CHAPTER

SIX

SHAPING THE DYNAMIC RESPONSE

6.1 INTRODUCTION

At last we have arrived at the point of using state-space methods for control system design. In this chapter we will develop a simple method of designing a control system for a process in which all the state variables are accessible for measurement—the method known as *pole-placement.* We will find that in a controllable system, with all the state variables accessible for measurement, it is possible to place the closed-loop poles anywhere we wish in the complex s plane. This means that we can, in principle, completely specify the closed-loop dynamic performance of the system. In principle, we can start with a sluggish open-loop system and force it to behave with alacrity; in principle, we can start with a system that has very little open-loop damping and provide any amount of damping desired. Unfortunately, however, what can be attained in principle may not be attainable in practice. Speeding the response of a sluggish system requires the use of large control signals which the actuator (or power supply) may not be capable of delivering. The consequence is generally that the actuator saturates at the largest signal that it can supply. In some instances the system behavior may be acceptable in spite of the saturation. But in other cases the effect of saturation is to make the closed-loop system unstable. It is usually not possible to alter open-loop dynamic behavior very drastically without creating practical difficulties.

Adding a great deal of damping to a system having poles near the imaginary axis is also problematic, not only because of the magnitude of the control signals needed, but also because the control system gains are very sensitive to the location of the open-loop poles. Slight changes in the open-loop pole location may cause the closed-loop system behavior to be very different from that for which it is designed.

We will first address the design of a regulator. Here the problem is to determine the gain matrix G in a linear feedback law

$$u = -Gx \tag{6.1}$$

which shapes the dynamic response of the process in the absence of disturbances and reference inputs. Afterward we shall consider the more general problem of determining the matrices G and G_0 in the linear control law

$$u = -Gx - G_0 x_0 \tag{6.2}$$

where x_0 is the vector of exogenous variables. The reason it is necessary to separate the exogenous variables from the process state x, rather than deal directly with the metastate

$$\mathbf{x} = \begin{bmatrix} x \\ \hdashline x_0 \end{bmatrix} \tag{6.3}$$

introduced in Chap. 5, is that in developing the theory for the design of the gain matrix, we must assume that the underlying process is *controllable.* Since the exogenous variables are not true state variables, but additional inputs that cannot be affected by the control action, they cannot be included in the state vector when using a design method that requires controllability.

The assumption that all the state variables are accessible to measurement in the regulator means that the gain matrix G in (6.1) is permitted to be any function of the state x that the design method requires. In most practical instances, however, the state variables are not all accessible for measurement. The feedback control system design for such a process must be designed to use only the measurable output of the process

$$y = Cx$$

where y is a vector of lower dimension than x. In some cases it may be possible to determine the gain matrix G_y for a control law of the form

$$u = -G_y y \tag{6.4}$$

which produces acceptable performance. But more often it is not possible to do so. It is then necessary to use a more general feedback law, of the form

$$u = -G\hat{x} \tag{6.5}$$

where $\hat{x}$ is the state of an appropriate dynamic system known as an "observer." The design of observers is the subject of Chap. 7. And in Chap. 8, we shall show that when a feedback law of the form of (6.5) is used with a properly designed observer, the dynamic properties of the overall system can be specified at will, subject to practical limitations on control magnitude and accuracy of implementation.

6.2 DESIGN OF REGULATORS FOR SINGLE-INPUT, SINGLE-OUTPUT SYSTEMS

The present section is concerned with the design of a gain matrix

$$G = g' = [g_1, g_2, \ldots, g_k] \tag{6.6}$$

for the single-input, single-output system

$$\dot{x} = Ax + Bu \tag{6.7}$$

where

$$B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \tag{6.8}$$

With the control law $u = -Gx = -g'x$ (6.7) becomes

$$\dot{x} = (A - bg')x$$

Our objective is to find the matrix $G = g'$ which places the poles of the closed-loop dynamics matrix

$$A_c = A - bg' \tag{6.9}$$

at the locations desired. We note that there are k gains $g_1, g_2, \ldots, g_k$ and k poles for a kth order system, so there are precisely as many gains as needed to specify each of the closed-loop poles.

One way of determining the gains would be to set up the characteristic polynomial for A_c:

$$|sI - A_c| = |sI - A + bg'| = s^k + \bar{a}_1 s^{k-1} + \cdots + \bar{a}_k \tag{6.10}$$

The coefficients $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$ of the powers of s in the characteristic polynomial will be functions of the k unknown gains. Equating these functions to the numerical values desired for $\bar{a}_1, \ldots, \bar{a}_k$ will result in k simultaneous equations the solution of which will yield the desired gains $g_1, \ldots, g_k$.

This is a perfectly valid method of determining the gain matrix g', but it entails a substantial amount of calculation when the order k of the system is higher than 3 or 4. For this reason, we would like to develop a direct formula for g in terms of the coefficients of the open-loop and closed-loop characteristic equations.

If the original system is in the companion form given in (3.90), the task is particularly easy, because

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6.11}$$

$$bg' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [g_1, g_2, \ldots, g_k] = \begin{bmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence

$$A_c = A - bg' = \begin{bmatrix} -a_1 - g_1 & -a_2 - g_2 & \cdots & -a_k - g_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The gains $g_1, \ldots, g_k$ are simply added to the coefficients of the open-loop A matrix to give the closed-loop matrix A_c. This is also evident from the block-diagram representation of the closed-loop system as shown in Fig. 6.1. Thus for a system in the companion form of Fig. 6.1, the gain matrix elements are given by

$$a_i + g_i = \hat{a}_i \qquad i = 1, 2, \ldots, k$$

or

$$g = \hat{a} - a \tag{6.12}$$

where

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \qquad \hat{a} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_k \end{bmatrix} \tag{6.13}$$

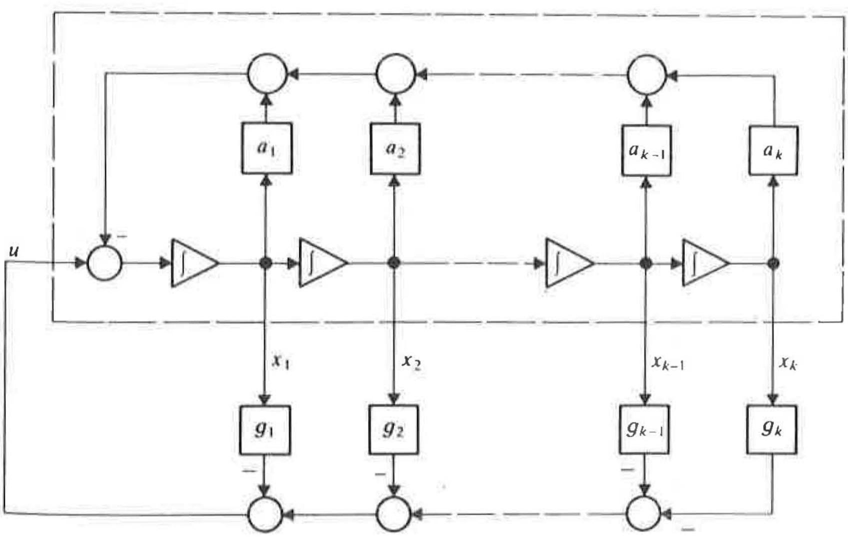

Figure 6.1 State variable feedback for system in first companion form.

are vectors formed from the coefficients of the open-loop and closed-loop characteristic equations, respectively.

The dynamics of a typical system are usually not in companion form. It is necessary to transform such a system into companion form before (6.12) can be used. Suppose that the state of the transformed system is $\bar{x}$, achieved through the transformation

$$\bar{x} = Tx \tag{6.14}$$

Then, as shown in Chap. 3,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \tag{6.15}$$

where

$$\bar{A} = TAT^{-1} \qquad \text{and} \qquad \bar{b} = Tb$$

For the transformed system the gain matrix is

$$\bar{g} = \hat{a} - \bar{a} = \hat{a} - a \tag{6.16}$$

since $\bar{a} = a$ (the characteristic equation being invariant under a change of state variables). The desired control law in the original system is

$$u = -g'x = -g'T^{-1}\bar{x} = -\bar{g}'\bar{x} \tag{6.17}$$

From (6.17) we see that

$$\bar{g}' = g'T^{-1}$$

Thus the gain in the original system is

$$g = T'\bar{g} = T'(\hat{a} - a) \tag{6.18}$$

In words, the desired gain matrix for a general system is the difference between the coefficient vectors of the desired and actual characteristic equation, premultiplied by the inverse of the transpose of the matrix T that transforms the general system into the companion form of (3.90), the A matrix of which has the form (6.11).

The desired matrix T is obtained as the product of two matrices U and V:

$$T = VU \tag{6.19}$$

The first of these matrices transforms the original system into an intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} \tag{6.20}$$

in the second companion form (3.107) and the second transformation U transforms the intermediate system into the first companion form.

Consider the intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u \tag{6.21}$$

with $\tilde{A}$ and $\tilde{b}$ in the form of (3.107). Then we must have

$$\tilde{A} = UAU^{-1} \qquad \text{and} \qquad \tilde{b} = Ub \tag{6.22}$$

The desired matrix U is precisely the inverse of the controllability test matrix Q of Sec. 5.4. To prove this fact, we must show that

$$U^{-1}\tilde{A} = AU^{-1} \tag{6.23}$$

or

$$Q\tilde{A} = AQ \tag{6.24}$$

Now, for a single-input system

$$Q = [b, Ab, \ldots, A^{k-1}b]$$

Thus, with $\tilde{A}$ given by (3.107), the left-hand side of (6.23) is

$$Q\tilde{A} = [b, Ab, \ldots, A^{k-1}b]\begin{bmatrix} 0 & 0 & \cdots & -a_k \\ 1 & 0 & \cdots & -a_{k-1} \\ 0 & 1 & \cdots & -a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_1 \end{bmatrix}$$

$$= [Ab, A^2b, \ldots, A^{k-1}b, -a_kb - a_{k-1}Ab - \cdots - a_kA^{k-1}b] \tag{6.25}$$

The last term in (6.25) is

$$(-a_kI - a_{k-1}A - \cdots - a_kA^{k-1})b \tag{6.26}$$

Now, by the Cayley-Hamilton theorem, (see Appendix):

$$A^k = -a_1A^{k-1} - a_2A^{k-2} - \cdots - a_kI$$

so (6.26) is A^kb. Thus the left-hand side of (6.24) as given by (6.25) is

$$Q\tilde{A} = [Ab, A^2b, \ldots, A^kb] = A[b, Ab, \ldots, A^{k-1}b] = AQ$$

which is the desired result.

If the system is not controllable, then Q^{-1} does not exist and there is no general method of transforming the original system into the intermediate system (6.21); in fact it is not possible to place the closed-loop poles anywhere one desires. Thus, controllability is an essential requirement of system design by pole placement. If the system is *stabilizable* (i.e., the uncontrollable part is asymptotically stable, as discussed in Chap. 5) a stable closed-loop system can be achieved by placing the poles of the controllable subsystem where one wishes and accepting the pole locations of the uncontrollable subsystem. In order to apply the formula of this section, it is necessary to first separate the uncontrollable subsystem from the controllable subsystem.

The control matrix $\tilde{b}$ of the intermediate system is given by

$$\tilde{b} = Ub \tag{6.27}$$

We now show that

$$\tilde{b} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6.28}$$

Multiply (6.28) by Q to obtain

$$Q\tilde{b} = [b, Ab, \ldots, A^{k-1}b]\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = b$$

which is the same as (6.27), since $Q^{-1} = U$.

