Applying the rotational version of Newton's second law to the motor shaft, we obtain

$$J\frac{d^2\theta(t)}{dt^2} + K_f\frac{d\theta(t)}{dt} = K_t i_a(t). \tag{2.13}$$

These two equations give the mathematical model of the DC motor system with input $v_a(t)$ and output $\theta(t)$.

In some applications, we are concerned with the angular velocity (speed) of the motor, instead of the angular position. Such cases are called **speed control cases**. Replacing $\frac{d\theta(t)}{dt}$ by $\omega(t)$ in (2.12) and (2.13), we get the differential equation model of a DC motor system in the speed control case.

$$R_a i_a(t) + L_a\frac{di_a(t)}{dt} + K_b\omega(t) = v_a(t) \tag{2.14}$$

$$J\frac{d\omega(t)}{dt} + K_f\omega(t) = K_t i_a(t). \tag{2.15}$$

These two equations give the mathematical model of the DC motor system with input $v_a(t)$ and output $\omega(t)$.

Another interesting electromechanical system is a magnetic-ball suspension system shown in Figure 2.8. The coil at the top, after being fed with current, produces a magnetic field. The magnetic field generates an attracting force on the steel ball.

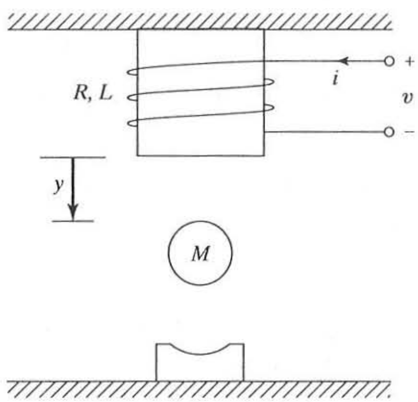

FIGURE 2.8: A magnetic-ball suspension system.

Here, the voltage applied to the coil $v(t)$ is the input, the distance from the ball to the coil $y(t)$ is the output, and the lifting force generated by the magnetic field on the ball is approximately given by $f(t) = K\frac{i^2(t)}{y(t)}$. The other parameters are mass of the ball M, winding resistance R, and winding inductance L.

Applying KVL to the coil, we obtain

$$Ri(t) + L\frac{di(t)}{dt} = v(t).$$

Applying Newton's second law to the ball, we obtain

$$M\frac{d^2y(t)}{dt^2} = -K\frac{i^2(t)}{y(t)} + Mg.$$

These two equations then give the mathematical model of the system. It is noted that the variables $y(t)$ and $i(t)$ are involved nonlinearly in the equations and this system is called a **nonlinear system**.

2.2 STATE SPACE MODEL AND LINEARIZATION

The mathematical models obtained in the previous sections consist of sets of differential equations with different orders and these equations involve variables other than the inputs and outputs. For the sake of systematic study, we need to put them in standard forms. One of the commonly used standard forms is the state space form.

Definition 2.1. The **state variables** of a system are a set of independent variables whose values at time t_0, together with input for all $t \geq t_0$, determine the behavior of the system for all $t \geq t_0$.

This definition looks very abstract but in many situations the state variables can be chosen intuitively. For electrical circuits, we can always choose the voltages across independent capacitors and the currents through independent inductors as state variables. For mechanical systems, we can always choose the positions and velocities of independent rigid bodies as state variables. Suppose that a differential equation model of a system is already obtained, the variables in the differential equations are the input $u(t)$ and the internal variables $v_1(t), \ldots, v_p(t)$, and the highest order of the derivatives of $v_i(t)$ in the differential equations is q_i. Then we can choose

$$v_i(t), \dot{v}_i(t), \ddot{v}_i(t), \ldots, v_i^{(q_i-1)}(t) \qquad i = 1, 2, \ldots, p$$

as the state variables. In this case, the total number of state variables is $\sum_{i=1}^{p} q_i$.

After the state variables are chosen, usually named $x_1(t), x_2(t), \ldots, x_n(t)$, we put them into a vector

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in \mathbb{R}^n.$$

This vector is called a **state vector**. Here and in the sequel, we use bold font letters to denote vectors (or matrices) and vector-valued (or matrix-valued) functions, whereas we use normal font letters to denote scalars and scalar-valued functions. Then the set of mixed ordered differential equations can be converted into a set of first-order differential equations plus an algebraic equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}[\boldsymbol{x}(t), u(t), t] \tag{2.16}$$

$$y(t) = g[\boldsymbol{x}(t), u(t), t] \tag{2.17}$$

EXAMPLE 2.3

The magnetic suspension system has the differential equation model

$$Ri(t) + L\frac{di(t)}{dt} = v(t)$$
$$M\frac{d^2y(t)}{dt^2} = -K\frac{i^2(t)}{y(t)} + Mg.$$

