = -1$ $p_H = -¼$, $z_H = 1$

Poles/zeros reflected in imaginary axis: $w_{CHP} = ½ - w_{CLP}$

Same gain @ $w_s/4$ (i.e., $\pi/4$)

$|H(w_{HP})| = |H(\pi/2 - w_{LP})|$

Discrete Filter Transformations

- Lowpass H(z') → highpass H(z)
  - Cut-off (3dB) frequency = wc (remains same)
- Lowpass H(z') → Bandpass H(z)
  - Centre frequency = w0 & 3dB bandwidth = wc

$$z' = \frac{\cos(w_c \Delta t) - z}{1 - \cos(w_c \Delta t) z}$$

$$z' = \frac{\alpha z - z^2}{-\alpha z + 1} \qquad \alpha = \frac{\cos(w_0 \Delta t)}{\cos(w_c \Delta t)}$$

Note: these are not the only possible BP and BS transformations!

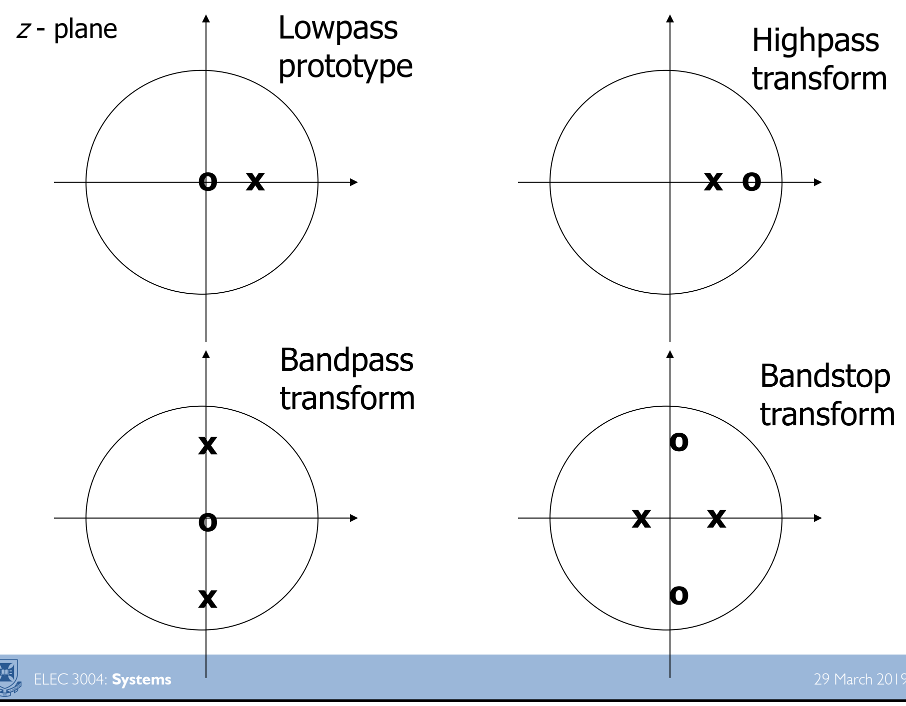
Discrete Filter Transformations

- Lowpass H(z') → Bandstop H(z)
  - Centre frequency = w0   3dB bandwidth = wc

$$z' = \frac{z^2 - \left(2\alpha/(k+1)\right)z + (1-k)/(1+k)}{1 + \left(2\alpha/(k+1)\right)z + \left((1-k)/(1+k)\right)z^2}$$

$$\alpha = \frac{\cos(w_0\Delta t)}{\cos(w_c\Delta t)} \qquad k = \tan^2(w_c\Delta t)$$

Note: order doubles for bandpass/bandstop transformations



Summary

- Digital Filter Structures
  - Direct form (simplest)
  - Canonical form (minimum memory)
- IIR filters
  - Feedback and/or feedforward sections
- FIR filters
  - Feedforward only
- Filter design
  - Bilinear transform (LP, HP, BP, BS filters)
  - Direct form (resonators and notch filters)
  - Filter transformations (LP $\rightarrow$ HP, BP, or BS)
- Stability & Precision improved
  - Using cascade of 1st/2nd order sections

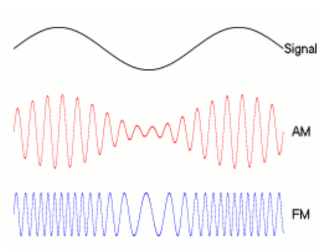
How to Beat the Noise?