The final step is to find the matrix V that transforms the intermediate system (6.21) into the final system (6.15). We must have

$$\bar{x} = V\tilde{x} \tag{6.29}$$

For the transformation (6.28) to hold, we must have

$$\bar{A} = V\tilde{A}V^{-1}$$

or

$$V^{-1}\bar{A} = \tilde{A}V^{-1} \tag{6.30}$$

The matrix V^{-1} that satisfies (6.30) is the transpose of the upper left-hand k-by-k submatrix of the (triangular Toeplitz) matrix appearing in (3.103)

$$V^{-1} = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = W \tag{6.31}$$

To prove this, we note that the left-hand side of (6.30) is

$$V^{-1}\bar{A} = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} -a_1 & -a_2 & \cdots & -a_k \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_k \\ 1 & a_1 & \cdots & a_{k-2} & 0 \\ 0 & 1 & \cdots & a_{k-3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6.32}$$

(Note that the zeros in the first row of $V^{-1}\bar{A}$ are the result of the difference of two terms $a_1 - a_1$, $a_2 - a_2$, etc.) and the right-hand side of (6.30) is

$$\tilde{A}V^{-1} = \begin{bmatrix} 0 & 0 & \cdots & -a_k \\ 1 & 0 & \cdots & -a_{k-1} \\ 0 & 1 & \cdots & -a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_k \end{bmatrix} \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ 0 & 0 & \cdots & a_{k-3} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_k \\ 1 & a_1 & \cdots & a_{k-2} & 0 \\ 0 & 1 & \cdots & a_{k-3} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

which is the same as (6.32). Thus (6.30) is proved.

We also need

$$\bar{b} = V\tilde{b}$$

We will show that

$$\bar{b} = \tilde{b}$$

Consider

$$\tilde{b} = V^{-1}\bar{b}$$

with

$$b = V^{-1}\bar{b} = \begin{bmatrix} 1 & a_1 & \cdots & a_{k-1} \\ 0 & 1 & \cdots & a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Thus $\tilde{b}$ and $\bar{b}$ are the same.

The result of this calculation is that the transformation matrix T whose transpose is needed in (6.18) is the inverse of the product of the controllability test matrix and the triangular matrix (6.31).

The above results may be summarized as follows. The desired gain matrix g, by (6.18) and (6.19), is given by

$$g = (VU)'(\hat{a} - a) \tag{6.33}$$

where

$$V = W^{-1} \qquad \text{and} \qquad U = Q^{-1}$$

Thus

$$VU = W^{-1}Q^{-1} = (QW)^{-1}$$

LQR

Hence (6.33) becomes

$$g = [(QW)']^{-1}(\hat{a} - a) \tag{6.34}$$

where Q is the controllability test matrix, W is the triangular matrix defined by (6.31), $\hat{a}$ is the vector of coefficients for the desired (closed-loop) characteristic polynomial, and a is the vector of coefficients of the open-loop system.

The basic pole-placement formula (6.34) was first stated by Bass and Gura.[1] It can be derived by other methods as discussed in Note 6.1.

Now that we have a specific formula for the gains of a controllable, single-input system that will place the poles at any desired location, several questions arise: If the closed-loop poles can be placed anywhere, where *should* they be placed? How can the technique be extended to multiple input systems? We shall address these questions and others after considering several examples.

Example 6A Instrument servo A dc motor driving an inertial load constitutes a simple instrument servo for keeping the load at a fixed position.

As shown in Chap. 2 (Example 2B), the state-space equations for the motor-driven inertia are

$$\dot{\theta} = \omega \tag{6A.1}$$

$$\dot{\omega} = -\alpha\omega + \beta u \tag{6A.2}$$

where θ is the angular position of the load, ω is the angular velocity, u is the applied voltage, and α and β are constants that depend on the physical parameters of the motor and load:

$$\alpha = -K^2/JR \qquad \beta = K/JR$$

If the desired position θ_r is a constant then we can define the servo error

$$e = \theta - \theta_r$$

Then

$$\dot{e} = \dot{\theta} - \dot{\theta}_r = \omega \qquad (\theta_r = \text{const}) \tag{6A.3}$$

and (6A.3) replaces (6A.1) to give

$$\begin{bmatrix} \dot{e} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} e \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u \tag{6A.4}$$

The angular position measurement can be instrumented by a potentiometer on the motor shaft and the angular velocity by a tachometer. Thus, the closed-loop system would have the configuration illustrated in Fig. 6.2. Note that the position gain is shown multiplying the negative of the system error which in turn is added to the control signal. This is consistent with the convention normally used for servos, wherein the position gain multiplies the difference $\theta_r - \theta$ between the reference and the actual positions. The quantity e defined above (6A.3) is the negative of the system error as normally defined in elementary texts.

The characteristic polynomial of the system is

$$|sI - A| = \begin{vmatrix} s & -1 \\ 0 & s + \alpha \end{vmatrix} = s^2 + \alpha s$$

Thus

$$a = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

The controllability test matrix Q and the matrix W are given respectively by

$$Q = [b, Ab] = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \qquad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

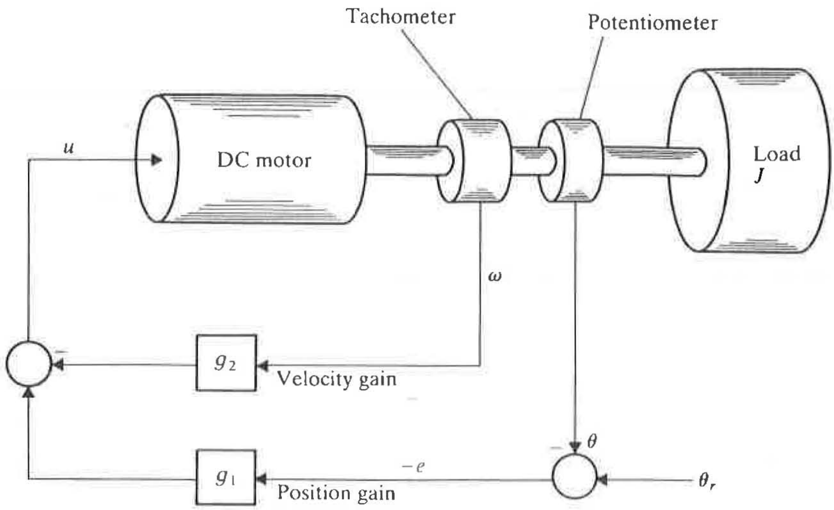

Figure 6.2 Implementation of an instrument servo.

Thus

$$QW = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} = (QW)'$$

and

$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Thus the desired gain matrix, by the Bass-Gura formula (6.34), is

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 - \alpha \\ \bar{a}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_2/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix} \tag{6A.5}$$

where $\bar{a}_1$ and $\bar{a}_2$ are the coefficients of the desired characteristic polynomial.

While the above calculation illustrates the general procedure, the gains could have been more easily computed directly. For a control law of the form

$$u = -g_1 e - g_2 \omega$$

(6A.4) becomes

$$\dot{e} = \omega$$

$$\dot{\omega} = -g_1 \beta e - (\alpha + \beta g_2)\omega$$

which has the closed-loop matrix

$$A_c = \begin{bmatrix} 0 & 1 \\ -g_1\beta & -(\alpha + g_2\beta) \end{bmatrix}$$

with the characteristic equation

$$|sI - A_c| = s^2 + (\alpha + g_2\beta)s + g_1\beta$$

Thus

$$\bar{a}_1 = \alpha + g_2\beta \qquad \bar{a}_2 = g_1\beta$$

LQR

CCACI EQN

or

$$g_1 = \bar{a}_2/\beta \qquad g_2 = (\bar{a}_1 - \alpha)/\beta$$

which is the same as (6A.5).

Note that the position and velocity gains g_1 and g_2, respectively, are proportional to the amounts we wish to move the coefficients from their open-loop positions. The position gain g_1 is necessary to produce a stable system: $\bar{a}_2 > 0$. But if the designer is willing to settle for $a_1 = \alpha$, i.e., to accept the open-loop damping, then the gain g_2 can be zero. This of course eliminates the need for a tachometer and reduces the hardware cost of the system. It is also possible to alter the system damping without the use of a tachometer, by using an estimate $\hat{\omega}$ of the angular velocity ω. This estimate is obtained by means of an observer as discussed in Chap. 7.

Example 6B Stabilization of an inverted pendulum An inverted pendulum can readily be stabilized by a closed-loop feedback system, just as a person of moderate dexterity can do it.

A possible control system implementation is shown in Fig. 6.3, for a pendulum constrained to rotate about a shaft at its bottom point. The actuator is a dc motor. The angular position of the pendulum, being equal to the position of the shaft to which it is attached, is measured by means of a potentiometer. The angular velocity in this case can be measured by a "velocity pick-off" at the top of the pendulum. Such a device could consist of a coil of wire

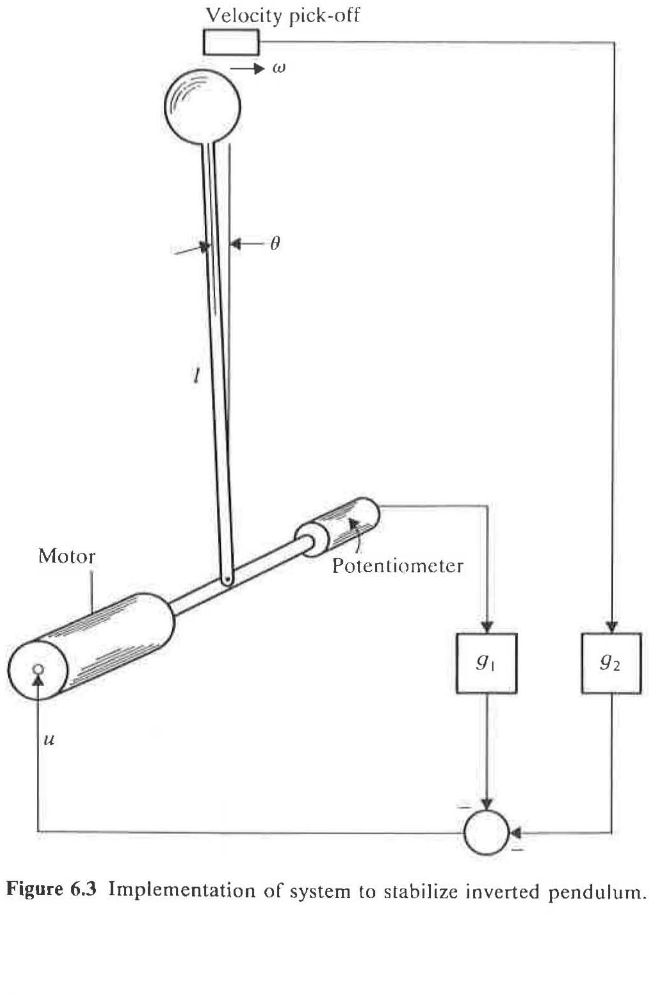

Figure 6.3 Implementation of system to stabilize inverted pendulum.

in a magnetic field created by a small permanent magnet in the pendulum bob. The induced voltage in the coil is proportional to the linear velocity of the bob as it passes the coil. And since the bob is at a fixed distance from the pivot point the linear velocity is proportional to the angular velocity. The angular velocity could of course also be measured by means of a tachometer on the dc motor shaft.

As determined in Prob. 2.2, the dynamic equations governing the inverted pendulum in which the point of attachment does not translate is given by

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \Omega^2\theta - \alpha\omega + \beta u \end{aligned} \tag{6B.1}$$

where α and β are given in Example 6A, with the inertia J being the total reflected inertia:

$$J = J_m + ml^2$$

where m is the pendulum bob mass and l is the distance of the bob from the pivot. The natural frequency Ω is given by

$$\Omega^2 = \frac{mgl}{J + ml^2} = \frac{g}{l + J/ml}$$

(Note that the motor inertia J_m affects the natural frequency.)

Since the linearization is valid only when the pendulum is nearly vertical, we shall assume that the control objective is to maintain $\theta = 0$. Thus we have a simple regulator problem.

The matrices A and b for this problem are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

The open-loop characteristic polynomial is

$$|sI - A| = \begin{vmatrix} s & -1 \\ -\Omega^2 & s + \alpha \end{vmatrix} = s^2 + \alpha s - \Omega^2$$

Thus

$$\begin{aligned} a_1 &= \alpha \\ a_2 &= -\Omega^2 \end{aligned}$$

The open-loop system is unstable, of course.

