Signals & Systems
Summary

ELEC3004 / ELEC7312
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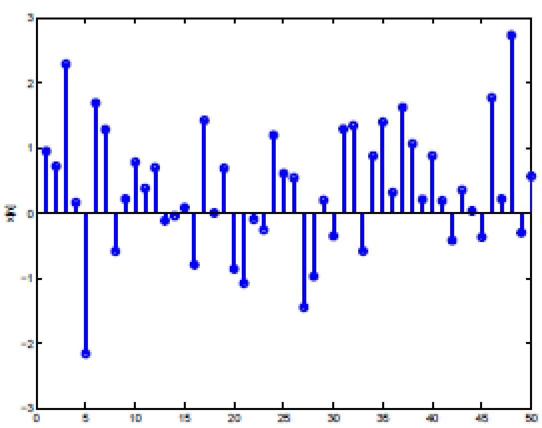
Vectors / Signals Can Be Multidimensional

A signal is a quantity that varies as a function of an index set

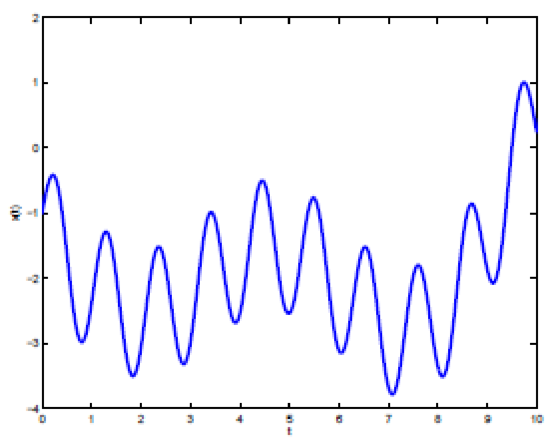
They can be multidimensional:

- 1-dim, discrete index (time): x[n]
- 1-dim, continuous index (time): x(t)
- 2-dim, discrete (e.g., a B/W or RGB image): x[j; k]
- 3-dim, video signal (e.g, video): x[j; k; n]

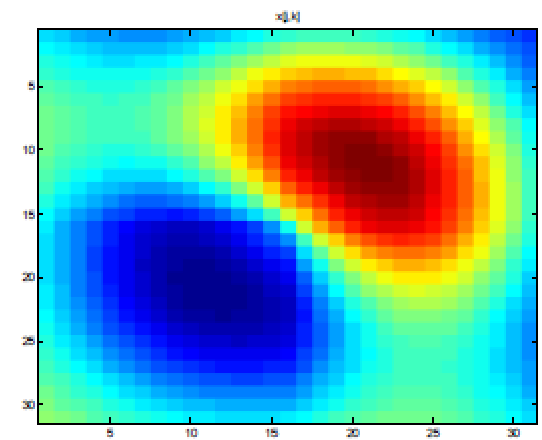
Discrete 1D



Continuous 1D



Discrete 2D



Sampling Theorem

The Nyquist criterion states:

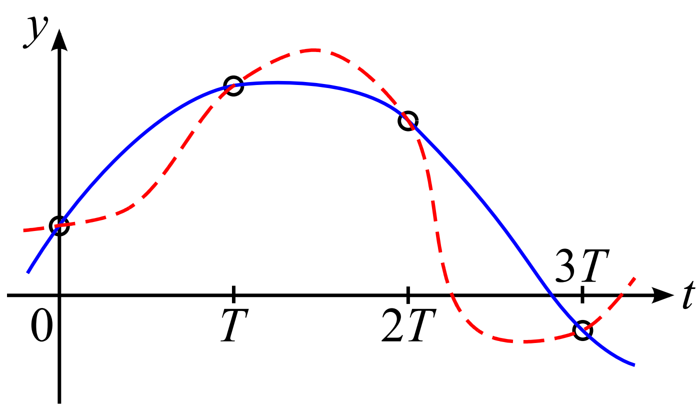
To prevent aliasing, a ***bandlimited*** *signal of bandwidth w_B rad/s must be sampled at a rate greater than $2w_B$ rad/s*

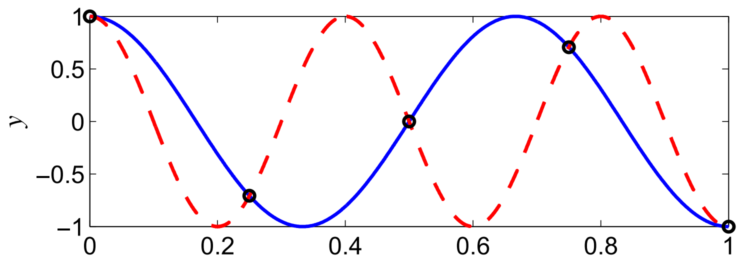
$$w_s > 2w_B$$

Note: this is a > sign not a ≥

Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

Nyquist Sampling Theorem and Aliasing

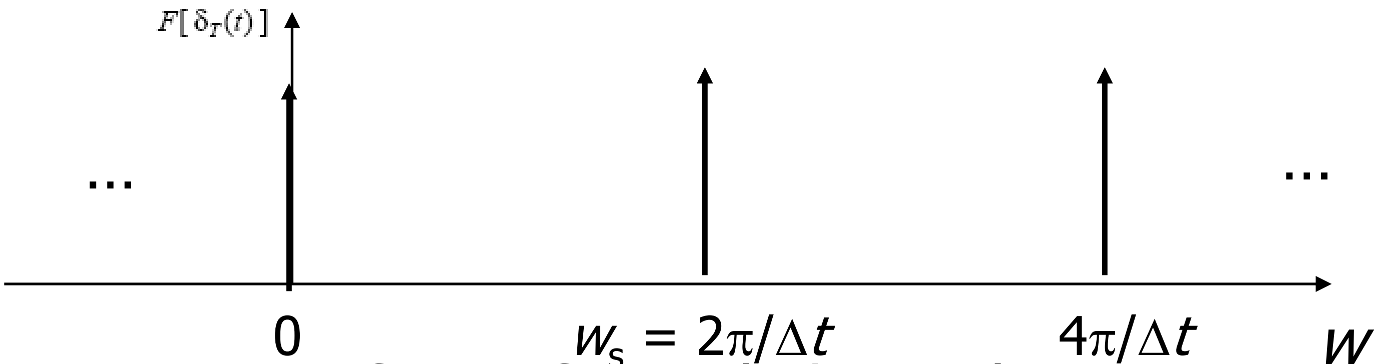




A signal $y(t)$ is uniquely defined by its samples $y(kT)$ if the sampling frequency is more than twice the bandwidth of $y(t)$.

