CHAPTER
FIVE

CONTROLLABILITY AND OBSERVABILITY

5.1 INTRODUCTION

Some state-space concepts can be regarded as reinterpretations of older, frequency-domain concepts; others are distinctive to state-space methods. Controllability and observability are in this latter category.

The ideas of controllability and observability were introduced by R. E. Kalman in the mid 1950s as a way of explaining why a method of designing compensators for unstable systems by cancelling unstable poles (i.e., poles in the right half-plane) by zeros in the right half-plane is doomed to fail even if the cancellation is *perfect.* (It was already known that this method of compensation was not feasible because perfect cancellation is not possible in practice. See Note 5.1.) Kalman showed that a perfect pole-zero cancellation would result in an unstable system with a stable transfer function. The transfer function, however, is of lower order than the system, and the unstable modes are either not capable of being affected by the input (uncontrollable) or not visible in the output (unobservable).

In frequency-domain analysis it is tacitly assumed that the dynamic properties of a system are completely determined by the transfer function of the system. That this is not always the case is illustrated by the following example.

Example 5A Hypothetical system Figure 5.1 shows the block-diagram of a hypothetical system contrived specifically to illustrate the concepts of controllability and observability. There is no reason, however, why it would not represent some physical process.

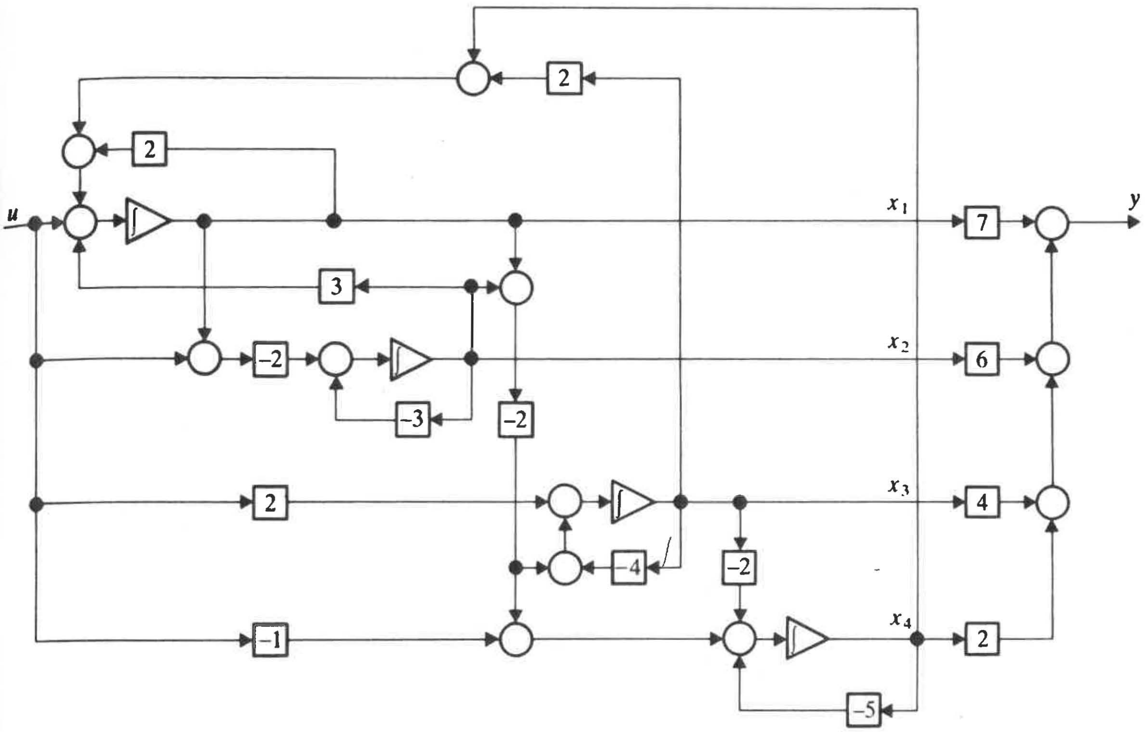

Figure 5.1 Hypothetical fourth-order system to illustrate controllability and observability.