The controllability test matrix and the W matrix are given respectively by

$$Q = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \qquad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

(which are the same as they were for the instrument servo). And

$$[(QW)']^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Thus the gain matrix required for pole placement using (6.34), is

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} (\bar{a}_1 - \alpha) \\ \bar{a}_2 + \Omega^2 \end{bmatrix} = \begin{bmatrix} (\bar{a}_2 + \Omega^2)/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix}$$

Example 6C Control of spring-coupled masses The dynamics of a pair of spring-coupled masses, shown in Fig. 3.7(a), were shown in Example 3I to have the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K/\bar{M} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The system has the characteristic polynomial

$$D(s) = s^4 + (K/\bar{M})s^2$$

Hence
$$a_1 = a_3 = a_4 = 0, \qquad a_2 = K/\bar{M}.$$

The controllability test and W matrices are given, respectively, by

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -K/\bar{M} \\ 1 & 0 & -K/\bar{M} & 0 \end{bmatrix} \qquad W = \begin{bmatrix} 1 & 0 & K/\bar{M} & 0 \\ 0 & 1 & 0 & K/\bar{M} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6C.1}$$

Multiplying we find that

$$QW = (QW)' = (QW)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{6C.2}$$

(This rather simple result is not really as surprising as it may at first seem. Note that A is in the first companion form but using the right-to-left numbering convention. If the left-to-right numbering convention were used the A matrix would already be in the companion form of (6.11) and would not require transformation. The transformation matrix T given by (6C.2) has the effect of changing the state variable numbering order from left-to-right to right-to-left, and vice versa.)

The gain matrix g is thus given by

$$g = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 - K/\bar{M} \\ \bar{a}_3 \\ \bar{a}_4 \end{bmatrix} = \begin{bmatrix} \bar{a}_4 \\ \bar{a}_3 \\ \bar{a}_2 - K/\bar{M} \\ \bar{a}_1 \end{bmatrix}$$

A suitable pole "constellation" for the closed-loop process might be a Butterworth pattern as discussed in Sec. 6.5. To achieve this pattern the characteristic polynomial should be of the form

$$\bar{D}(s) = s^4 + (1+\sqrt{3})\Omega s^3 + (2+\sqrt{3})\Omega^2 s^2 + (1+\sqrt{3})\Omega^3 s + \Omega^4$$

Thus

$$\begin{aligned} \bar{a}_1 &= (1+\sqrt{3})\Omega \\ \bar{a}_2 &= (2+\sqrt{3})\Omega^2 \\ \bar{a}_3 &= (1+\sqrt{3})\Omega^3 \\ \bar{a}_4 &= \Omega^4 \end{aligned}$$

Thus the gain matrix g is given by

$$g = \begin{bmatrix} \Omega^4 \\ (1+\sqrt{3})\Omega^3 \\ (2+\sqrt{3})\Omega^2 - K/\bar{M} \\ (1+\sqrt{3})\Omega \end{bmatrix}$$

6.3 MULTIPLE-INPUT SYSTEMS

If the dynamic system under consideration

$$\dot{x} = Ax + Bu$$

has more than one input, that is, B has more than one column, then the gain matrix G in the control law

$$u = -Gx$$

has more than one row. Since each row of G furnishes k gains that can be adjusted, it is clear that in a controllable system there will be more gains available than are needed to place all of the closed-loop poles. This is a benefit: the designer has more flexibility in the design than in the case of a single-input system; it is possible to specify all the closed-loop poles and still be able to satisfy other requirements. How should these other requirements be specified? The answer to this question may well depend on the circumstances of the particular application. One possibility might be to set some of the gains to zero. For example, it is sometimes possible to place the closed-loop poles at locations desired with a gain matrix which has a column of zeros. This means that the state variable corresponding to that column is not needed in the generation of any of the control signals in the vector u, and hence there is no need to measure (or estimate) that state variable. This simplifies the resulting control system structure. If all the state variables, except those corresponding to columns of zeros in the gain matrix, are accessible for measurement then there is no need for an observer to estimate the state variables that cannot be measured. A very simple and robust control system is the result.

Another possible method of selecting a particular structure for the gain matrix is to make each control variable depend on a different group of state variables which are physically more closely related to that control variable than to the other control variables.

Still another possibility arises in systems which have a certain degree of structural symmetry and in which it is desired to retain the symmetry in the closed-loop system by an appropriate feedback structure.

The following example illustrates one method of selecting the gain matrix.

Example 6D Distillation column For the distillation column of Example 4A, having the block-diagram of Fig. 4.2, we saw in Example 5G that both inputs are needed in order for the system to be controllable, because there are redundant poles at the origin (due to the integrators) from either Δu_1 or Δu_2. If there were only one integrator present, it is easy to see that the system would be controllable from Δu_1 alone. This suggests a gain structure in which Δu_1 depends on x_1, x_2, and x_3, and Δu_2 depends on x_4. This gives four adjustable gains for the closed-loop fourth-order system and we would expect to be able to locate the closed-loop poles at whatever locations are desired.

Thus we use a gain matrix of the form

$$G = \begin{bmatrix} g_1 & g_2 & g_3 & 0 \\ 0 & 0 & 0 & g_4 \end{bmatrix} \tag{6D.1}$$

With the A and B matrices as given by (2G.5) it is found that the closed-loop dynamics matrix is

$$A_c = A - BG = \begin{bmatrix} a_{11} - b_{11}g_1 & -b_{11}g_2 & -b_{11}g_3 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & 0 & -b_{32}g_4 \\ 0 & 0 & 0 & -b_{42}g_4 \end{bmatrix}$$

Thus the closed-loop characteristic equation is

$$|sI - A_c| = \left|\begin{array}{ccc:c} s - a_{11} + b_{11}g_1 & b_{11}g_2 & b_{11}g_3 & 0 \\ -a_{21} & s - a_{22} & 0 & 0 \\ 0 & -a_{32} & s & 0 \\ \hdashline 0 & 0 & 0 & s + b_{42}g_4 \end{array}\right|$$

$$= (s + b_{42}g_4)\begin{vmatrix} s - a_{11} + b_{11}g_1 & b_{11}g_2 & b_{11}g_3 \\ -a_{21} & s - a_{22} & 0 \\ 0 & -a_{32} & s \end{vmatrix}$$

$$= (s + b_{42}g_4)(s^3 + \bar{a}_1 s^2 + \bar{a}_2 s + \bar{a}_3) \tag{6D.2}$$

Note that $|sI - A_c|$ factors into two terms, a first-order term giving a pole at $s = -b_{42}g_4$ and a third-order term. The third-order term is the same as would result for a third-order system having dynamics and control matrices given respectively by

$$A_3 = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & 0 \end{bmatrix} \qquad B_3 = \begin{bmatrix} b_{11} \\ 0 \\ 0 \end{bmatrix}$$

with a state variable feedback of the form

$$G_3 = [g_1, g_2, g_3]$$

Thus we can adjust g_1, g_2, and g_3 to achieve any desired location of the roots of the third-order factor in (6D.2) and use g_4 to adjust the location of the pole at $s = -b_{42}g_4$.

Note that if the gains are real numbers, as they must be in a physical system, then one pole must be the real pole at $s = -b_{42}g_4$, and hence one of the poles arising from the cubic factor in (6D.2) must also be real when the gains g_1, g_2, and g_3 are real. Thus, by using a gain matrix having the structure of (6D.1), we do not have freedom to place the closed-loop poles anywhere in the complex plane. This is not a contradiction of controllability, because (6D.1) is not the most general form that the gain matrix G can take: four of the possible gains have been set to zero. Since a very satisfactory transient response can be achieved, however, with two real poles, the gain matrix structure of (6D.1) is, in the practical sense, perfectly acceptable.

6.4 DISTURBANCES AND TRACKING SYSTEMS: EXOGENOUS VARIABLES

In the previous section we considered the design of regulators in which the performance objective is to achieve a specified closed-loop dynamic behavior (pole locations) of the system in response to arbitrary initial disturbances. A more general design objective is to control the system error not only for initial disturbances, but also for persistent disturbances, and also to track reference inputs.

In Chap. 5 the general problem was set up by defining the system error

$$e = x - x_r \tag{6.35}$$

where x_r is assumed to satisfy a differential equation

$$\dot{x}_r = A_r x_r \tag{6.36}$$

In addition to the reference input we also have a disturbance x_d, so that the error is given by

$$\dot{e} = Ae + (A - A_r)x_r + Fx_d + Bu = Ae + Bu + Ex_0 \tag{6.37}$$

In Chap. 5 we defined the metastate

$$\mathbf{x} = \begin{bmatrix} e \\ \hline x_0 \end{bmatrix}$$

which makes it possible to regard the design problem, including reference and disturbance inputs, as a regulator problem. As was shown in Chap. 5, however, the metasystem is not controllable, so it is not possible to apply the pole-placement design technique to the metasystem. (Since controllability is not a requirement for formulating the optimum control problem, as discussed in Chap. 9, we will be able to use the metasystem formulation in connection with optimum control system design.)

Instead of working with the metasystem, we work directly with the error differential equation (6.37). The exogenous vector x_0 is treated as an input just like u. The design problem is really to arrange matters so that the control input u counteracts the effects of the exogenous variables. The control that we seek should be effective not only for a specific exogenous input, but rather for an entire class of inputs. Only the characteristics of the *class* are known to the designer; the specific member of the class is determined by measurements on x_0 while the process is in operation.

Since we are limiting our attention to linear systems, we consider only a linear control law, which now takes the general form

$$u = -Ge - G_0x_0 = -Ge - G_rx_r - G_dx_d \tag{6.38}$$

The closed-loop system using a control of the form (6.38) has the appearance shown in Fig. 6.4. Note the presence of two paths in addition to the feedback

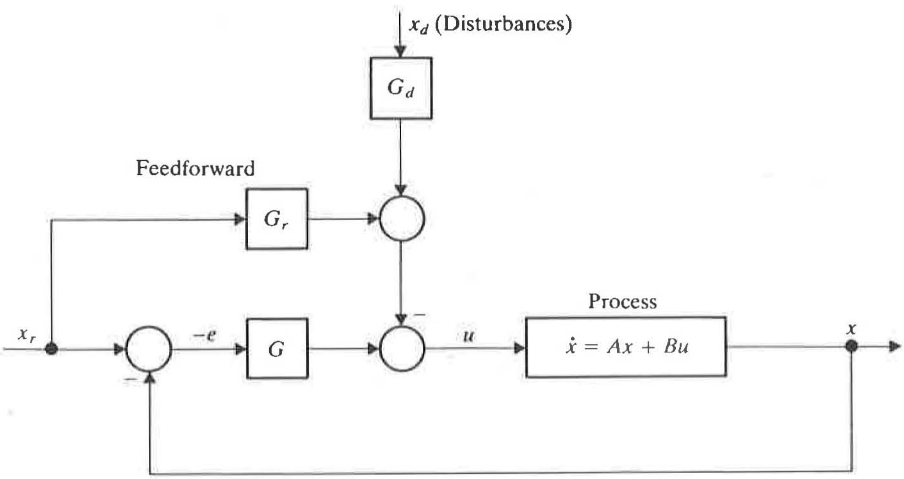

Figure 6.4 Schematic of feedback system for process with reference state and disturbance input.

loop in which the system error appears. There is a "feedforward" path with a gain G_r and a path through the gain G_d, the purpose of which is to counteract the effect of these disturbances.

The design is based, as already mentioned, on the assumption that the exogenous input vector $x_0 = [x_r', x_d']'$ as well as the system error e are accessible for measurement during the operation of the control system (i.e., in "real time"). Since x_r is a reference input, one might think that it is always accessible. The instrumentation, however, might be such that only the system error can be measured; it may be difficult (hence costly in terms of hardware) to measure x_d independent of the system error. It is noted that reference input x_r appears in (6.37) through the difference $A - A_r$ between the dynamics matrix A and the matrix A_r used to model the reference input. If $A_r = A$, that is, if the reference input can be generated as the solution of the unforced differential equation of the open-loop process, then no feedforward path is needed. If the open-loop process is stable, then the only reference inputs that can be generated are decaying exponentials which go to zero in the steady state. Thus if we need to track steps, ramps, etc., in the steady state, we cannot have $A_r = A$ for an asymptotically stable open-loop system. On the other hand if A has a pole at the origin of order ν, then by proper choice of initial conditions x_r can include a polynomial in time of order $\nu - 1$; we can still have $A_r = A$ and hence not require feedforward. (Recall from Chap. 5 that the presence of a pole of order ν in the open-loop system makes it a "type ν" system. We thus see again that a type ν system can follow reference inputs containing polynomials of degree up to and including $\nu - 1$ without use of feedforward.)