Choose $x_1(t) = i(t), x_2(t) = y(t), x_3(t) = \dot{y}(t), u(t) = v(t)$, and we get the state space model

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \dfrac{u(t) - Rx_1(t)}{L} \\ x_3(t) \\ -\dfrac{Kx_1^2(t)}{Mx_2(t)} + g \end{bmatrix}$$
$$y(t) = x_2(t)$$

with

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} i(0) \\ y(0) \\ \dot{y}(0) \end{bmatrix}.$$

Definition 2.4. A system is said to be **linear** if it can be described by linear differential equations, in particular, if the functions $\boldsymbol{f}$ and g in its state space model are linear functions of $\boldsymbol{x}(t)$ and $u(t)$.

For a linear system, the state space model takes the following matrix form:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{b}(t)u(t)$$
$$y(t) = \boldsymbol{c}(t)\boldsymbol{x}(t) + d(t)u(t)$$

where $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$ is an $n \times n$ matrix, possibly depending on time t, and $\boldsymbol{b}(t) \in \mathbb{R}^{n\times 1}$ and $\boldsymbol{c}(t) \in \mathbb{R}^{1\times n}$ are, respectively, column and row vectors depending possibly on time t. For example, the RLC circuit in Section 2.1 is a linear system and its state space equations were already in the matrix form as in (2.7) and (2.8)

Theorem 2.5 (Superposition Principle). Assume that a linear system has zero initial condition. If input $u_1(t)$ produces output $y_1(t)$ and input $u_2(t)$ produces output $y_2(t)$, then input $\alpha_1 u_1(t) + \alpha_2 u_2(t)$ produces output $\alpha_1 y_1(t) + \alpha_2 y_2(t)$ for all $\alpha_1, \alpha_2 \in \mathbb{R}$.

Proof. With zero initial condition, if input $u_1(t)$ produces output $y_1(t)$ and input $u_2(t)$ produces output $y_2(t)$, then there are $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ with $\boldsymbol{x}_1(0) = \boldsymbol{0}$ and $\boldsymbol{x}_2(0) = \boldsymbol{0}$ satisfying

$$\dot{\boldsymbol{x}}_1(t) = \boldsymbol{A}(t)\boldsymbol{x}_1(t) + \boldsymbol{b}(t)u_1(t)$$
$$y_1(t) = \boldsymbol{c}(t)\boldsymbol{x}_1(t) + d(t)u_1(t)$$

where

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}, \quad \frac{\partial \boldsymbol{f}}{\partial u} = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_n}{\partial u} \end{bmatrix}, \quad \frac{\partial g}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix}.$$

Since $\tilde{u}(t), \tilde{\boldsymbol{x}}(t), \tilde{y}(t)$ are small, we can neglect the high-order terms and approximate the original system by the following linear system:

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{b}\tilde{u}(t)$$
$$\tilde{y}(t) = \boldsymbol{c}\tilde{\boldsymbol{x}}(t) + d\tilde{u}(t)$$

where

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_0 \\ u=u_0}}, \qquad \boldsymbol{b} = \left.\frac{\partial \boldsymbol{f}}{\partial u}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_0 \\ u=u_0}},$$
$$\boldsymbol{c} = \left.\frac{\partial g}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_0 \\ u=u_0}}, \qquad d = \left.\frac{\partial g}{\partial u}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_0 \\ u=u_0}}.$$

This linear system is called a **linearized system** of the original nonlinear system.

EXAMPLE 2.10

The magnetic suspension system is a nonlinear system described by state space equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} \frac{u(t) - Rx_1(t)}{L} \\ x_3(t) \\ -\frac{x_1^2(t)}{Mx_2(t)} + g \end{bmatrix}$$
$$y(t) = x_2(t).$$

A usual control problem is to lift the ball to a certain height and suspend it at that height. Hence we wish to linearize it around an operating point with $y(t) = y_0$.