The differential equations of the system, obtained by inspection of the block-diagram are

$$\begin{aligned}\dot{x}_1 &= 2x_1 + 3x_2 + 2x_3 + x_4 + u \\ \dot{x}_2 &= -2x_1 - 3x_2 - 2u \\ \dot{x}_3 &= -2x_1 - 2x_2 - 4x_3 + 2u \\ \dot{x}_4 &= -2x_1 - 2x_2 - 2x_3 - 5x_4 - u\end{aligned} \qquad (5A.1)$$

and the observation equation is

$$y = 7x_1 + 6x_2 + 4x_3 + 2x_4 \qquad (5A.2)$$

Thus, the matrices of the state-space representation are:

$$A = \begin{bmatrix} 2 & 3 & 2 & 1 \\ -2 & -3 & 0 & 0 \\ -2 & -2 & -4 & 0 \\ -2 & -2 & -2 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ -2 \\ 2 \\ -1 \end{bmatrix} \qquad C = [7 \quad 6 \quad 4 \quad 2]$$

The resolvent corresponding to A is given by

$(sI - A)^{-1} =$

$$\frac{1}{\Delta(s)}\begin{bmatrix} s^3 + 12s^2 + 47s + 6 & 3s^2 + 21s + 36 & 2s^2 + 14s + 24 & s^2 + 7s + 12 \\ -2s^2 - 18s - 40 & s^3 + 7s^2 + 8s - 16 & -4s - 16 & -2s - 8 \\ -2s^2 - 12s - 10 & -2s^2 - 12s - 10 & s^3 + 6s^2 + 7s + 2 & -2s - 2 \\ -2s^2 - 6s - 4 & -2s^2 - 6s - 4 & -2s^2 - 6s - 4 & s^3 + 5s^2 + 8s + 4 \end{bmatrix}$$

where

$$\Delta(s) = |sI - A| = s^4 + 21s^3 + 35s^2 + 50s + 24$$

Thus the transfer function from the input u to the output y is given by

$$H(s) = C(sI - A)^{-1}B = \frac{s^3 + 9s^2 + 26s + 24}{s^4 + 21s^3 + 35s^2 + 50s + 24} \tag{5A.3}$$

which is the ratio of a third-degree polynomial to a fourth-degree polynomial—quite as expected. On factoring the numerator and denominator, however, we discover that

$$H(s) = \frac{(s+2)(s+3)(s+4)}{(s+1)(s+2)(s+3)(s+4)} = \frac{1}{s+1} \tag{5A.4}$$

Thus, three of the poles (at $s = -2, -3$, and -4) are cancelled by zeros at exactly the same locations, and what seems to be a fourth-order transfer function is actually only first-order.

To help explain this rather remarkable behavior, the following change of state variables is performed:

$$\bar{x} = Tx$$

where

$$T = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

The matrix T happens to be a diagonalizing transformation

$$TAT^{-1} = \Lambda = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

and the corresponding control and observation matrices are

$$\bar{B} = TB = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \bar{C} = CT^{-1} = [1 \quad 1 \quad 0 \quad 0]$$

Hence the corresponding state equations are

$$\begin{aligned} \dot{\bar{x}}_1 &= -\bar{x}_1 + u \\ \dot{\bar{x}}_2 &= -2\bar{x}_2 \\ \dot{\bar{x}}_3 &= -3\bar{x}_3 + u \\ \dot{\bar{x}}_4 &= -4\bar{x}_4 \end{aligned} \tag{5A.5}$$

and the observation equation is

$$y = \bar{x}_1 + \bar{x}_2 \tag{5A.6}$$

A block-diagram representation of (5A.5) and (5A.6) is shown in Fig. 5.2. Clearly, the input u affects only the state variables $\bar{x}_1$ and $\bar{x}_3$; $\bar{x}_2$ and $\bar{x}_4$ are unaffected by the input. The output y depends only on $\bar{x}_1$ and $\bar{x}_2$; $\bar{x}_3$ and $\bar{x}_4$ do not contribute to the output. Thus, in the transformed coordinates, the system has four different subsystems. (In this case each subsystem is only first-order.)