Sometimes the disturbance x_d can be measured easily, sometimes not. In a temperature control system, for example, in which x_d is the ambient temperature of the environment of the process, it is not too difficult to accomplish this measurement with an extra thermometer. In an aircraft autopilot design, on the other hand, in which the disturbances may consist of wind-induced forces, it may be all but impossible to instrument the required measurements. In cases where the required quantities, or some of them, are not accessible for measurement, an observer, as discussed in Chap. 7, is used to infer estimates of these quantities, based on the assumed dynamic model, using the quantities that are accessible for measurement.

For the present, our objective is to design the gain matrices G and G_0 in (6.38). When the control given by (6.38) is used in the general process (6.37) the closed-loop dynamics are

$$\dot{e} = Ae + Ex_0 - B(Ge + G_0x_0) \tag{6.39}$$

which is the differential equation of a linear system excited by x_0.

If it were possible, it would be desirable to choose the gains G and G_0 to keep the system error zero. But this is not possible: system errors may be present initially that cannot instantly be reduced to zero. And even when initial errors are zero, there are usually not enough control variables (i.e., columns in the B matrix) to make the coefficients of x_0 vanish, as they must in order that the error be zero for *any* x_0 and e.

More reasonable performance objectives are the following:

(*a*) The closed-loop system should be asymptotically stable.
(*b*) A linear combination of the error state variables (rather than the entire state vector) is to be zero in the steady state.

In order for the closed-loop system to be asymptotically stable the closed-loop dynamics matrix $A_c = A - BG$ must have its characteristic roots in the left half-plane. If the system is controllable, this can be accomplished by a suitable choice of the gain matrix.

The steady-state condition is characterized by a constant error state vector, i.e., in the steady state

$$\dot{e} \equiv 0$$

which, from (6.39), means that

$$(A - BG)e = (BG_0 - E)x_0$$

If the closed-loop system is asymptotically stable, $A - BG = A_c$ has no characteristic roots at the origin, and hence its inverse exists. Thus the steady state error is given by

$$e = (A - BG)^{-1}(BG_0 - E)x_0 \tag{6.40}$$

As noted before it is not reasonable to expect that e be zero. Instead we require that

$$y = Ce \equiv 0 \tag{6.41}$$

where C is a singular matrix of suitable dimension. We'll see shortly what a "suitable" dimension is. If (6.41) holds, then from (6.40)

$$C(A - BG)^{-1}(BG_0 - E)x_0 = 0 \tag{6.42}$$

Remember that we want (6.42) to hold for any x_0. This can be achieved if and only if the coefficient matrix multiplying x_0 vanishes:

$$C(A - BG)^{-1}(BG_0 - E) = 0 \tag{6.43}$$

The matrix G_0 which satisfies (6.43) will meet the requirement of (6.41). We note that (6.43) can be written

$$C(A - BG)^{-1}BG_0 = C(A - BG)^{-1}E \tag{6.44}$$

We examine the possibility of solving (6.44) for the required gain matrix G_0. Here is where the dimension of C becomes significant. Suppose that the dimension of y is j. Then C is a $j \times k$ matrix, $(A - BG)^{-1}$ is a $k \times k$ matrix, and B is a $k \times m$ matrix, where m is the number of control variables. The product of the three matrices multiplying G_0 is thus a $j \times m$ matrix. If $j > m$, then (6.44) is "overdetermined": there are too many conditions to be satisfied by G_0 and, except for special values of E, no solution to (6.44) for G_0 exists. If $j < m$, then (6.44) is "underdetermined": G_0 is not uniquely specified by (6.44). This poses

LQR

no problem; it only means that G_0 can be chosen to satisfy not only (6.41), but also to satisfy other conditions.

Analytically the "cleanest" case is when the number of inputs m is equal to the dimension of y. (If y is regarded as the system output, then we can say that the process is "square," having the same number of inputs as outputs.) In this case, when the matrix multiplying G_0 is nonsingular, the desired gain matrix is given by

$$G_0 = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1}E \tag{6.45}$$

The big, messy matrix

$$B^{\#} = [C(A - BG)^{-1}B]^{-1}C(A - BG)^{-1} \tag{6.46}$$

that multiplies E in (6.45) has the property that

$$B^{\#}B = I \tag{6.47}$$

A matrix having this property is called a left inverse (or left "pseudoinverse") of B. Matrices of this type are encountered frequently in linear systems analysis. In terms of $B^{\#}$, (6.45) can be written

$$G_0 = B^{\#}E \tag{6.48}$$

Under what circumstances does the matrix $C(A - BG)^{-1}B$ possess an inverse? One might think that the existence of an inverse depends on the stabilizing gain matrix G. In fact, this is not the case. The existence of an inverse depends only on the open-loop dynamics: it can be shown that $C(A - BG)^{-1}B$ possesses an inverse if and only if

$$\lim_{s \to 0} \mathrm{H}_0(s) = |C(sI - A)^{-1}B| \neq 0 \tag{6.49}$$

If A is nonsingular (6.49) reduces to the requirement that $|CA^{-1}B| \neq 0$. The reason that invertability of $C(A - BG)^{-1}B$ is independent of G is related to the fact that state-variable feedback does not alter the transmission zeros of a process, as discussed in Prob. 4.1. (See also Note 6.2.)

The specific value of the inverse, however, does in general depend on G. Nevertheless, one can safely choose any feedback gain matrix G without being concerned about the possibility that this choice of gain will compromise the invertability of $C(A - BG)^{-1}B$.

In most cases, the reference state x_r does not need to have all of its components specified. In other words, the error that the control system must be designed to reduce to zero may be of lower dimension that the state vector. The other components of the state vector may be unspecified. In these cases, the component of the exogenous vector corresponding to the reference state may be of lower dimension that the state x and the corresponding submatrix of E will not be $A - A_d$ but a different matrix with fewer than k columns. Rather than try to express this in general terms, we illustrate it by the example that follows.

Example 6E Temperature control Consider the temperature control problem having the electrical network analog shown in Fig. 6.5. The voltage u may be regarded as the analog of the temperature of a heater and x_0 as the ambient temperature. Since there is only one heater (i.e., one input) then we can in general control only a single quantity, perhaps v_1 or v_2, or a linear combination of the two, such as their average $y = (v_1 + v_2)/2$.

The dynamic equations, in state-space form, are (see Example 2C)

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} -a_{11} & a_{12} \\ a_{21} & -a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} x_0 \tag{6E.1}$$

with

$$a_{11} = -\frac{1}{C_1}[R_3^{-1} + D_1(R_2^{-1} + R_0^{-1} + R^{-1})] \qquad a_{12} = \frac{1}{C_1}[R_3^{-1} + R_2^{-1}D_1]$$

$$b_1 = R_0^{-1}D_1 \qquad e_1 = R^{-1}D_1$$

$$a_{21} = \frac{1}{C_2}[R_3^{-1} + R_1^{-1}D_2], \qquad a_{22} = -\frac{1}{C_2}[R_3^{-1} + D_2(R_1^{-1} + R_0^{-1} + R^{-1})]$$

$$b_2 = R_0^{-1}D_2, \qquad e_2 = R^{-1}D_2$$

where

$$D_1 = [1 + R_1(R_0^{-1} + R_2^{-1} + R^{-1})]^{-1} \qquad D_2 = [1 + R_2(R_0^{-1} + R_1^{-1} + R^{-1})]^{-1}$$

We assume that the desired state is

$$x_{1d} = \bar{v}_1 = \text{const}$$

$$x_{2d} = \bar{v}_2 = \text{const}$$

Thus

$$A_0 = 0 \tag{6E.2}$$

We take as the output matrix

$$C = [c_1, c_2] \tag{6E.3}$$

Assume the feedback gain matrix is

$$G = [g_1, g_2]$$

Then

$$A_c^{-1} = (A - BG)^{-1} = \begin{bmatrix} -a_{11} - b_1 g_1 & a_{12} - b_1 g_2 \\ a_{21} - b_2 g_1 & -a_{22} - b_2 g_1 \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} -a_{22} - b_2 g_2 & -a_{12} + b_1 g_2 \\ -a_{21} + b_2 g_1 & -a_{11} - b_1 g_1 \end{bmatrix}}{(a_{11} + b_1 g_1)(a_{22} + b_2 g_2) - (a_{21} - b_2 g_1)(a_{12} - b_1 g_2)} \tag{6E.4}$$

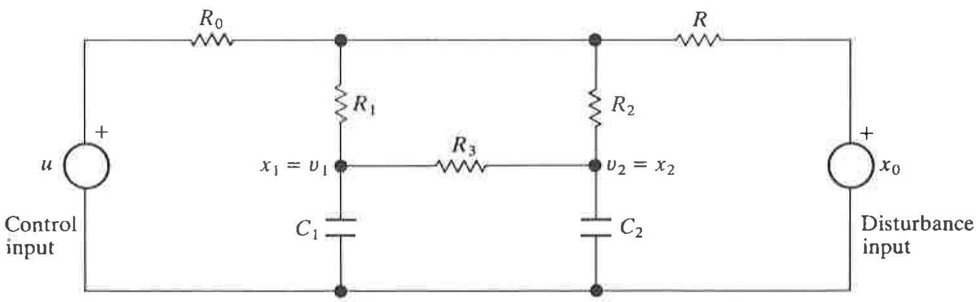

Figure 6.5 Electric network analog of temperature control problem.

and

$$C(A - BG)^{-1}B = \frac{[c_1 \quad c_2]\begin{bmatrix} -a_{22}b_1 - a_{12}b_2 \\ -a_{21}b_1 - a_{11}b_2 \end{bmatrix}}{a_{11}a_{22} - a_{21}a_{12} + g_1(b_1 a_{22} + b_2 a_{12}) + g_2(b_1 a_{21} + b_2 a_{11})} \tag{6E.5}$$

Note that the numerator of (6E.5) is independent of the feedback gains, irrespective of c_1 and c_2.

In this example $C(A - BG)^{-1}B$ is a scalar (i.e., a 1-by-1 matrix) given by

$$p = C(A - BG)^{-1}B = \frac{-c_1(a_{22}b_1 + a_{12}b_2) - c_2(a_{21}b_1 + a_{11}b_2)}{a_{11}a_{22} - a_{21}a_{12} + g_1(b_1 a_{22} + b_2 a_{12}) + g_2(b_1 a_{21} + b_2 a_{11})}$$

Thus, from (6.47)

$$B^{\#} = [q_1, q_2]/p$$

where

$$q_1 = -c_1(a_{22} + b_2 g_2) + c_2(-a_{21} + b_2 g_1)$$
$$q_2 = c_1(-a_{12} + b_1 g_2) - c_2(a_{11} + b_1 g_1)$$

and

$$G_0 = B^{\#}E = [e_1 q_1 + e_2 q_2]/p \qquad \text{(a scalar)}$$

The implementation of the control law is illustrated in Fig. 6.6. It is noted that even though the performance criterion $y = Ce = c_1(v_1 - \bar{v}_1) + c_2(v_2 - \bar{v}_2)$ is a scalar combination of the two errors e_1 and e_2, the feedforward signal $g_{r1}\bar{v}_1 + g_{r2}\bar{v}_2$ is not expressible as a function of the difference between $\bar{v}_1$ and $\bar{v}_2$; both $\bar{v}_1$ and $\bar{v}_2$ are required in the control law implementation.