Fourier transform of original signal X(ω) (signal spectrum)

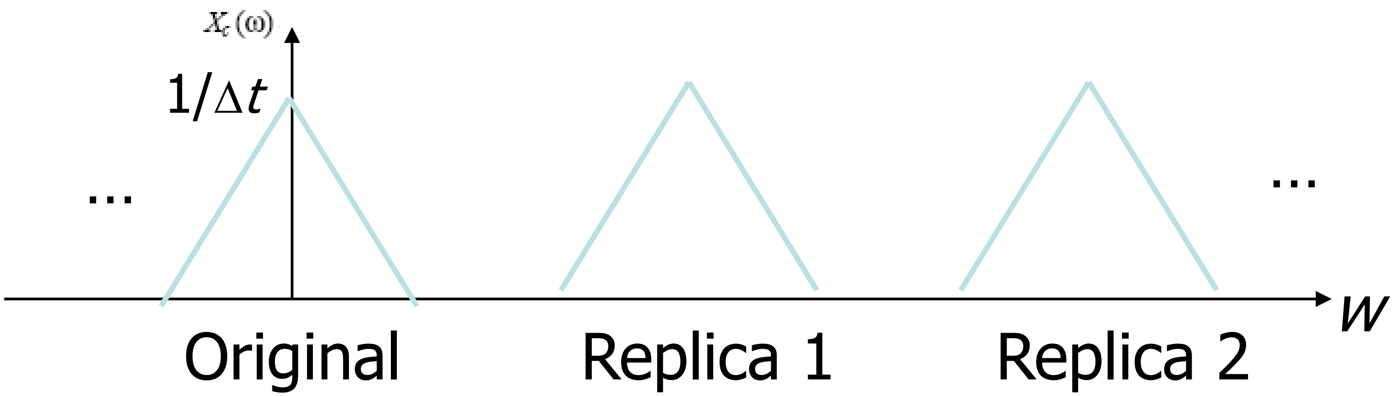
Fourier transform of impulse train $\delta_T(\omega/2\pi)$ (sampling signal)



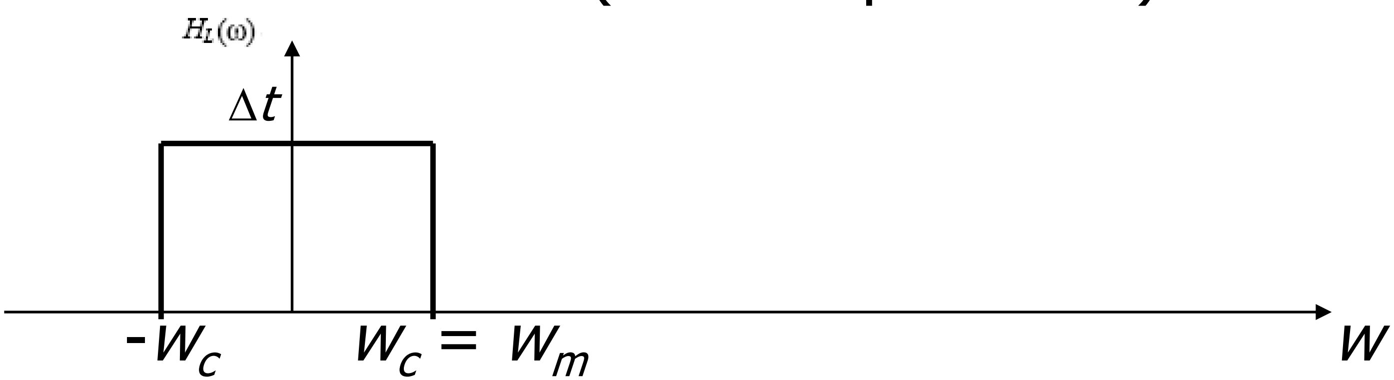
Fourier transform of sampled signal

Original spectrum convolved with spectrum of impulse train

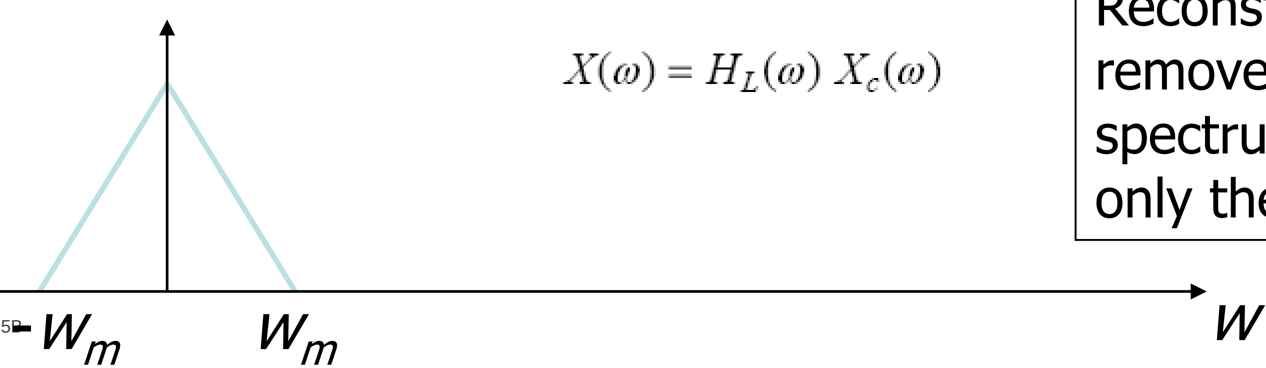
Spectrum of sampled signal



Reconstruction filter (ideal lowpass filter)



Spectrum of reconstructed signal

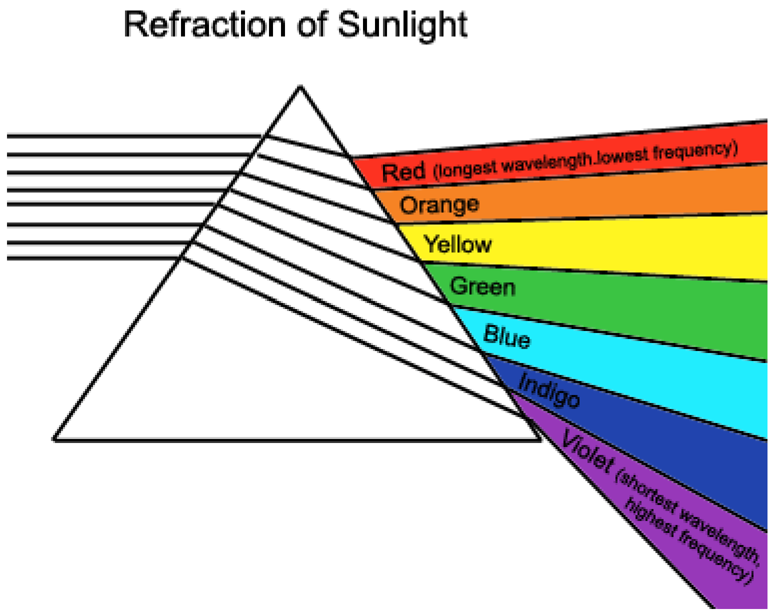


$$X(\omega) = H_L(\omega)\, X_c(\omega)$$

Reconstruction filter removes the replica spectrums & leaves only the original

Refraction Analogy

White light
(all wavelengths)

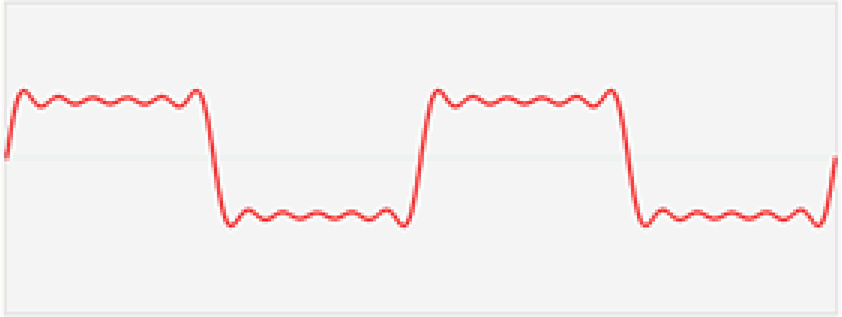


Colour
spectrum

$f = 1/\lambda$

Think of a Fourier Transform like a prism:

"Destructs a source signal into its constituent frequencies"



By: Andrés Cabrera and Karl Yerkes.
http://w2.mat.ucsb.edu/201A/nb/Sinusoids%20and%20Phasors.html

Complex Fourier Coefficients

Again, X_n calculated from $x(t)$

$$X_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\exp(-jnw_0 t)\,dt$$

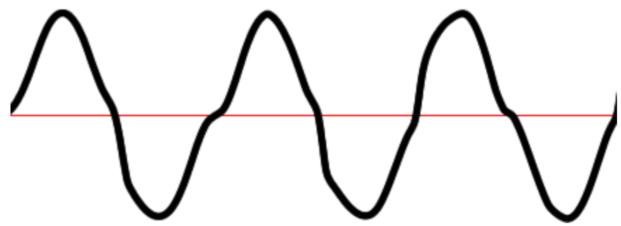
Only one set of coefficients, X_n

- but, generally they are complex

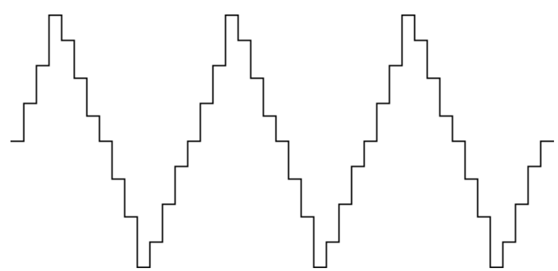
Remember: fundamental $w_0 = 2\pi/T$!

Analog vs Digital

Analog Signal: An analog or analogue signal is any variable signal **continuous** in both time and amplitude



Digital Signal: A digital signal is a signal that is both **discrete** and quantized



E.g. Music stored in a CD: 44,100 Samples per second and 16 bits to represent amplitude

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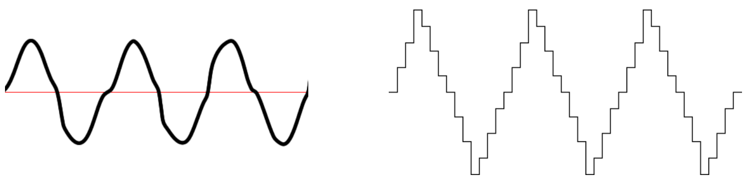
➔ Digital Systems ∵

Better SNR

We trade-off
"**certainty in time**" for "**signal noise/uncertainty**"

Analog: ∞ time resolution

- *Digital* has fixed time steps



This avoids the noise and uncertainty in component values that affect analogue signal processing.

Better Processing

Digital microprocessors are in a range of objects, from obvious (e.g. phone) to disposable (e.g. Go cards).
(what doesn't have one?)

Compared to antilog computing (op-amp):

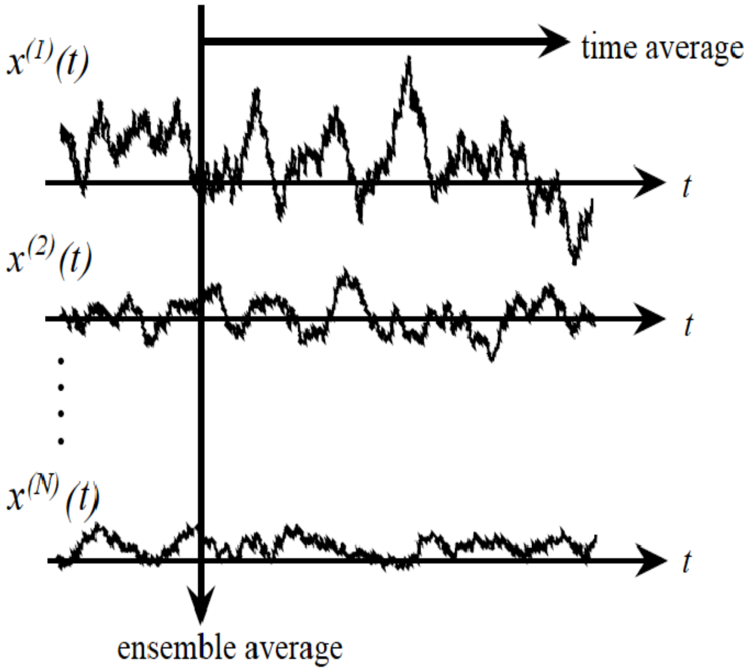
Accuracy: digital signals are usually represented using 12 bits or more.

Reliability: The ALU is stable over time.

Flexibility: limited only programming ability!

Cost: advances in technology make microcontrollers economical even for small, low cost applications.
(Raspberry Pi 3: US$35)

Treating Uncertainty with Multiple Measurements



1. **Over time:** multiple readings of a quantity over time
   - "stationary" or "ergodic" system
   - Sometimes called "integrating"

2. **Over space:** single measurement (summed) from multiple sensors each distributed in space

3. **Same Measurand**: multiple measurements take of the **same observable quantity** by multiple, related instruments

   e.g., measure position & velocity simultaneously

→ Basic "sensor fusion"

$$\sigma_{\mathsf{final}} = \left[\sigma_1^{-1} + \sigma_2^{-1} + \cdots + \sigma_n^{-1}\right]^{-1}$$

Interpolation

Increase sample rate

- Up-sampling by factor *L*

$$\longrightarrow \boxed{\uparrow L} \longrightarrow$$

$$x_u[n] = \begin{cases} x[n/L], & \text{if } n = 0, L, 2L, 3L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

e.g., $x[n] = \{1\ 2\ 4\ 3\ 5\ 6\ 7\ 2\ 4\ 3\}$

$\uparrow 2 \quad x_d[n] = \{1\ 0\ 2\ 0\ 4\ 0\ 3\ 0\ 5\ 0\ 6\ 0\ 7\ 0\ 2\ 0\ 4\ 0\ 3\ 0\}$

Insert zeros between each sample

Zero Order Hold (ZOH)

Known as nearest neighbour (NN)

Previous sample is ‘held’