$\bar{x}_1$: affected by the input; visible in the output

$\bar{x}_2$: unaffected by the input; visible in the output

$\bar{x}_3$: affected by the input; invisible in the output

$\bar{x}_4$: unaffected by the input; invisible in the output

Only the first subsystem $\bar{x}_1$ contributes to the transfer function $H(s)$, which clearly is $1/(s+1)$.

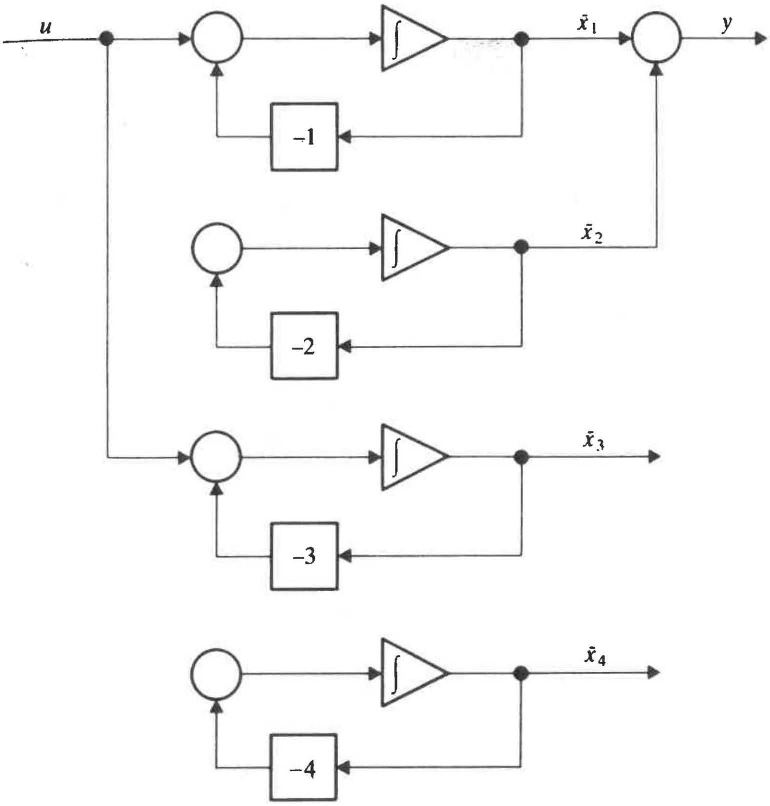

Figure 5.2 System equivalent of Fig. 5.1 showing separation into controllable and observable subsystems.

Example 5A is a microcosm of the general case. As Kalman has shown,[1] every system of the generic form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

can be transformed into the four subsystems of Fig. 5.2. The first subsystem is both controllable and observable: the second is uncontrollable but observable; the third is controllable but unobservable; and the fourth is neither observable nor controllable. The transfer function of the system is determined only by the controllable and observable subsystem. It thus follows that if the transfer function of a single-input, single-output system is of lower degree than the dimension of the state-space, then the system must contain an uncontrollable subsystem, or an unobservable subsystem, or possibly both. By convention, if a system contains an uncontrollable subsystem it is said to be *uncontrollable*; likewise, if it contains an unobservable subsystem it is said to be *unobservable.* (See Note 5.2.)

The system in the foregoing example is asymptotically stable: all its poles are in the left half-plane, so the consequences of the system being unobservable and uncontrollable are innocuous. Any initial conditions on the uncontrollable and unobservable states decay harmlessly to zero. But suppose that one of the uncontrollable or unobservable subsystems were *unstable*. The resulting behavior could well be disastrous: a random disturbance, no matter how small, which establishes a nonzero initial state will send the subsystem off to infinity. Murphy's law par excellence!

There is a distinction between an uncontrollable system in which the uncontrollable part is stable and one in which the uncontrollable part is unstable. A system of the former type is said to be ***stabilizable***, and the uncontrollable part often can be safely ignored by the control engineer.

Similarly, there is a distinction between an unobservable system in which the unobservable subsystem is stable and one in which it is unstable. The former type is said to be ***detectable***, and the unobservable part usually can be safely ignored in the control system design.

5.2 WHERE DO UNCONTROLLABLE OR UNOBSERVABLE SYSTEMS ARISE?

The example of an uncontrollable and unobservable system that was given in the previous section is highly contrived. One might suspect that such systems are academic curiosities and do not arise in the real world. But in fact uncontrollable and unobservable systems are not at all uncommon, as the illustrations of the present section will reveal.