To get the operating point, solve equations

$$0 = \frac{u_0 - Rx_{10}}{L}$$
$$0 = x_{30}$$
$$0 = -\frac{x_{10}^2}{Mx_{20}} + g$$
$$y_0 = x_{20}.$$

This gives

$$u_0 = R\sqrt{Mgy_0}, \quad \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \end{bmatrix} = \begin{bmatrix} \sqrt{Mgy_0} \\ y_0 \\ 0 \end{bmatrix}, \quad y_0 = y_0$$

which means that to suspend the ball at height y_0 in the steady state, one need to apply a constant voltage $u(t) = R\sqrt{Mgy_0}$ to the coil. Denote the deviations o the input, state, and output variables from the operating point by

$$\tilde{u}(t) = u(t) - u_0 = u(t) - R\sqrt{Mgy_0}$$

$$\tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(t) - \boldsymbol{x}_0 = \begin{bmatrix} x_1(t) - \sqrt{Mgy_0} \\ x_2(t) - y_0 \\ x_3(t) \end{bmatrix}$$

$$\tilde{y}(t) = y(t) - y_0.$$

Now, the linearized model of the deviation variables is

$$\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{A}\tilde{\boldsymbol{x}}(t) + \boldsymbol{b}\tilde{u}(t)$$

$$\tilde{y}(t) = \boldsymbol{c}\tilde{\boldsymbol{x}}(t) + d\tilde{u}(t)$$

where

$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_0 \\ u=u_0}} = \begin{bmatrix} -\frac{R}{L} & 0 & 0 \\ 0 & 0 & 1 \\ -2\sqrt{\frac{g}{My_0}} & \frac{g}{y_0} & 0 \end{bmatrix}, \quad \boldsymbol{b} = \left.\frac{\partial \boldsymbol{f}}{\partial u}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_0 \\ u=u_0}} = \begin{bmatrix} \frac{1}{L} \\ 0 \\ 0 \end{bmatrix},$$

$$\boldsymbol{c} = \left.\frac{\partial g}{\partial \boldsymbol{x}}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_0 \\ u=u_0}} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}, \qquad d = \left.\frac{\partial g}{\partial u}\right|_{\substack{\boldsymbol{x}=\boldsymbol{x}_0 \\ u=u_0}} = 0.$$

2.3 TRANSFER FUNCTIONS AND IMPULSE RESPONSES

Consider an LTI system described by state space equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t)$$

$$y(t) = \boldsymbol{c}\boldsymbol{x}(t) + du(t).$$

Take the Laplace transform with zero initial conditions:

$$s\boldsymbol{X}(s) = \boldsymbol{A}\boldsymbol{X}(s) + \boldsymbol{b}U(s) \tag{2.18}$$

$$Y(s) = \boldsymbol{c}\boldsymbol{X}(s) + dU(s). \tag{2.19}$$

Now a set of differential equations in the time domain becomes a set of algebraic equations in the frequency domain. There are a total of $n+1$ equations in (2.18)–(2.19) and we can use them to eliminate the n variables in $\boldsymbol{X}(s)$ to obtain an equation relating the input $U(s)$ and the output $Y(s)$. Linear algebra now gives

For systems with a small number of state variables, it is probably more convenient to obtain the transfer function by directly manipulating the Laplace transform of the state space model (2.18) and (2.19). For example, in the speed control case, the Laplace transform of the state space model is

$$\begin{aligned} sX_1(s) &= -\frac{R_a}{L_a}X_1(s) - \frac{K_b}{L_a}X_2(s) + \frac{1}{L_a}U(s) \\ sX_2(s) &= \frac{K_t}{J}X_1(s) - \frac{K_f}{J}X_2(s) \\ Y(s) &= X_2(s). \end{aligned}$$

Substitute $X_1(s)$ from the second equation into the first equation and note that $Y(s) = X_2(s)$ from the third equation. We then get

$$\left[\frac{J}{K_t}\left(s+\frac{R_a}{L_a}\right)\left(s+\frac{K_f}{J}\right)+\frac{K_b}{L_a}\right]Y(s) = \frac{1}{L_a}U(s).$$

Consequently,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K_t}{L_aJs^2 + (R_aJ + K_fL_a)s + (R_aK_f + K_tK_b)},$$

the same result as the one obtained by matrix inversion.

EXAMPLE 2.13

Let us continue with Example 2.10, the magnetic suspension system. The transfer function of the linearized model is

$$G(s) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} s+\dfrac{R}{L} & 0 & 0 \\ 0 & s & -1 \\ 2\sqrt{\dfrac{g}{My_0}} & -\dfrac{g}{y_0} & s \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{1}{L} \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{-2\sqrt{\dfrac{g}{My_0}}\dfrac{1}{L}}{\left(s+\dfrac{R}{L}\right)\left(s^2-\dfrac{g}{y_0}\right)} = \frac{-2\sqrt{gy_0}}{\sqrt{M}(Ls+R)(y_0s^2-g)}.$$

The transfer function of an LTI system with a state space model is always a ratio of two polynomials

$$G(s) = \frac{b(s)}{a(s)}$$

where $b(s)$ is called the numerator polynomial and $a(s)$ is called the denominator polynomial. We assume that polynomials $b(s)$ and $a(s)$ are **coprime**, i.e., they do not have common factors.