- interpolated value is nearest original sample
- impulse response $h_{nn}[n]$

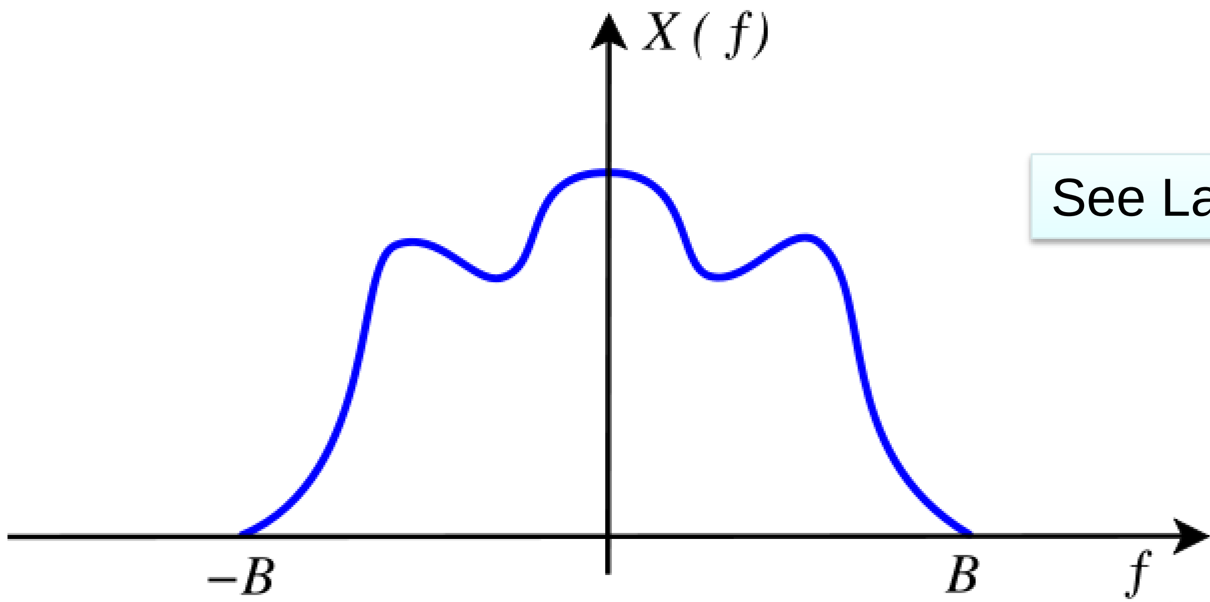
$$h_{nn}[n] = \begin{cases} 1, & -L/2 \leq n < L/2, \\ 0, & \text{otherwise.} \end{cases}$$

Approximation is discontinuous ☹
Efficient as no computation required, just repeat values
best suited to signals that are discontinuous, e.g., text

Reconstruction

Whittaker–Shannon (sinc) interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$



See Lathi 5.1-1

The DFT

Discrete Fourier Transform (DFT)

- samples of DTFT, $X^{\wedge}_{c}(w)|_{w = k\,\Delta w}$

$$\hat{X}(k\Delta\omega) = X[k] = \sum_{n=0}^{N-1} x[n]\exp\left(\frac{-jnk2\pi}{N}\right)$$

where $0 \le n, k \le N-1$

Interpretation,

- N equally spaced samples of $x(t)|_{t = n\,\Delta t}$
- Calculates N equally spaced samples of $X(w)|_{w = k\Delta w}$
- k often referred to a frequency ‘bin’: $X[k] = X(w_k)$

Inverse DFT

Relates frequency domain samples to

- time domain samples

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\exp\left(\frac{jnk2\pi}{N}\right)$$

Note, differences to forward DFT

- 1/N scaling and sign change on exponential
- DFT & IDFT implemented with same algorithm
  - i.e., Fast Fourier Transform (FFT)

Require both DFT and IDFT to implement (fast)

- convolution as multiplication in frequency domain

Note, 1/N scaling can be on DFT only OR as 1/sqrt(N) on both DFT and IDFT

Fourier Transforms

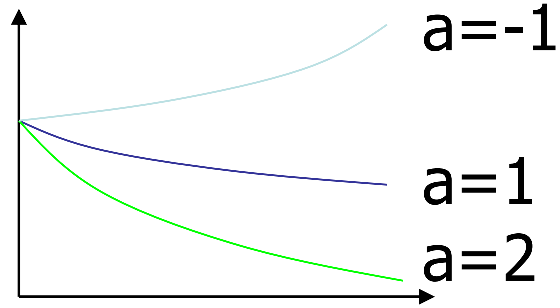
Transform	**Time Domain**	**Frequency Domain**
Fourier Series (FS)	Continuous & Periodic	Discrete
Fourier Transform (FT)	Continuous	Continuous
Discrete-time Fourier Transform (DTFT)	Discrete	Continuous & Periodic
Discrete Fourier Transform (DFT)	Discrete & Periodic	Discrete & Periodic

FT Convergence: Example

Consider signal

$$x(t) = \exp(-at)u(t)$$

Where u(t) = unit step fctn



Signal only finite energy if a > 0

- ∴ FT only exists (converges) if a > 0

$$X(w) = \frac{1}{a + jw}, \quad a > 0$$

More on this later…

Laplace Transform

Problem: FT of a signal may not always exist!

- finite power (and not periodic),
- e.g., $x(t) = u(t)\exp(-at))$ with $a < 0$
- Or $x(t) = u(t)\cos(5t)$!

Solution: Force signal to have finite energy

- Multiply by convergence factor ($\exp(-\sigma t)$)
- i.e., new signal $x_\sigma(t) = \exp(-\sigma t)x(t)$
- Therefore, FT of $x_\sigma(t)$ exists

$$X_\sigma(w) = \int_{-\infty}^{\infty} x_\sigma(t)\exp(-jwt)dt$$

Rearranging…

$$X_\sigma(w) = \int_{-\infty}^{\infty} x(t)\exp(-(\sigma + jw)t)dt$$

Fourier

$\Downarrow$

Laplace

Unilateral Laplace Transform

One-sided Laplace transform

$$X(s) = \int_{0^-}^{\infty} x(t)\exp(-st)dt$$

$\boxed{0^- \text{ indicates origin is included in integration } 0 \le t < \infty}$

Laplace transform

- $X(s) = L\{x(t)\}$

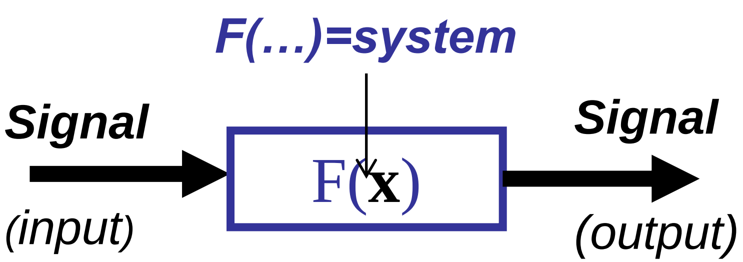
Inverse Laplace transform

- $x(t) = L^{-1}\{X(s)\}$

What is a System?

≡ A **process** (function) by which information (signals) are modified so as to extract additional information from them

- Systems modify the signal(s) to yield a new result (also a signal)
- Can be of various forms: electrical, mechanical, etc.