Redundant state variables One common source of uncontrollable systems arises when redundant state variables are defined. Consider, for example, the dynamic system

$$\dot{x} = Ax + Bu$$

and suppose, for some reason, more state variables, proportional to those already present in the state vector x are defined:

$$z = Fx \tag{5.1}$$

where F is an $n \times k$ matrix. Then

$$\dot{z} = F\dot{x} = F(Ax + Bu)$$

is a true differential equation, so we can define a "metastate" vector

$$\mathbf{x} = \begin{bmatrix} x \\ z \end{bmatrix}$$

which satisfies the differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{5.2}$$

where

$$\mathbf{A} = \begin{bmatrix} A & 0 \\ FA & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} B \\ FB \end{bmatrix}$$

The system characterized by (5.2) has the block diagram shown in Fig. 5.3(*a*). There is a path from the input u to the state x and to the (redundant) state z; superficially the system seems to be controllable. But consider the change of variable

$$\bar{\mathbf{x}} = \mathbf{T}\mathbf{x} \tag{5.3}$$

where

$$\mathbf{T} = \begin{bmatrix} I_k & 0 \\ -F & I_n \end{bmatrix} \qquad \mathbf{T}^{-1} = \begin{bmatrix} I_k & 0 \\ F & I_n \end{bmatrix} \tag{5.4}$$

where I_l ($l = k, n$) is an l-by-l identity matrix. (Multiply $\mathbf{T}$ by $\mathbf{T}^{-1}$ to verify (5.4).)

The dynamics matrix of the transformed system is given by

$$\mathbf{A} = \mathbf{TAT}^{-1} = \begin{bmatrix} I_k & 0 \\ -F & I_n \end{bmatrix}\begin{bmatrix} A & 0 \\ FB & 0 \end{bmatrix}\begin{bmatrix} I_k & 0 \\ F & I_n \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

and the control matrix is given by

$$\bar{\mathbf{B}} = \mathbf{TB} = \begin{bmatrix} I_k & 0 \\ -F & I_n \end{bmatrix} = \begin{bmatrix} B \\ FB \end{bmatrix} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

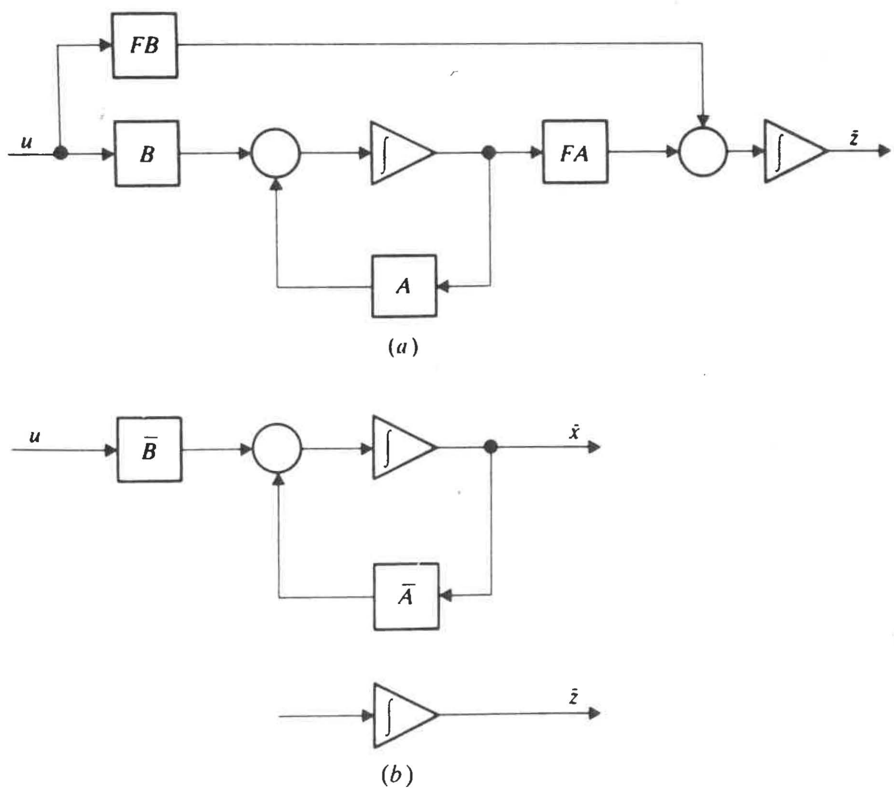

Figure 5.3 Redundant state produces an uncontrollable system. (*a*) System with redundant state $z = Fx$; (*b*) System of (*a*) after being transformed by $\bar{x} = Tx$.