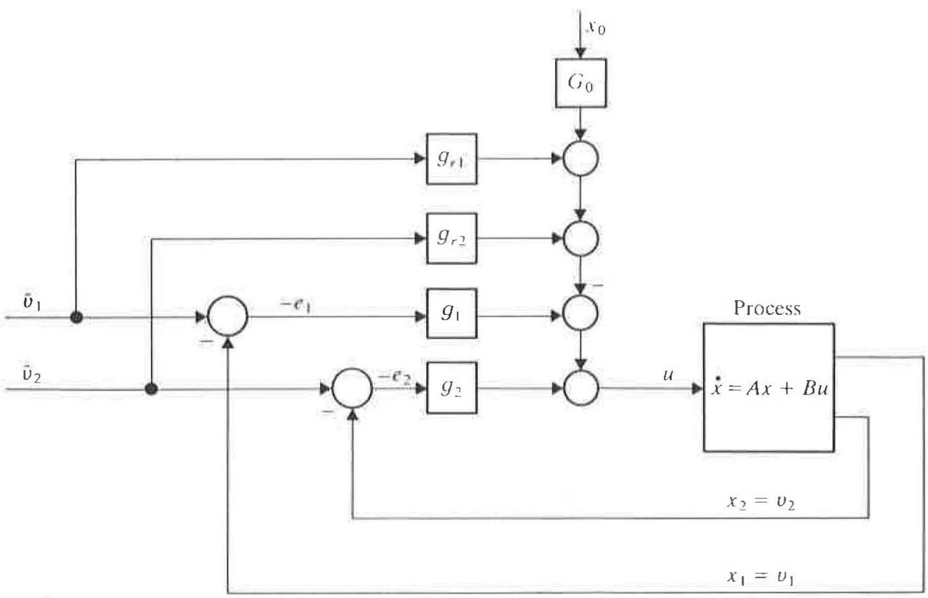

Figure 6.6 Control law implementation.

6.5 WHERE SHOULD THE CLOSED-LOOP POLES BE PLACED?

Having determined that the closed-loop poles of a controllable system can be placed anywhere, it is natural to ask where the poles *should* be placed. To assert that they should be placed to meet the performance requirements is begging the question, which is how to relate the performance requirements to the gain matrix G that is used in the implementation of the feedback law. A systematic method of selecting the gains by minimizing a quadratic performance integral is given in Chap. 9. That method has many advantages but it is by no means the only method available. Among the concerns that the designer might wish to address are those to be discussed in this section.

The control law for a regulator $u = -Gx$ implies that for a given state x the larger the gain, the larger the control input. There are limits on the control input in practical systems: The actuator which supplies the control u cannot be arbitrarily large without incurring penalties of cost and weight. Other reasons for limiting the control may be to avoid the potential damaging effects of stresses on the process that large inputs might cause. If the control signal generated by the linear feedback law $u = -Gx$ is larger than possible or permissible for reasons of safety, the actuator will "saturate" at a lower input level. The effect of occasional control saturation is usually not serious: in fact a system which never saturates is very likely overdesigned, having a larger and less efficient actuator than is needed to accomplish even the most demanding tasks. On the other hand, if the control signals produced by the linear control law are so large that the actuator is almost always saturated, it is not likely that the system behavior will be satisfactory, unless the actuator saturation is explicitly accounted for in an intentionally nonlinear (e.g., "bang-bang") control law design. If such a design is not intended, the gain matrix should be selected to avoid excessively large control signals for the range of states that the control system can encounter during operation.

The effect of control system gain on pole locations can be appreciated by considering the Bass-Gura formula (6.34) for a single-input system. (Qualitatively, similar considerations apply to multiple-input systems.) Note first that the gains are proportional to the amounts that the poles are to be moved, i.e., to the distance that the coefficients of the characteristic polynomial must move between the open-loop and the closed-loop system. The less the poles are moved, the smaller the gain matrix. Thus, large system gains are avoided by limiting the changes in the coefficients of the characteristic equation. It is also noted that the control system gains are inversely proportional to the controllability test matrix. The less controllable the system, the larger the gains that are needed to effect a change in the system poles. There is nothing surprising about this.

The inference that may be reasonably drawn from this is that the designer should not attempt to alter the dynamic behavior of the open-loop process more than is required. One reason for trying to alter the behavior of a process is to stabilize it, if it is unstable, or to increase its stability by moving its poles into the interior of the left half of the s plane. Although stability is the most important consideration it is not the only consideration. Speed of response (i.e., bandwidth) is also important. Fast response—high bandwidth—of the closed-loop system is often sought after, since the errors in following rapidly changing inputs will be smaller. There may be instances, however, in which the bandwidth of the closed-loop system is intentionally not made as high as it can be. If the reference input contains a good deal of noise, it might be desirable to reduce the bandwidth to prevent the system from becoming excessively agitated by following the noise.

Another reason for limiting the bandwidth of the closed-loop system is the uncertainty of the high-frequency dynamics of the process. A mechanical system, for example, has resonance effects (modes) due to the elasticity of the structural members. The dynamic model used for design ignores many if not all of these effects: their magnitudes are small; the exact frequencies are not easy to determine; the effort required to include them in the model is not justified. Other types of processes (thermal, chemical, etc.) also have uncertain behavior at high frequencies. If the uncertain high-frequency poles are included within the bandwidth of the closed-loop process, these resonances may be excited and result in unexpected high-frequency oscillation, or even instability. A prudent design requires that the loop transmission be well below unity at the frequencies where these resonances may occur.

The bandwidth of a system is governed primarily by its *dominant* poles, i.e., the poles with real parts closest to the origin. To see this, visualize the partial-fraction expansion of the transfer function of the system. Terms corresponding to poles whose (negative) real parts are farthest from the origin have relatively high decay rates (damping) and hence, after an initial transient period, they will contribute less to the total response than terms corresponding to poles with real parts close to the origin. (While this behavior is typical of physical processes, there is no theoretical reason why the residues at poles with high damping cannot be much greater than the residues at the poles with less damping. If the highly damped poles have large residues, their effects may persist simply because they start out much larger.)

In order for the transient to decay as rapidly as is required by the poles that are far from the origin, it is necessary to change the energy in the system rapidly; this would require the use of large control inputs. If there are some poles that are far from the origin and others that are close to the origin, the maximum control amplitudes will be governed by the former, but the system speed of response is slowed by the latter. This behavior suggests that the feedback gains are such that the available control is not efficiently used. Efficient use of the control signal would require that all the closed-loop poles be about the same distance from the origin.

Having reasoned that it is imprudent to try to move the open-loop poles farther than is necessary (obviously it is necessary to move them to the left half-plane if the open-loop process is unstable) and inefficient to make some poles much more highly damped (farther from the origin) than the other poles, one might seek to *optimize* the closed-loop pole locations. How to accomplish this in general is the subject of Chap. 9. One result of optimization theory that can be used here concerns "asymptotic pole location": As control effort becomes increasingly less "expensive," the closed-loop poles tend to radiate out from the origin along the spokes of a wheel in the left half-plane as given by the roots of

$$\left(\frac{s}{\omega_0}\right)^{2k} = (-1)^{k+1} \tag{6.50}$$

where k is the number of poles in the left half-plane. (Fig. 6.7.) Poles located in

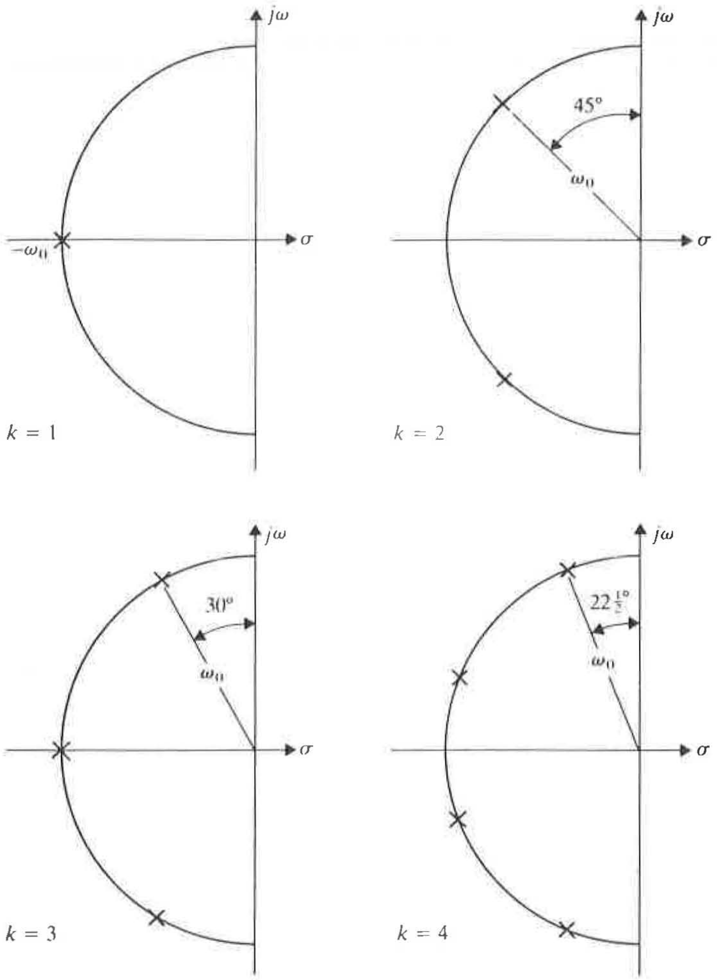

Figure 6.7 Butterworth pole configurations.

accordance with (6.50) are said to have a "Butterworth configuration," a term that originated in communication networks.

The polynomial having as its factors the zeros of (6.50) in the *left half-plane only* are known as *Butterworth polynomials* $B_k(z)$, $z = s/\omega_0$, the first few of which are:

$$B_1(z) = z + 1$$

$$B_2(z) = z^2 + \sqrt{2}z + 1$$

$$B_3(z) = z^3 + 2z^2 + 2z + 1$$

$$B_4(z) = z^4 + 2.613z^3 + (2 + \sqrt{2})z^2 + 2.613z + 1$$

Some of the properties of transfer functions having Butterworth polynomials for their denominators are given in Note 6.3 and Prob. 6.10.

In the absence of any other consideration, a Butterworth configuration is often suitable. Note, however that as the order k becomes high, one pair of poles come precariously close to the imaginary axis. It might be desirable to move these poles farther into the left half-plane.

The considerations that govern the choice of closed-loop poles that were discussed above may be summarized as follows:

Select a bandwidth high enough to achieve the desired speed of response.

Keep the bandwidth low enough to avoid exciting unmodeled high-frequency effects and undesired response to noise.

Place the poles at approximately uniform distances from the origin for efficient use of the control effort.

These broad guidelines allow plenty of latitude for special needs of individual applications.

Example 6F Missile autopilot As noted in Example 4F, the usual function of an autopilot in a missile is to make the normal component of acceleration a_N track a commanded acceleration signal a_{NC} which is produced by the missile guidance system. This example illustrates the design of such an autopilot.

Open-loop dynamics A high-performance missile, when provided with a suitable autopilot, is capable of achieving a relatively high bandwidth. This bandwidth may be comparable to that of the actuator which drives the control surface. Thus it may be necessary to include the dynamics of the actuator in order to have an adequate model of the process. We assume this to be the case in the present example, and use the first-order dynamic model for the actuator that was used in Example 4F, namely

$$\dot{\delta} = \frac{1}{\tau}(u - \delta) \tag{6F.1}$$

where u is the input to the actuator and τ is its time constant.

The complete dynamic model of the missile (airframe and actuator) is thus given by (4F.1) and (4F.2) in Example 4F. In this application, however, we are interested in tracking an acceleration command and hence prefer to use the acceleration error

$$e = a_{NC} - a_N \tag{6F.2}$$

as a state variable instead of the angle of attack. The derivative of the acceleration error is

$$\dot{e} = \dot{a}_{NC} - \dot{a}_N$$

Now, although the commanded acceleration is not constant, we can approach the design problem on the assumption that it is: $\dot{a}_{NC} \approx 0$. (A better design might be achieved by making use of the actual rate of change of normal acceleration command, a signal that might be available from the missile guidance system.)