ODE's and Linear Systems

Linear system described by differential equation

$$a_0 y + a_1 \frac{dy}{dt} + \Lambda + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \Lambda + b_m \frac{d^m x}{dt^m}$$

Which using Laplace Transforms can be written as

$$a_0 Y(s) + a_1 s Y(s) + \Lambda + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \Lambda + b_m s^m X(s)$$

$$A(s) Y(s) = B(s) X(s)$$

where *A(s)* and *B(s)* are polynomials in *s*

Transfer Function

Transfer function can be written as

$$H(s) = \frac{Y(s)}{X(s)} = \frac{B(s)}{A(s)}$$

$$= \frac{b_0 + b_1 s + \Lambda + b_m s^m}{a_0 + a_1 s + \Lambda + a_n s^n}$$

Transfer functions have:

- **Poles**
  - Infinite value of *H(s)*, i.e., when *A(s)* = 0 (roots of *A(s)*)
- **Zeros**
  - Zeros value of *H(s)*, i.e., when *B(s)* = 0 (roots of *B(s)*)

The poles & zeros of H(s) define freq response & stability

The z - transform

$$X_c(s) = \int_0^\infty x_c(t)\exp(-st)dt$$

Start with sampled signal

- and then find its Laplace Transform

$$x_c(t) = \sum_{n=0}^{\infty} x(n\Delta t)\delta(t - n\Delta t)$$

$$X_c(s) = \sum_{n=0}^{\infty} x(n\Delta t)\exp(-n\Delta t s)$$

Discrete time so integral becomes summation &

$$L\{\delta(t-a)\} = e^{-as}$$

Unfortunately as $x_c(t)$ is discrete in time

- $X_c(s)$ is also periodic (as sampled F(w) was)

$$X_c(s) = X_c\left(s + \frac{j2\pi}{\Delta t}\right)$$

Remember e^{-as} is a rotating phasor

- This means it has an infinite number of poles!
  - So problems with PFE when calculating L^{-1} ☹

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The *z* - transform

The solution is to make a substitution

$$z = \exp(s\Delta t)$$

This can be viewed in two ways:

1. Mapping all points in *s* –plane to a *z* –plane, or
2. Convenient short-hand for exp(sΔt)

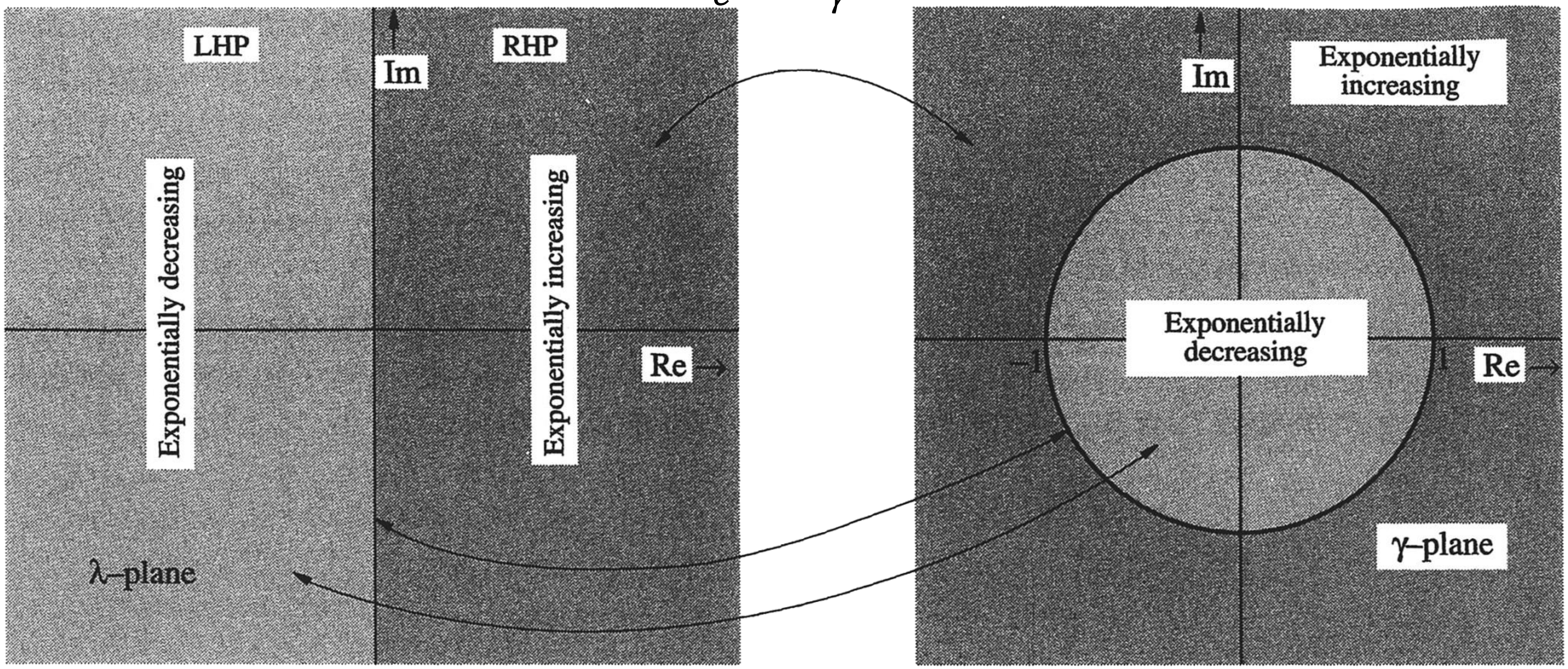
The *z* – transform is defined as

$$X(z) = X_c(s)\Big|_{\exp(s\Delta t)=z}$$

$$X(z) = \sum_{n=0}^{\infty} x(n\Delta t) z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n}$$