Thus, in the transformed system,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$
$$\dot{\bar{z}} = 0 \tag{5.5}$$

Differential equation (5.5) represents k integrators with no inputs connected to them (Fig. 5.3(b)) and hence the substate z is uncontrollable.

All the algebra used above is really quite superfluous. The transformation T of (5.3) merely asserts that

$$\bar{z} = z - Fx$$

and, by virtue of (5.1), $\bar{z} \equiv 0$, so surely (5.5) must hold.

Now of course no one would intentionally use more state variables than the minimum number needed to characterize the behavior of the dynamic process. In a complicated process with unfamiliar physics, however, the control system engineer may be tempted to write down differential equations for everything in sight and in so doing, may write down more equations than are necessary. This will invariably result in the model for an uncontrollable system.

Physically uncontrollable system Another instance of an uncontrollable system is one in which the only forces and torques are *internal* to the system. For example, as a consequence of Newton's law of action and reaction, the location of the center of mass of a closed system cannot be changed by use of forces within the system. This is illustrated by the following example.

Example 5B Motion of coupled masses with internal force Consider the system comprising two carts coupled by a (passive) spring, as shown in Fig. 5.4. In addition to the spring force, an

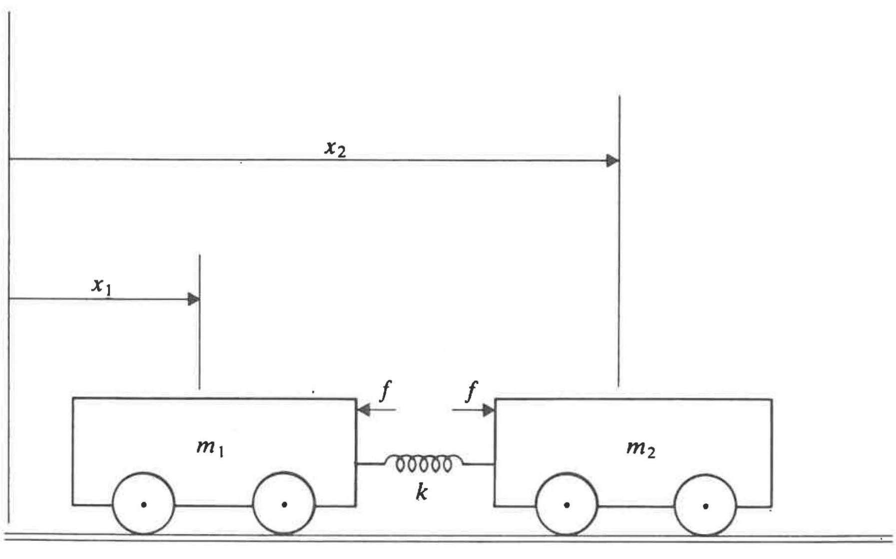

Figure 5.4 Center of mass of system cannot be moved by internal force.

active control force f is to be provided by some means within the system, so that whatever the force on cart 1, an equal and opposite reaction force from that source must push on cart 2. Thus the differential equations of the pair of carts are

$$\frac{dx_1}{dt} = \dot{x}_1 \tag{5B.1}$$

$$\frac{dx_2}{dt} = \dot{x}_2 \tag{5B.2}$$

$$\frac{d\dot{x}_1}{dt} = -\frac{k}{m_1}(x_1 - x_2) - \frac{f}{m_1} \tag{5B.3}$$

$$\frac{d\dot{x}_2}{dt} = -\frac{k}{m_2}(x_2 - x_1) + \frac{f}{m_2} \tag{5B.4}$$

From (5B.3) and (5B.4)

$$m_1\frac{dx_1}{dt} + m_2\frac{dx_2}{dt} = \frac{d}{dt}(m_1x_1 + m_2x_2) = 0$$

Thus

$$m_1x_1 + m_2x_2 = mx_c = \text{const}$$

where

$$m = m_1 + m_2 \qquad x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \text{center of mass}$$

Thus the center of mass of the system cannot be moved by the internal force f.