In addition to approximating the commanded acceleration by a constant, we also assume that the aerodynamic coefficients Z_α and Z_δ and the missile speed V are approximately constant. Using all these approximations

$$\dot{e} = -\dot{a}_N = -Z_\alpha \dot{\alpha} - Z_\delta \dot{\delta} \tag{6F.3}$$

But, from (3F.1) and (3F.4),

$$\dot{\alpha} = q + \frac{a_N}{V} = q + \frac{1}{V}(a_{NC} - e)$$

Thus, by (6F.2) and (6F.1), we obtain from (6F.3)

$$\dot{e} = -Z_\alpha\left[q + \frac{1}{V}(a_{NC} - e)\right] + \frac{Z_\delta}{\tau}(\delta - u) \tag{6F.4}$$

The angle of attack α, by (4F.3), is

$$\alpha = \frac{1}{Z_\alpha}(a_N - Z_\delta \delta) = \frac{1}{Z_\alpha}(a_{NC} - e - Z_\delta \delta) \tag{6F.5}$$

Thus the differential equation for the pitch rate, using (4F.1), is

$$\dot{q} = \frac{M_\alpha}{Z_\alpha}(a_{NC} - e - Z_\delta \delta) + M_q q + M_\delta \delta \tag{6F.6}$$

A single third-order vector-matrix equation defining the system is obtained from (6F.1), (6F.4), and (6F.6). Defining the state vector by

$$x = [e, q, \delta]' \tag{6F.7}$$

we obtain the state-space equations

$$\dot{x} = Ax + Bu + Ea_{NC} \tag{6F.8}$$

where

$$A = \begin{bmatrix} Z_\alpha/V & -Z_\alpha & Z_\delta/\tau \\ -M_\alpha/Z_\alpha & M_q & \bar{M}_\delta \\ 0 & 0 & -1/\tau \end{bmatrix} \quad B = \begin{bmatrix} -Z_\delta/\tau \\ 0 \\ 1/\tau \end{bmatrix} \quad E = \begin{bmatrix} -Z_\alpha/V \\ M_\alpha/Z_\alpha \\ 0 \end{bmatrix} \tag{6F.9}$$

where

$$\bar{M}_\delta = M_\delta - \frac{M_\alpha}{Z_\alpha} Z_\delta \tag{6F.10}$$

The following numerical data were obtained for a representative highly maneuverable tactical missile:

$V = 1253$ ft/s (Mach 1.1)

$Z_\alpha = -4170$ ft/s^2 (per radian of angle of attack)

$$Z_\delta = -1115 \text{ ft/s}^2 \text{ (per radian of surface deflection)}$$

$$M_\alpha = -248 \text{ rad/s}^2 \text{ (per radian of angle of attack)}$$

$$M_q \approx 0$$

$$M_\delta = -662 \text{ rad/s}^2 \text{ (per radian of surface deflection)}$$

$$\tau = .01 \text{ s}$$

The characteristic equation of this system (with $M_q = 0$) is

$$\left(s + \frac{1}{\tau}\right)\left(s^2 + \frac{Z_\alpha}{V}s - M_\alpha\right) = 0 \tag{6F.11}$$

and, using the numerical data given above, (6F.11) becomes

$$(s + 100)(s^2 + 3.33s + 248) = 0$$

with roots at

$$s = -100 \qquad \text{(due to actuator)}$$

and at

$$s = -1.67 \pm j15.65 \qquad \text{(due to airframe)}$$

as shown in Fig 6.8. The open loop thus has very little damping and a natural frequency ω of approximately 15.65 rad/s = 2.49 Hz.

Design considerations If the damping factor were raised to a more suitable value (say $\zeta \approx 0.707$) the natural frequency of 2.49 Hz would result in a time constant of about 0.4 s. A shorter closed-loop time constant would be desirable for a high-performance missile: about 0.2 s would be more appropriate. Thus we should seek a natural frequency of $\omega \approx 30$ and $\zeta \approx 0.707$. This suggests a quadratic factor in the closed-loop characteristic polynomial of

$$s^2 + 30\sqrt{2}s + (30)^2 \tag{6F.12}$$

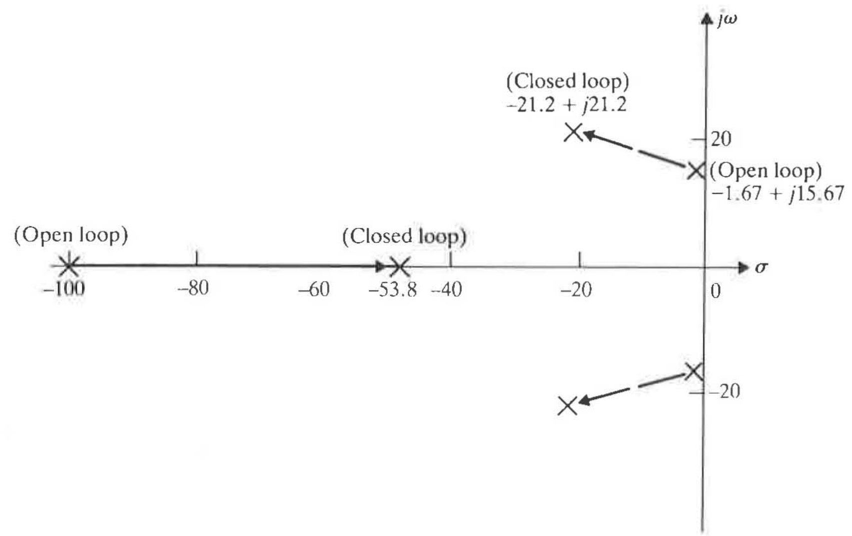

Figure 6.8 Open- and closed-loop poles for missile autopilot.

The location of the real pole at $s = -100$ due to the actuator is satisfactory: it is far enough away from the origin so as not to add substantially to the autopilot lag. We shall shortly discover, however, that to keep a closed-loop pole at $s = -100$ entails measuring (or estimating) and feeding back the actuator output δ. To simplify the implementation of the autopilot it might be desirable to permit the open-loop actuator pole to move to a different location provided that the overall system performance is not degraded. This is a design option we wish to explore.

The autopilot design will be done in two steps, as described earlier in the text. First we will design a regulator for a commanded normal acceleration of zero, then we will compute the feedforward gain to eliminate the steady state error for a nonzero commanded acceleration.

Regulator design To apply the Bass–Gura formula we need the open-loop characteristic equation: From (6F.11) this is

$$s^3 + 103.33s^2 + 581.s + 24\,800. = 0$$

Thus the open-loop coefficient vector is

$$a = \begin{bmatrix} 103.33 \\ 581. \\ 24\,800. \end{bmatrix}$$

thus

$$W = \begin{bmatrix} 1 & 103.33 & 581. \\ 0 & 1 & 103.33 \\ 0 & 0 & 1 \end{bmatrix}$$

We also find

$$Q = \begin{bmatrix} 111\,500. & -11.5 \times 10^6 & 8.77 \times 10^8 \\ 0. & -66.2 \times 10^3 & 6.64 \times 10^6 \\ 100. & -10^4 & 10^6 \end{bmatrix}$$

and

$$QW = \begin{bmatrix} 11\,500. & 0. & -0.248 \times 10^9 \\ 0. & -66\,204. & 0.198 \times 10^6 \\ 100. & 333.0 & 24\,800. \end{bmatrix}$$

from which:

$$(QW)^{-1} = \begin{bmatrix} 0.8657 \times 10^{-6} & 0.4544 \times 10^{-4} & 0.9035 \times 10^{-2} \\ 0.1090 \times 10^{-7} & -0.1517 \times 10^{-4} & -0.1215 \times 10^{-4} \\ -0.3637 \times 10^{-8} & 0.2040 \times 10^{-7} & 0.4055 \times 10^{-5} \end{bmatrix}$$

For any choice of closed-loop poles, the feedback gain matrix is given by:

$$g = G' = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = [(QW)']^{-1} \begin{bmatrix} \hat{a}_1 - a_1 \\ \hat{a}_2 - a_2 \\ \hat{a}_3 - a_3 \end{bmatrix} \qquad (6F.13)$$

As discussed earlier, practical implementation is simplified by omitting the feedback from the control surface deflection. This is achieved by having $g_3 = 0$. From (6F.13), this requirement is satisfied by making

$$0.9035 \times 10^{-2}(\hat{a}_1 - a_1) - 0.1215 \times 10^{-4}(\hat{a}_2 - a_2) + 0.4055 \times 10^{-5}(\hat{a}_3 - a_3) = 0 \qquad (6F.14)$$

We already decided that one factor of the characteristic polynomial be given by (6F.12). Thus

LQR

A2 Try

the complete characteristic polynomial is chosen to be

$$(s + \omega_c)(s^2 + 30\sqrt{2}s + 900) = s^3 + \hat{a}_1 s^2 + \hat{a}_2 s + \hat{a}_3$$

where

$$\begin{aligned}\hat{a}_1 &= \omega_c + 30\sqrt{2}\\ \hat{a}_2 &= 30\sqrt{2}\omega_c + 900\\ \hat{a}_3 &= 900\omega_c\end{aligned} \tag{6F.15}$$

with ω_c as yet undetermined. Equations (6F.14) and (6F.15) constitute four linear equations in the four unknowns $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, and ω_c. These are solved to yield

$$\begin{aligned}\hat{a}_1 &= 96.24\\ \hat{a}_2 &= 3182.\\ \hat{a}_3 &= 48\,419.\\ \omega_c &= 53.8\end{aligned}$$

The location of the real pole at $s = -\omega_c = -53.8$ is satisfactory, so no feedback gain from the surface deflection is necessary. Thus the gain matrix contains only two nonzero elements:

$$G = [-0.6366 \times 10^{-4}, -0.3929 \times 10^{-1}, 0] \tag{6F.16}$$

Feedforward gain Having decided that no feedback of the control surface deflection is necessary, and having adjusted the gains from the acceleration error $a_{NC} - a_N$ and the pitch rate q to provide the desired closed-loop poles, it now remains to set the feedforward gain G_0 to eliminate the steady-state error for a step input of acceleration.

The C matrix for the scalar error is

$$C = [1 \quad 0 \quad 0]$$

and the closed-loop A matrix is

$$A_c = A - BG = \begin{bmatrix} 3.767 & -8550.3 & -111\,500. \\ -0.0595 & 0. & -595.7 \\ 0.006\,366 & 3.929 & -100. \end{bmatrix}$$

and

$$A_c^{-1} = -\begin{bmatrix} 0.048\,33 & 8.613 & -105.20 \\ -0.000\,201 & 0.006\,88 & 0.1834 \\ -0.000\,005 & 0.000\,82 & 0.0105 \end{bmatrix}$$

Thus

$$CA_c^{-1} = [-0.048\,33 \quad -8.613 \quad 105.20]$$

and

$$CA_c^{-1}B = 5130.$$

Hence

$$B^{\#} = (CA_c^{-1}B)^{-1}CA_c^{-1} = [-9.42 \times 10^{-6} \quad -1.68 \times 10^{-3} \quad 2.05 \times 10^{-2}]$$

and, finally,

$$G_0 = B^{\#}E = -1.313 \times 10^{-4} \tag{6F.17}$$

The autopilot can be implemented as shown in Fig. 6.9. A body-mounted accelerometer measures the actual normal acceleration and a rate gyro measures the actual body pitch rate.