S-Plane to z-Plane [1/2]: Discrete-Time Exponential γ^k

$$e^{\lambda k} = \gamma^k$$

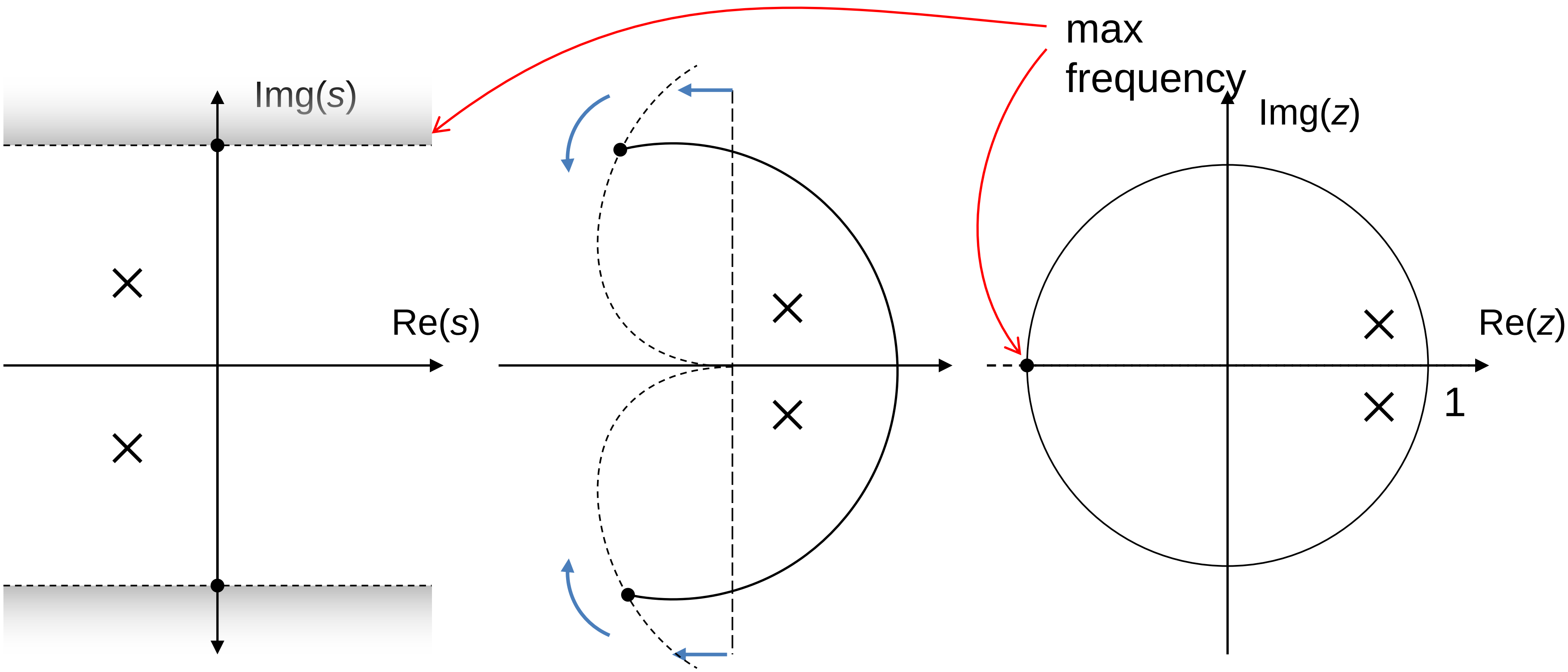


(a)

(b)

Deep insight

The mapping between continuous and discrete poles and zeros acts like a distortion of the plane



Inverse *z* - transform

$x[n] = \mathcal{Z}^{-1}\{X(z)\}$

As with inverse Laplace transform:

1. Perform a partial fraction expansion (PFE)
2. Look up in tables

There are some subtle differences

z-Transform of a Sequence

Given a sequence $\{x(n\Delta t)\}_0^{\infty} = \{x[n]\}_0^{\infty}$

- i.e., n = 0, 1, 2,…

z-Transform is given by

- $X(z) = Z\{x(n\Delta t)\}$

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x(n\Delta t) z^{-n} \\ &= \sum_{n=0}^{\infty} x[n] z^{-n} \end{aligned}$$

Therefore, z-transform exists for ALL discrete sequences
Not just sampled analogue ones, e.g., hours daylight per day

z – transform: Examples

Unit impulse, δ[n]

- z - transform defined as

$$X(z) = \sum_{n=0}^{\infty} \delta[n] z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} \delta[0] z^{-0} = 1(1) = 1$$

Unit step, u[n]

$$X(z) = \sum_{n=0}^{\infty} (1) z^{-n} = \sum_{n=0}^{\infty} \left(z^{-1}\right)^{n} = \frac{1}{1 - z^{-1}}$$

Remember:
δ[n] = 1
at n = 0
z = exp(sΔt)
&
u[n] = 1
For n ≥ 0

sum of geometric series

Discrete Transfer Function

Consider a simple difference equation

- and its *z* - transform

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + b_1 y[n-1] + b_2 y[n-2]$$

$$Y(z) = Z\{y[n]\}$$

$$Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z) + b_1 z^{-1} Y(z) + b_2 z^{-2} Y(z)$$

Rearranging to give transfer function, H(z)

$$Y(z)\left[1 - b_1 z^{-1} - b_2 z^{-2}\right] = X(z)\left[a_0 + a_1 z^{-1} + a_2 z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 - b_1 z^{-1} - b_2 z^{-2}}$$

General form of 2nd order H(z)

See also examples 11.5 and 12.9 in Lathi

Ideal "Brick wall" Filters

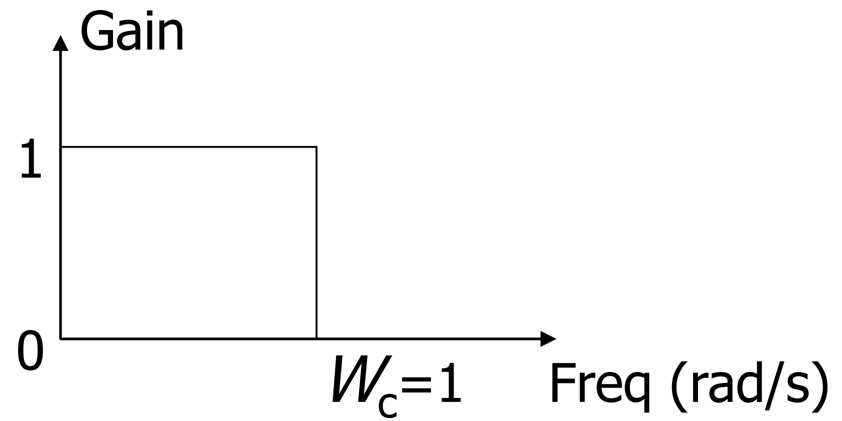
Often modelled by squared magnitude transfer function

$$H_I(\omega)H_I^*(\omega) = \left|H_I(\omega)\right|^2 = \frac{1}{1+F_I(\omega^2)}$$

Where, for $w_c = 1$

$$F_I(\omega^2) = \begin{cases} 0, & 0 < \omega < 1; \\ \infty, & \omega > 1. \end{cases}$$

Lowpass Filter



$H_I(w)$ not realisable because

- Infinite order filter required (∞ reative components)
  - F(w) is a polynomial in w^2
- H(w) finite in frequency therefore
  - h(t) infinite in time

Note: implication here is that filters are designed by specifying their (magnitude) frequency response. From this we can calculate H(s)

Butterworth: magnitude response

Approximation is

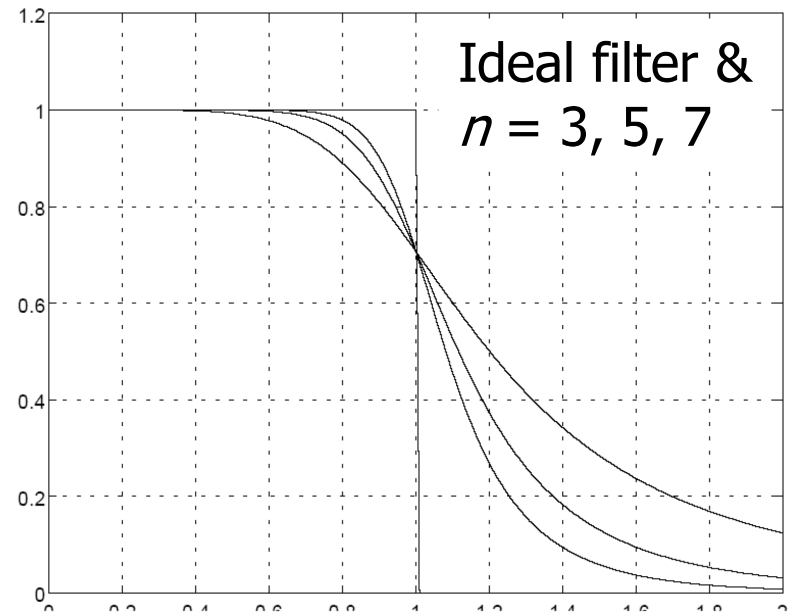
$$F(\omega^2) = \omega^{2n}$$

$$\left|H_n(\omega)\right| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

All orders (n) have same cut-off (half power) frequency

$$\left|H_n(\omega_c)\right|^2 = \frac{1}{2}, \forall n$$

Approximation to ideal improves as n increases



Magnitude response
Error from ideal response:

$$e(\omega) = \left|H_I(\omega)\right| - \left|H(\omega)\right|$$

Largest error close to w_c

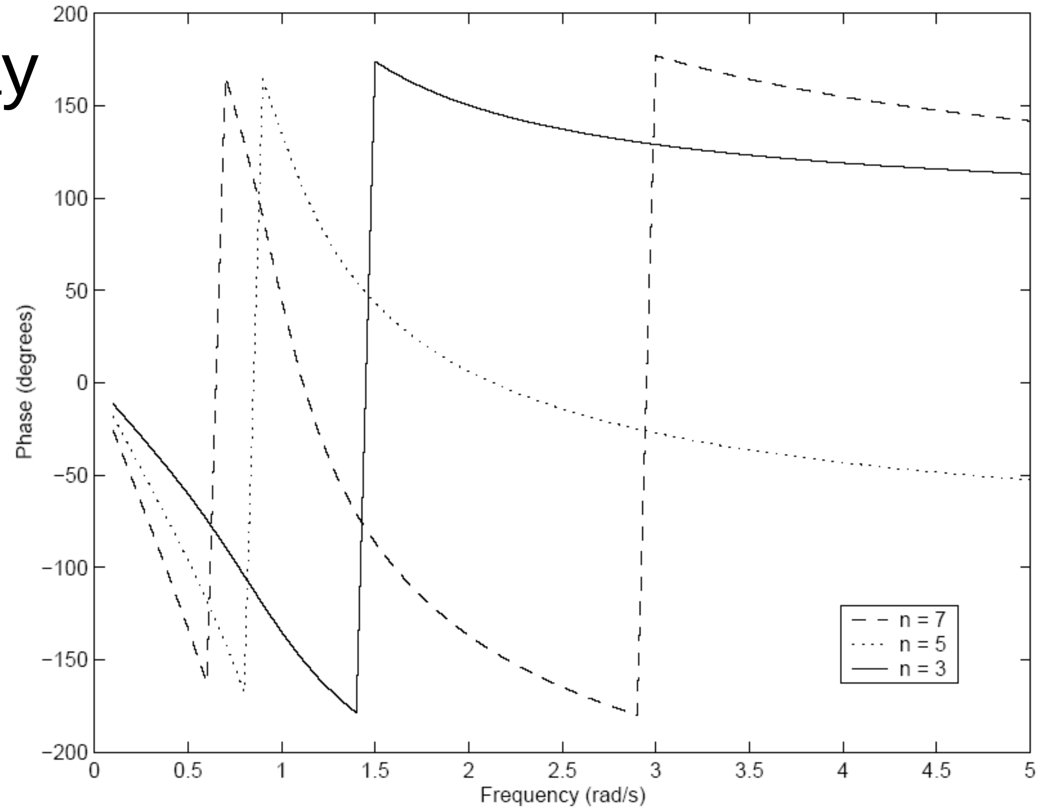
Butterworth: phase response

Increased group delay as n increases

Maximum delay at cut-off frequency

$$\tau_g(\omega) = -\frac{d}{d\omega}\angle\{H(\omega)\}$$

Max change in phase at w_c



Phase response, order n = 3, 5, 7

Butterworth Properties

DC gain ($s = jw = 0$) of 1

No significant deviation in pass band (maximally flat)

- First $2n - 1$ derivatives of $|H(w)|$
  - are zero at $w = 0$ (DC)
- Rolls-off: ~$20n$ dB/dec (~$6n$ dB/oct)

See section 7.5 and example 7.6 in Lathi

Filter Design

Previously we have analysed

- difference equations ($y[n]$)
- transfer functions ($H(z)$)

To obtain time/frequency domain response

- Impulse ($h[n]$) or frequency ($H(w)$) response

Now we have a specification

- frequency response (filters)
- time response (control)

Goal to design a filter that meets specification

- i.e., determine transfer function
- and therefore difference equation (implementation)

IIR Filter Design Methods

Normally based on analogue prototypes

- Butterworth, Chebyshev, Elliptic etc

Then transform $H(s) \rightarrow H(z)$

Three popular methods:

1. Impulse invariant
   - produces H(z) whose impulse response is a sampled version of h(t) (also step invariant)
2. Matched *z* – transform
   - poles/zeros H(s) directly mapped to poles/zeros H(z)
3. Bilinear *z* – transform
   - left hand *s* – plane mapped to unit circle in *z* - plane

Introduction

IIR filters efficient low transition bandwidth

- however, non-linear phase response
- especially higher orders

FIR are non-recursive (convolution machines)

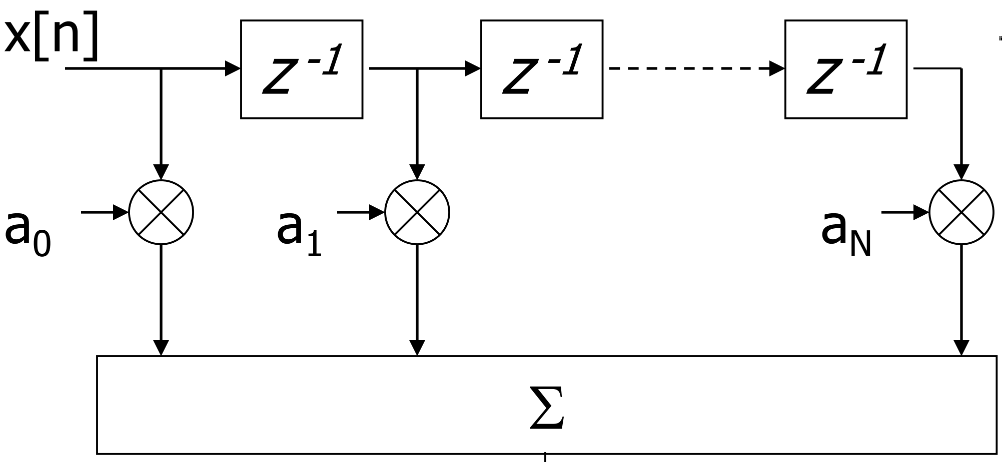
- always stable (poles at the origin)
- can offer linear phase (i.e., no phase distortion)
- but higher order than IIR required to achieve same transition bandwidth
- Therefore, not as computationally efficient as IIR
- But more tolerant of finite precision effects