This physical fact is formally illustrated by matrix analysis. For the original system (5B.1)-(5B.4) the state vector is

$$x = [x_1, x_2, \dot{x}_1, \dot{x}_2]'$$

and the corresponding matrices are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ -1/m_1 \\ 1/m_2 \end{bmatrix} \tag{5B.5}$$

We make the change of state variables

$$x_c = \frac{m_1}{m}x_1 + \frac{m_2}{m}x_2$$

$$\delta = x_1 - x_2$$

$$\dot{x}_c = \frac{m_1}{m}\dot{x}_1 + \frac{m_2}{m}\dot{x}_2$$

$$\dot{\delta} = \dot{x}_1 - \dot{x}_2$$

Then

$$\bar{x} = \begin{bmatrix} x_c \\ \delta \\ \dot{x}_c \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} m_1/m & m_2/m & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & m_1/m & m_2/m \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$$

Thus

$$T = \begin{bmatrix} m_1/m & m_2/m & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & m_1/m & m_2/m \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 1 & m_2/m & 0 & 0 \\ 1 & -m_1/m & 0 & 0 \\ 0 & 0 & 1 & m_2/m \\ 0 & 0 & 1 & -m_1/m \end{bmatrix}$$

Thus we find that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -k(1/m_1 + 1/m_2) & 0 & 0 \end{bmatrix} \qquad \bar{B} = TB = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -(1/m_1 + 1/m_2) \end{bmatrix}$$

Hence, as expected, the differential equations of the transformed system are

$$\frac{dx_c}{dt} = \dot{x}_c$$

$$\frac{d\delta}{dt} = \dot{\delta}$$

$$\frac{d\dot{x}_c}{dt} = 0$$

$$\frac{d\dot{\delta}}{dt} = -k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\delta - \left(\frac{1}{m_1} + \frac{1}{m_2}\right)f$$

The internal force f can change the distance δ between x_1 and x_2 but not the coordinates x_1 and x_2 independently. To do that, an external force is needed.

This example illustrates that the mathematical model must be consistent with the physics of the system. The A and B matrices must be exactly as given by (5B.5). If an error in calculation were made, for example, such that the fourth element in the B matrix were not $1/m_2$ but some other number, the system would seem to be controllable and one might try to move the center of mass by using the force f. But of course no matter how large we make f, the center of mass won't move.

Too much symmetry Another situation that results in an uncontrollable system arises when the system in question has too much symmetry. This typically arises in electrical networks that contain balanced bridges, and in mechanical systems which have similar symmetry. This is illustrated by the following example.

Example 5C Balanced bridges are uncontrollable The differential equations of the electrical network, or its thermal analog, shown in Fig. 5.5, were found in Chap. 2 to be

$$\dot{v}_1 = -\frac{1}{C_1}\left(\frac{1}{R_1} + \frac{1}{R_3}\right)v_1 + \frac{1}{C_1R_3}v_2 + \frac{1}{C_1R_1}e_0 \tag{5C.1}$$

$$\dot{v}_2 = \frac{1}{C_2R_3}v_1 - \frac{1}{C_2}\left(\frac{1}{R_2} + \frac{1}{R_3}\right)v_2 + \frac{1}{C_2R_2}e_0 \tag{5C.2}$$

Consider the difference voltage

$$\bar{v}_1 = v_1 - v_2$$

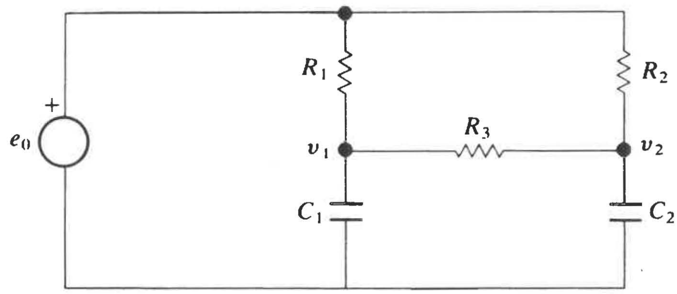

Figure 5.5 Electrical bridge network.