Idea 2: **Modulation**

Modulation

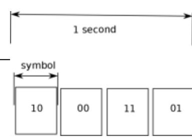
Analog Methods:

- AM - Amplitude modulation
  - Amplitude of a (carrier) is modulated to the (data)
- FM - Frequency modulation
  - Frequency of a (carrier) signal is varied in accordance to the amplitude of the (data) signal
- PM – Phase Modulation



Source: http://en.wikipedia.org/wiki/Modulation

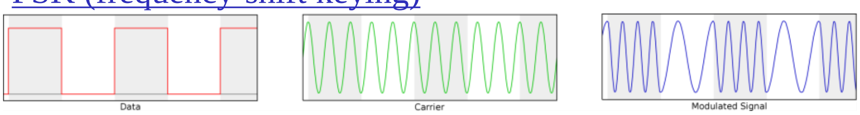
Modulation [Digital Methods]



Start with a "**symbol**" & place it on a channel

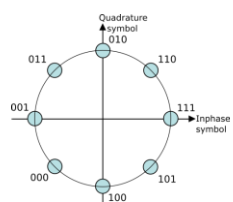
- ASK (amplitude-shift keying)

- FSK (frequency-shift keying)



- PSK (phase-shift keying)
- QAM (quadrature amplitude modulation)

$s(t) = A \cdot cos(\omega_c + \phi_i(t))$
$= x_i(t)\cos(\omega_c t) + x_q(t)\sin(\omega_c t)$



Source: http://en.wikipedia.org/wiki/Modulation | http://users.ecs.soton.ac.uk/sqc/EL334 | http://en.wikipedia.org/wiki/Constellation_diagram

Modulation [Example – V.32bis Modem]

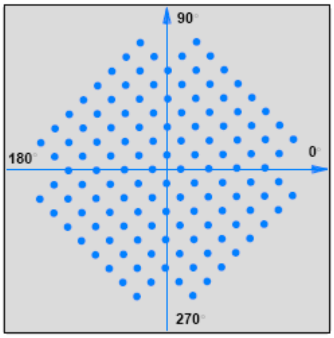
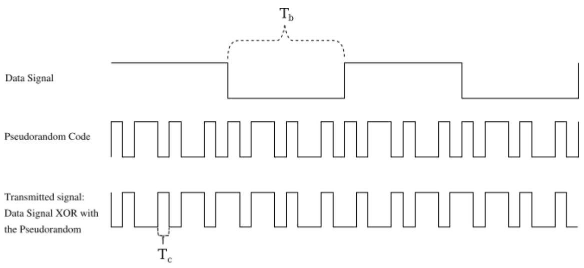


Figure 10.13 Illustration of the QAM constellation for a V.32bis dialup modem.

Source: Computer Networks and Internets, 5e, Douglas E. Comer

Multiple Access (Channel Access Method)

- Send multiple signals on 1 to N channel(s)
  - Frequency-division multiple access (FDMA)
  - Time-division multiple access (TDMA)
  - Code division multiple access (CDMA)
  - Space division multiple access (SDMA)
- CDMA:
  - Start with a pseudorandom code (the noise doesn't know your code)



Source: http://en.wikipedia.org/wiki/Code_division_multiple_access

BREAK

Next Time…

- **Digital Filters**

- Review:
  – Chapter 10 of Lathi

- A signal has many signals ☺
  [Unless it's bandlimited. Then there is the one ω]