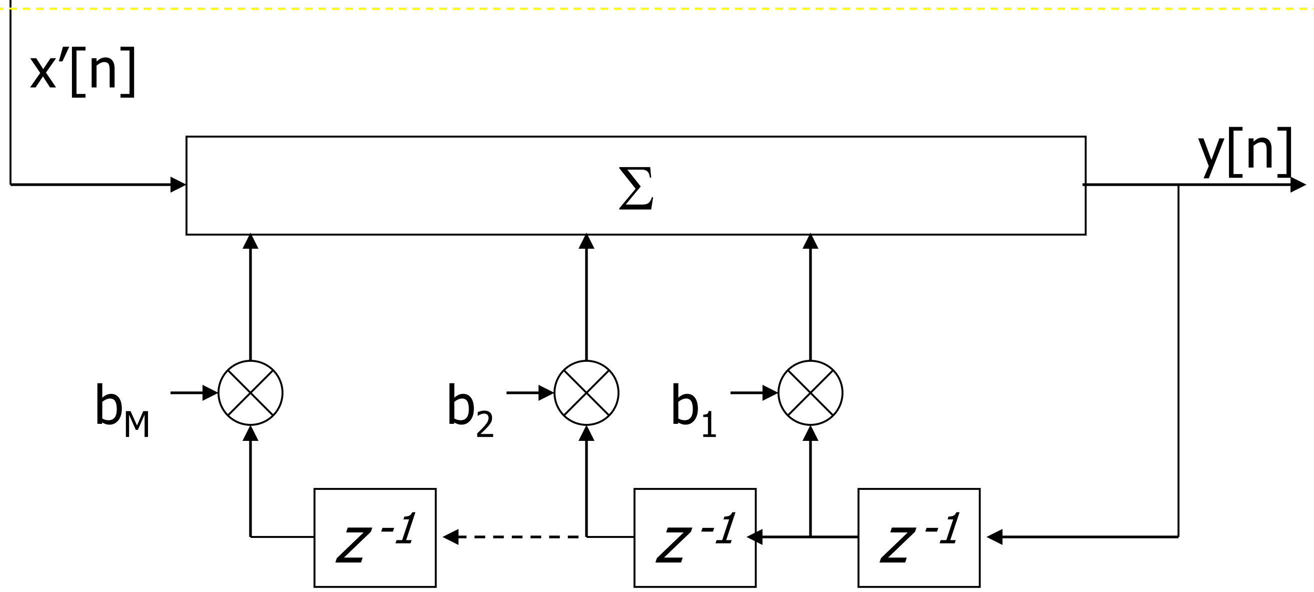
Direct realisation of digital filter (Direct Form I)

$$y(n) = \sum_{i=0}^{N} a_i \; x(n-i) + \sum_{i=1}^{M} b_i \; y(n-i)$$

Two LTI filters in cascade:

1. feedforward (a_i)
   - forms x′[n]
2. feedback (b_i)
   - forms y[n]





FIR Filters

Feedforward section (a_i coefficients) only

- no feedback (no b_i coefficients)

$$y[n] = \sum_{i=0}^{N-1} a_i x[n-i]$$

$$Y(z) = \sum_{i=0}^{N-1} a_i X(z) z^{-i}$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{i=0}^{N-1} a_i z^{-i}$$

$$H(z) = a_0\, z^{0} + a_1\, z^{-1} + \ldots. + a_{N-1}\, z^{-(N-1)}$$

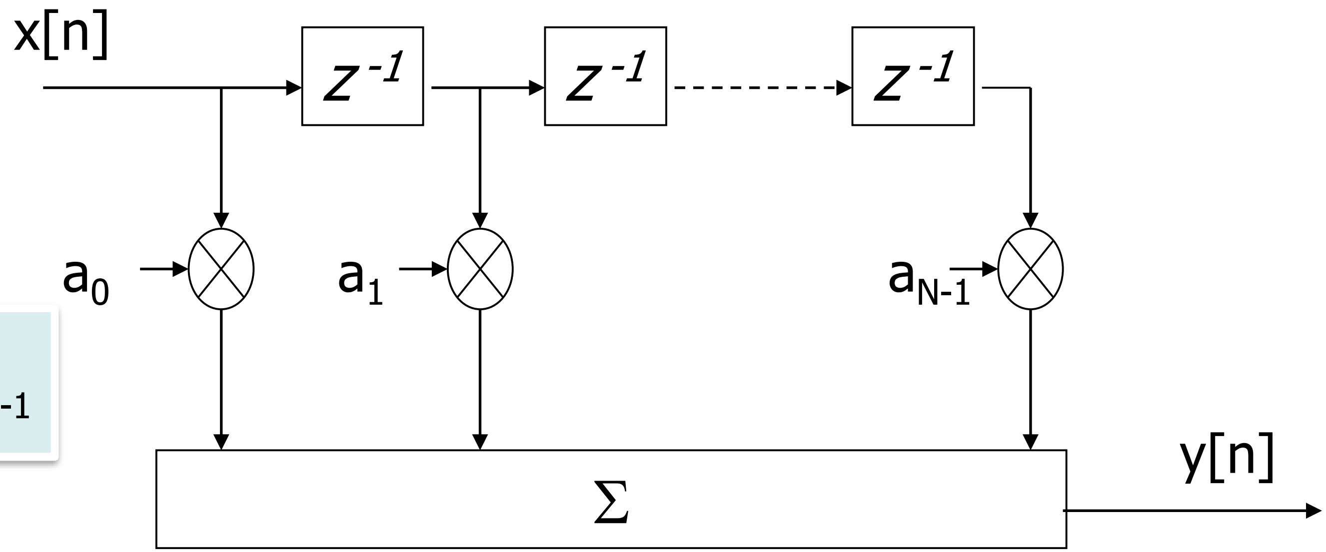
$$= \frac{z^{N-1}}{z^{N-1}} \left(a_0\, z^{0} + a_1\, z^{-1} + \ldots. + a_{N-1}\, z^{-(N-1)} \right)$$

$$= \frac{a_0\, z^{N-1} + a_1\, z^{N-2} + \ldots. + a_{N-1}\, z^{0}}{z^{N-1}}$$

N coefficient FIR (order *N* - 1) has
- N-1 zeros
- N-1 poles (at origin)

$\therefore$ always stable

FIR Filter Structure $y[n]=\sum_{i=0}^{N-1} x[n-i]a[i] = x[n]*a[n]$



for FIR
$h[n] \equiv \{a_i\}_0^{N-1}$

$$y[n]=\sum_{i=0}^{N-1} a_i x[n-i]$$

Output, $y[n]$, is convolution of input, $x[n]$, with filter coefficients, a_i

Matrix Formulation of Convolution

$$y = \mathbf{H}x$$

Where **H** is a Toeplitz Matrix

$$\begin{bmatrix} 0.75 \\ 2 \\ 4.25 \\ 2 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.25 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.5 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.5 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0.5 & 0.25 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 & 0.25 \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Concise maths, but not how you implement it!

Convolution Theorem

Important: used extensively!

Convolution in the time-domain is equivalent to multiplication in the Laplace (s) domain (or frequency domain)

$$x_1(t) * x_2(t) = L^{-1}\{X_1(s) X_2(s)\}$$
$$= F^{-1}\{X_1(w) X_2(w)\}$$

and vice versa: Multiplication in time equals convolution in *s* (or frequency)

$$L\{x_1(t) x_2(t)\} = \frac{1}{2\pi j} X_1(s) * X_2(s)$$

e.g., frequency modulation

$$F\{x_1(t) x_2(t)\} = \frac{1}{2\pi} X_1(w) * X_2(w)$$

Convolution

Continuous-time

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$y(t) = h(t) * x(t)$$

Discrete-time

$$y[n] = \sum_{m=0}^{\infty} h[n-m]x[m]$$

$$y[n] = \sum_{m=0}^{\infty} x[n-m]h[m]$$

$$y[n] = h[n] * x[n]$$

Impulse Response

Lets do a Worked Example!

Sampling Theory to the Next Level

Creating a slow motion camera without a special camera!

https://youtu.be/dw7U3BYMs4U?t=446

https://en.wikipedia.org/wiki/Stroboscopic_effect