The time-derivative of $\bar{v}_1$ using (5C.1) and (5C.2) is

$$\frac{d\bar{v}_1}{dt} = -\left[\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_3}\right)+\frac{1}{C_2R_3}\right]v_1 + \left[\frac{1}{C_1R_3}+\frac{1}{C_2}\left(\frac{1}{R_2}+\frac{1}{R_3}\right)\right]v_2 + \frac{R_2C_2 - R_1C_1}{C_1C_2R_1R_2}e_0 \tag{5C.3}$$

If the bridge is *balanced*, i.e.,

$$R_1C_1 = R_2C_2 \tag{5C.4}$$

then the coefficient of the input voltage e_0 vanishes. And moreover, the bracketed coefficients of v_1 and v_2 become equal. Thus (5C.3) reduces to

$$\frac{d\bar{v}_1}{dt} = -\frac{R_1+R_2+R_3}{C_1R_1R_3}\bar{v}_1$$

This means that the voltage $\bar{v}_1 = v_1 - v_2$ between the terminals of R_3 cannot be influenced by the input e_0; the voltage $\bar{v}_1$ decays from whatever initial voltage it starts with to zero with the time constant $\tau = C_1R_1R_3/(R_1+R_2+R_3)$ irrespective of the input.

If the only observation is the voltage $\bar{v}_1 = v_1 - v_2$, then the system is also *unobservable.* To see this we define the transformed state

$$\bar{v}_1 = v_1 - v_2$$
$$\bar{v}_2 = v_2$$

To this definition of transformed voltages there corresponds the transformation matrix

$$T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The transformed differential equations are

$$\dot{\bar{v}}_1 = -\left[\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+\frac{1}{C_2R_3}\right]\bar{v}_1 + \left(\frac{1}{C_2R_2}-\frac{1}{C_1R_1}\right)\bar{v}_2 + \left(\frac{R_2}{C_1}-\frac{R_1}{C_2}\right)\frac{e_0}{R_1R_2}$$

$$\dot{\bar{v}}_2 = \frac{1}{C_2R_3}\bar{v}_1 - \frac{1}{C_2R_2}\bar{v}_2 + \frac{R_1}{C_1}\frac{e_0}{R_1R_2}$$

and the observation is given by

$$y = \bar{v}_1$$

When the bridge is balanced there is no path from $\bar{v}_2$ to the output, hence $\bar{v}_2$ cannot be observed. (See Fig. 5.6.)

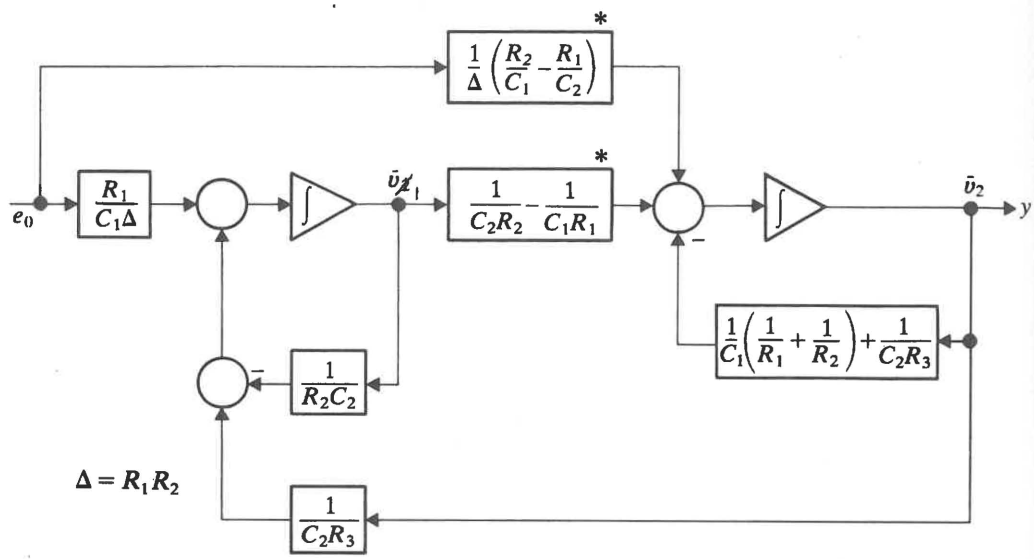

Figure 5.6 Block diagram showing that balanced bridge is neither controllable nor observable. (Elements with * open when bridge is balanced.)