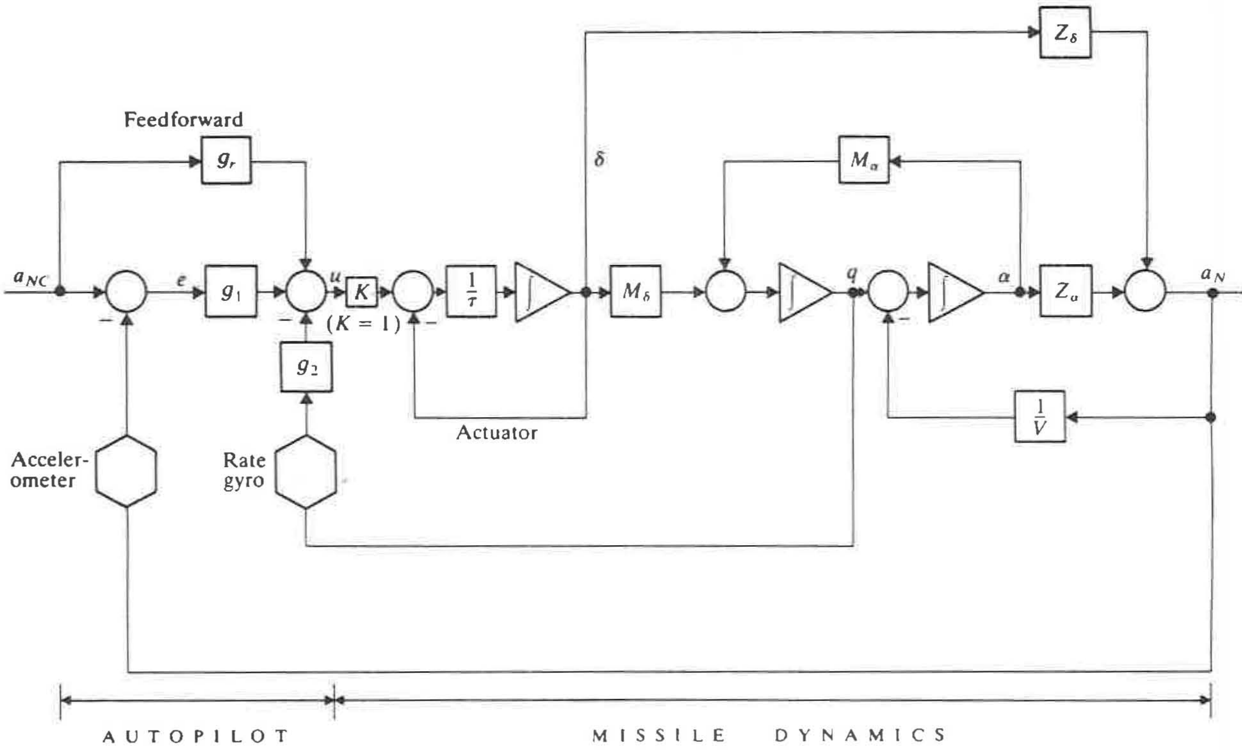

Figure 6.9 Dynamics of missile with autopilot.

Robustness of design The "robustness" of the design, i.e., its ability to withstand parameter variations, is of interest. It is not likely that the gain of the accelerometer or the gyro will vary by more than a fraction of a percent. The actuator and airframe dynamics are much more liable to change. In a careful performance evaluation, one would study the effect of parameter variations one at a time and in combination. Possibly the most likely change would be the dc transmission through the actuator to the output acceleration. This could be the result of an actuator gain change or the result of variations of airframe parameters from the values used in the design. Regardless of the true cause of the change, it can be represented by a gain K (with a nominal value of unity) multiplying the control signal u as shown in Fig. 6.9.

The return difference for the loop containing the gain K is

$$I + KG(sI - A)^{-1}B$$

The forward loop transmission

$$G_0(s) = G(sI - A)^{-1}B = \frac{G(s^3I + E_1s^2 + E_2s + E_3)B}{|sI - A|} = \frac{N(s)}{D(s)}$$

Using the above numerical data we find that

$$\begin{aligned} N(s) &= -7.09s^2 + 2601.s + 23\,608 \\ &= 7.09(-s + 376.)(s + 8.86) \end{aligned}$$

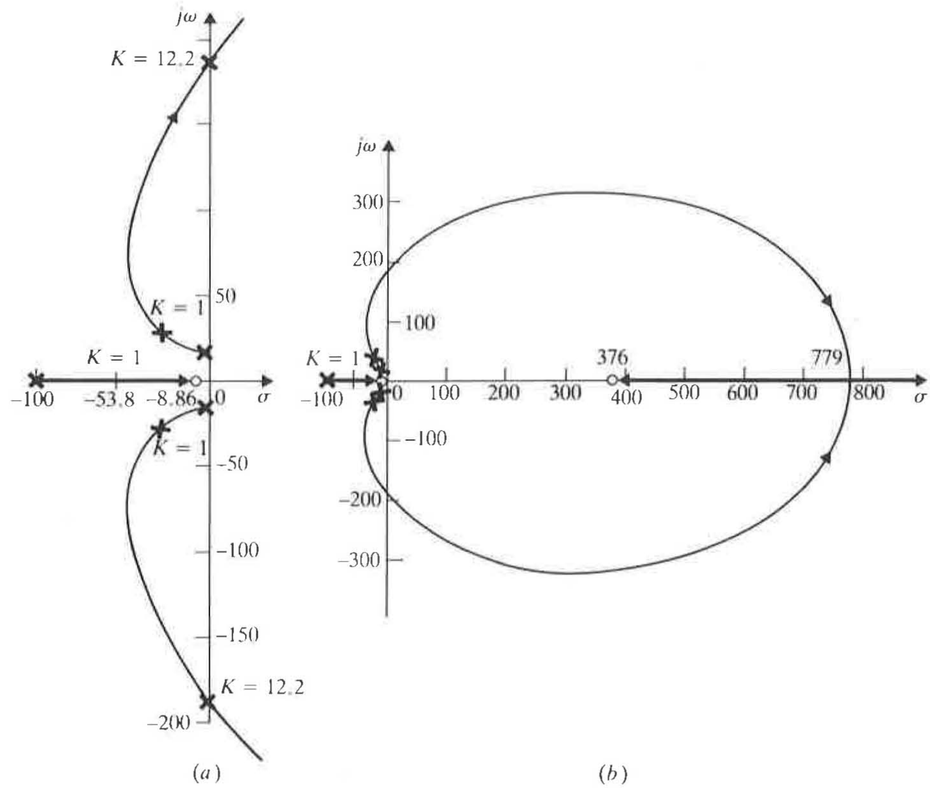

Figure 6.10 Root-locus plot for missile autopilot

$$G(s) = \frac{7.09(-s + 376)(s + 8.86)}{(s + 53.8)(s^2 + 30\sqrt{2}s + 900)}$$

which results in apparent zeros at $s = 376.$ and $s = -8.86$. (These are not zeros of the open-loop process, but are created by the use of the sensors of acceleration and pitch rate.)

The root locus has the appearance shown in Fig. 6.10. The root locus starts at the open-loop poles at $s = -100$ and $s = -1.67 \pm j15.65$ and goes to the apparent zeros. At a nominal gain of $K = 1$, the loci pass through the poles for which the operation was designed ($s = -15\sqrt{2} \pm j15\sqrt{2}$, and $s = -53.8$) and then continue toward the imaginary axis and ultimately into the right half-plane. Because of the nonminimum phase zero at $s = 376$, the locus has a branch that goes out along the *positive* real axis as $K \to \infty$, as was discussed in Chap. 5.

The range of gain K for which the system is stable can be found using the Routh or Hurwitz algorithm of Chap. 5 and is

$$-1.14 < K < 12.21$$

The gain margin is thus 12.2 (or 22 dB) which is more than ample. The frequency at which the root locus crosses the imaginary axis is found to be $\omega = 187$. The right half-plane root-locus plot is shown in Fig. 6.10(b). It should be noted that the loci, after crossing the imaginary axis,

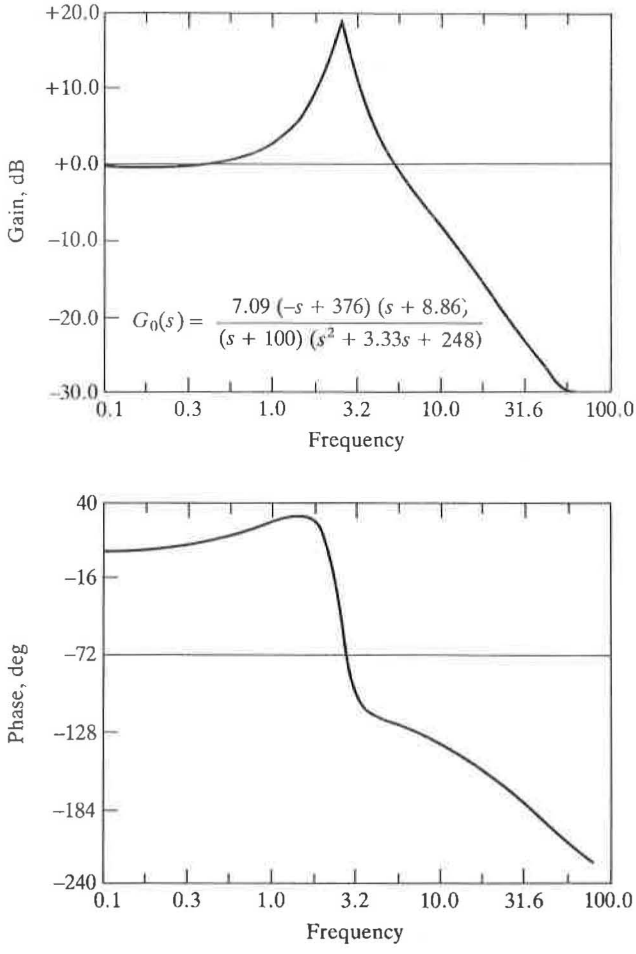

Figure 6.11 Bode plot for open-loop transmission for missile autopilot.

bend over toward the positive real axis and reach it at some positive real value of $s > 376$, the positive zero of $N(s)$. Then one branch goes to the zero and the other goes to $+\infty$.

The Bode plot for the open-loop transmission $G_0(s)$ is shown in Fig. 6.11.

PROBLEMS

Problem 6.1 Inverted pendulum on cart: state variable feedback

Consider the inverted pendulum on the motor-driven cart of Prob. 2.1 with numerical data as given in Prob. 3.6. It is desired to place the dominant poles (in a Butterworth configuration) at

$$s = -4 \qquad \text{and} \qquad s = -2 \pm j2\sqrt{3}$$

and to leave the pole at $s = -25$ unchanged.

(*a*) Find the gain matrix that produces this set of closed-loop poles.

(*b*) It is desired to move the cart from one position to another without causing the pendulum to fall. How must the control law of part *a* be modified to account for a reference input x_r?

Problem 6.2 Hydraulically actuated gun turret

It is desired to increase the bandwidth of the hydraulically actuated gun turret of Example 4E by use of state-variable feedback.

The dominant poles, i.e., those closest to the origin, are to be moved to $s = -10\sqrt{2}(1 \pm j1)$. The other poles (at $s = -64.5 \pm j69.6$) are already in suitable locations, but they can be moved in the interest of simplifying the feedback law by eliminating feedback paths.

(*a*) Determine the regulator gains for which the closed-loop poles are at $s = -10\sqrt{2}(1 \pm j1)$ and at $s = -64.5 \pm j69.6$.

(*b*) For simplicity, only two nonzero regulator gains are permitted: the gain from $x_1 = \theta$ and one other gain, either from $x_2 = \omega$ or from $x_3 = p$. Is it possible with a gain matrix of the form

$$g_A = [g_1, g_2, 0, 0]$$

or

$$g_B = [g_1, 0, g_3, 0]$$

to place the dominant poles at $s = -10\sqrt{2}(1 \pm j1)$ and still keep the "fast" poles at their approximate locations? If both g_A and g_B can achieve this requirement, which is the better choice? Explain.

(*c*) Let the tracking error e be defined by

$$e = \theta - \theta_0 = x_1 - \theta_0$$

where θ_0 is a constant reference angle. Show that a feedforward gain is not needed to achieve zero steady state error in tracking a constant reference. (Note that the open-loop system is "type 1.")

(*d*) There are three possible disturbances d_q, d_p, and d_τ as shown in Fig. 4.4. Since, by part *a*, it is unnecessary to estimate the reference input θ_0, we can define the exogenous vector as

$$x_0 = [d_\tau, d_p, d_q]'$$

and the distribution matrix as

$$E = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For each of the sets of gains in parts *a* and *b*, find the feedforward gains for the exogenous variables which will ensure zero steady state error.

Problem 6.3 Two-car train

It is desired to bring the two-car train of Probs. 2.5 and 3.9 to rest at the origin using only the motor on car 1. Find the gain matrix $G = g'$ in the control law $u = -g'x$ which places the poles at $s = -1 \pm j1$ and at $s = -100 \pm j100$.

Problem 6.4 Two-car train (continued)

Modify or redesign the control law obtained in Prob. 6.3, so that the train maintains a constant velocity $V = \text{const.}$

Problem 6.5 Aircraft longitudinal motion with simplified dynamics

The speed variations in aircraft longitudinal motion are often "trimmed" by a separate throttle control so that Δu can be assumed negligible. Thus we can use a simplified dynamic model in which the state variables are

$$x_1 = \alpha \qquad x_2 = q \qquad x_3 = \theta$$

Using these state variables and the aerodynamic coefficients of Prob. 4.5, find the gains that place the closed-loop poles in the Butterworth pattern: $s = -2$, $s = -1 \pm j\sqrt{3}$.

Problem 6.6 Constant-altitude autopilot

The altitude h of an aircraft is given by

$$\dot{h} = V\gamma = V(\theta - \alpha)$$

where γ is the flight-path angle. (See Fig. P6.6.) Hence the aircraft altitude can be maintained constant by keeping the flight-path angle $\gamma = \theta - \alpha$ zero. Add a state variable

$$x_4 = (h - h_0)/V$$

where h_0 is a reference altitude, to the state variables used in Prob. 6.5.

(*a*) Draw the block diagram of the closed-loop system.

(*b*) Find the gains for which the closed-loop poles lie in the Butterworth pattern:

$$s = 2.5(-1/2 \pm j\sqrt{3}/2) \qquad s = 2.5(-\sqrt{3}/2 \pm j1/2)$$

Problem 6.7 Aircraft lateral dynamics: turn coordination

When an aircraft executes a perfectly coordinated turn the sideslip angle β is zero. (When this occurs, the net force vector acting on the aircraft lies in the vertical plane of the aircraft so the occupant has the same kinesthetic sensation as when the aircraft is flying without banking.)

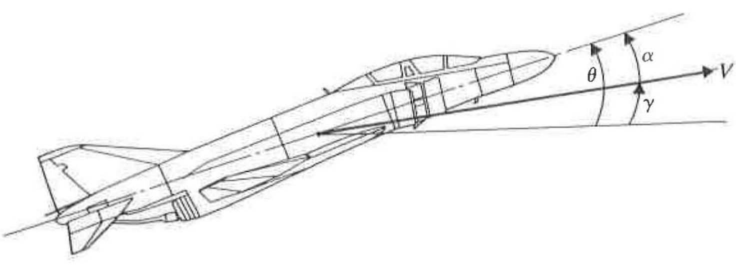

Figure P6.6 Aircraft longitudinal dynamics.

(*a*) The rudder is often used for turn coordination. We may thus assume a control law for the rudder, using (2.41)

$$\frac{Y_R}{V}\delta_R = -\frac{Y_A}{V}\delta_A - \frac{g}{V}\phi + \left(1 - \frac{Y_r}{V}\right)r - \frac{Y_p}{V}p - \left(\frac{Y_\beta}{V} + \frac{1}{T}\right)\beta \tag{P6.7a}$$

when (P6.7*a*) is substituted into the first equation of (2.41) we obtain

$$\dot{\beta} = -\frac{1}{T}\beta \tag{P6.7b}$$

Hence any sideslip that may be initially present will be reduced to zero with a time constant of T. When (P6.7*a*) is substituted into the next three equations of (2.41) a third-order system with a single control δ_A and a disturbance β is obtained. The poles of that system may be placed by use of the Bass-Gura formula. Using the data of Prob. 4.4, find the control law for the ailerons that makes the sideslip decay time constant $T = 0.2$ s and places the remaining poles at $s = -1$ and $s = -1 \pm j3$. Combine the result with (P6.7*a*) to obtain the entire control law.

(*b*) As the aircraft makes a constant-radius turn the bank angle ϕ becomes constant. Thus if a constant radius turn is desired, a constant bank angle ϕ_0 is commanded. Modify the control law of part *a* so that the aircraft error $e = \phi - \phi_0$ is reduced to zero in the steady state. (Let e be a state variable in place of ϕ.)

Problem 6.8 Three-capacitance thermal system

A state-variable feedback control law is to be designed for the thermal control system considered in Prob. 3.7, et seq.

(*a*) Find the control gains that place the regulator poles in a third-order Butterworth configuration of radius 2, i.e., the characteristic equation of the closed-loop system is to be

$$D(s) = \left(\frac{s}{2}\right)^3 + 2\left(\frac{s}{2}\right)^2 + 2\left(\frac{s}{2}\right) + 1 = 0$$

(*b*) It is desired to keep point 3 (i.e., v_3) at a constant temperature $\bar{v}$ in the presence of an external temperature v_0. Let the state be defined as $x = [v_1, v_2, e]'$ where $e = v_3 - \bar{v}$, and the exogenous vector as $x_0 = [\bar{v}, v_0]'$. Find the matrix E for the system, and, using the gain matrix from part *a*, find the feedforward gain matrix $G_0 = B^{\#}E$.

(*c*) Draw a block diagram of the control law showing the feedback and feedforward paths. Does anything seem unusual about this structure?

Problem 6.9 Two-axis gyro: gains by pole placement

A control law such as shown in Fig. 2.15 is to be designed for a two-axis gyro described in Example 2F (et seq.). The design will be accomplished in a number of steps which will encompass several problems.

The present problem is to design a deterministic control law under the assumption that all the state variables are measurable. The dynamic model to be used for the design is summarized in Example 3. The following data, typical of a small gyro, may be used for numerical calculations:

$$\frac{H}{J_d} = 3000\ \text{sec}^{-1} \qquad \frac{K_Q}{J_d} = 60\ \text{sec}^{-2}$$

$$\frac{B}{J_d} \approx 0 \qquad \frac{K_D}{J_d} = 30\ \text{sec}^{-2}$$

For this stage of the design it is assumed that the state variables δ_x, δ_y, ω_{xB}, ω_{yB}, *and* the external angular velocity components ω_{xE}, ω_{yE} are all measurable. (The external angular velocity components are *not* measurable, of course. If they were, there would be no need for the gyro!) In subsequent problems we shall consider the design of observers to measure those state variables, namely ω_{xB}, ω_{yB}, ω_{xE}, ω_{yE} that cannot be measured, using only observations given the measurements of δ_x and δ_y.

A linear control law of the form

$$u = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \tau_x/J_d \\ \tau_y/J_d \end{bmatrix} = -G_\delta \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} - G_\omega \begin{bmatrix} \omega_{xB} \\ \omega_{yB} \end{bmatrix} - G_0 \begin{bmatrix} \omega_{xE} \\ \omega_{yE} \end{bmatrix}$$

The matrices G_δ and G_ω are the "regulator" gains, to keep the gyro wheel stable in the absence of external angular velocity components, and G_0 is the gain matrix for the exogenous inputs, in this case the external angular velocity components ω_{xE} and ω_{yE}.

(*a*) Considerations of symmetry suggest that the regulator gain matrices should be of the form

LQR

$$G_\delta = \begin{bmatrix} g_1 & g_2 \\ -g_2 & g_1 \end{bmatrix} \qquad G_\omega = \begin{bmatrix} g_3 & g_4 \\ -g_4 & g_3 \end{bmatrix}$$

This means that there are four parameters for a fourth-order system, and a unique design can be achieved by pole placement. Determine the regulator closed-loop characteristic equation in terms of g_1, g_2, g_3, g_4. Does this place any restriction on the closed-loop pole locations?

(*b*) Using the theory developed in Sec. 6.4, find the matrix G_0 that maintains δ_x, δ_y at zero in the steady state, given that the exogenous input angular velocity components ω_{xE} and ω_{yE} are constants.

(*c*) In the steady state with $\omega_{xE} \neq 0$ and $\omega_{yE} \neq 0$ the control vector $u = [\tau_x/J_d, \tau_y/J_d]'$ is not zero. How does it depend on the input angular velocity components? Does this suggest a method for determining the input angular velocity?

Problem 6.10 Properties of Butterworth filters

Let

$$\mathsf{H}(s) = \frac{1}{B_k(s/\omega_0)}$$

where $B_k(z)$ is a Butterworth polynomial of order k.

(*a*) Show that

$$|\mathsf{H}(j\omega)| = \frac{1}{[1 + (\omega/\omega_0)^{2k}]^{1/2}}$$

(*b*) Sketch the amplitude plot corresponding to $\mathsf{H}(j\omega)$.

(*c*) Explain why the Butterworth polynomial is said to have a "maximally flat" amplitude response as compared with other systems of the same order.

NOTES

Note 6.1 Bass-Gura formula

The Bass-Gura formula[1] was originally derived by a method that closely resembles that used in this book. A simpler but less intuitive derivation may be found in Chap. 3 of Kailath's book[2] which contains several other formulas for the feedback gains.

Note 6.2 Zeros of closed-loop system

The matrix of transfer functions for the m-input, m-output system

$$\dot{x} = Ax + Bu \qquad y = Cx \qquad (l = m)$$

is

$$\mathsf{H}(s) = C(sI - A)^{-1}B$$

In accordance with the definition of transmission zeros given in Sec. 4.10, the transmission zeros of $\mathrm{H}(s)$ are the zeros of $|C(sI - A)^{-1}B|$.

As revealed in the analysis of Prob 4.1, the transmission zeros of a system in which state-variable feedback is used are not altered by the use of such feedback, i.e., the transmission zeros of $\mathrm{H}_c(s) = C(sI - A + BG)^{-1}B$ are the zeros of $\mathrm{H}(s)$.

For $B^{\#}$ as given by (6.46) to exist, it is necessary that $|C(A - BG)B| \neq 0$, which is the same as requiring that $\mathrm{H}_c(s)$ have no transmission zeros at the origin ($s = 0$). Since the transmission zeros of $\mathrm{H}_c(s)$ coincide with those of $\mathrm{H}(s)$, however, we conclude that the necessary and sufficient condition for $B^{\#}$ to exist is that $\mathrm{H}(s)$ have no transmission zeros at the origin.

Note 6.3 Butterworth polynomials

Butterworth polynomials have found extensive application in communication networks for their "maximally flat" frequency response characteristics. (See Problem 6.10.) They have also occurred in control system design by classical methods. (See [3], for example.) That the optimum closed-loop pole locations tend to a Butterworth configuration as the control cost decreases was first pointed out by Kalman[4] and subsequently studied in considerable detail by Kwakernaak.[5] (See Note 9.4 for further discussion of asymptotic behavior.)

REFERENCES

1. Bass, R. W., and Gura, I., "High-Order System Design Via State-Space Considerations," *Proc. Joint Automatic Control Conf.*, Troy, NY, June 1965, pp. 311-318.
2. Kailath, T., *Linear Systems*, Prentice-Hall Inc., Englewood Cliffs, NJ, June 1980.
3. Chang, S. S. L., *Synthesis of Optimum Control Systems*, McGraw-Hill Book Co., New York, June 1961.
4. Kalman, R. E., "When Is a Linear Control System Optimal?," *Trans. ASME (J. Basic Engineering)*, vol. 86D, no. 1, March 1964, pp. 51-60.
5. Kwakernaak, H., and Sivan, R., *Linear Optimal Control Systems*, Wiley-Interscience, New York, 1972.

CHAPTER
SEVEN
LINEAR OBSERVERS

7.1 THE NEED FOR OBSERVERS

In Chap. 6 we studied methods for shaping the dynamic response of a closed-loop system by selecting the feedback gains to "place" the resulting poles at desired locations. In order to place the poles at arbitrary locations, it is generally necessary to have all the state variables available for feedback. There are many systems, of course, such as those illustrated in Examples 4D and 4E, in which acceptable performance can be achieved by feeding back only those state variables that are accessible to measurement. But often it is not possible to achieve acceptable performance using only those state variables that can be measured. Must we abandon the hope of controlling such systems? Fortunately not. If the system is observable, it is possible to estimate those state variables that are not directly accessible to measurement using the measurement data from those state variables that are accessible. And by use of these state-variable estimates rather than their measured values one can usually achieve acceptable performance. State-variable estimates may in some circumstances be even preferable to direct measurements, because the errors introduced by the instruments that provide these measurements may be larger than the errors in estimating these variables.

A dynamic system whose state variables are the estimates of the state variables of another system is called an *observer* of the latter system. This term was introduced into linear system theory by D. Luenberger in 1963[1, 2, 3] (see Note 7.1). Luenberger showed that, for any observable linear system, an observer can be designed having the property that the estimation error (i.e., the difference between the state of the actual system and the state of the observer)