When numerical values are inserted for the physical parameters in the systems of Examples 5B and 5C there is no way of distinguishing between the qualitative nature of the uncontrollability of the two systems: they are both simply uncontrollable. But physically there is a very important distinction between the two systems. The two-mass mechanical system is uncontrollable for *every* value of the parameters (masses, spring rates); the only way to control the position of the center of mass is to add an external force. This necessitates a *structural* change to the system. The balanced bridge, however, is uncontrollable only for one specific relationship between the parameters, namely the balance condition (5C.4). In other words, the system is *almost always controllable.* (As a practical matter, it will be difficult to control v_1 and v_2 independently when (5C.4) is nearly true. This raises the issue of degree of controllability, a topic discussed in Note 5.3.)

It is important for the control system engineer to recognize this distinction, particularly when dealing with an unfamiliar process for which the state-space representation is given only by numerical data. A numerical error in calculating the elements of the A and B matrices, or an experimental error in measuring them, may make an uncontrollable system seem controllable. A control system designed with this data may seem to behave satisfactorily in simulation studies based on the erroneous design data, but will fail in practice. On the other hand, a process that appears to be uncontrollable (or nearly uncontrollable), but which is not structurally uncontrollable, may be rendered more tractable by changing some parameter of the process—by "unbalancing the bridge."

Example 5D How not to control an unstable system (inverted pendulum) There are many ways of designing perfectly fine control systems for unstable processes such as the inverted pendulum of Examples 2E and 3D. These will be discussed at various places later on in this

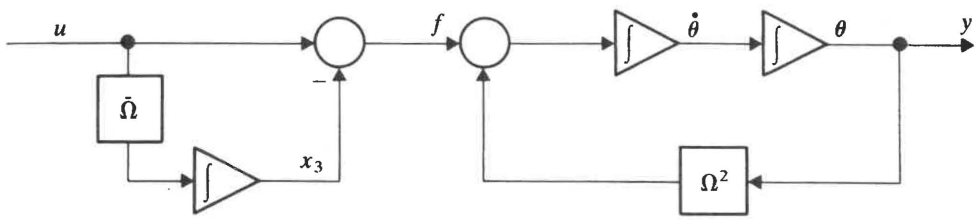

Figure 5.7 Unstabilizable compensation of inverted pendulum.

book. But one way guaranteed to be disastrous is to try to cancel the unstable pole with a zero in the compensator. The reason for the disaster is the subject of this example.

Consider the inverted pendulum of Example 3D with the output being the measured position. The transfer function from the input (force) to the output (position) is

$$H(s) = \frac{y(s)}{f(s)} = \frac{1}{s^2 - \Omega^2} = \frac{1}{(s+\Omega)(s-\Omega)} \tag{5D.1}$$

This is obviously unstable. A much better transfer function would be

$$H(s) = \frac{1}{s(s+\Omega)} \tag{5D.2}$$

which is stable and, because of the pole at the origin, would be a "type-one" system, with zero steady state error. Thus, one might be tempted to "compensate" the unstable transfer function by means of a compensator having the transfer function (Fig. 5.7)

$$G(s) = \frac{s - \bar{\Omega}}{s} = 1 - \frac{\bar{\Omega}}{s} \tag{5D.3}$$

with

$$\bar{\Omega} = \Omega$$

Of course it will not be possible to make $\bar{\Omega}$ precisely equal to Ω so the compensation will not be perfect. But that is not the trouble, as we shall see.

The compensator transfer function (5D.3) represents "proportional plus integral" compensation which is quite customary in practical process control systems. The transfer function of the compensated system is now

$$H_c(s) = G(s)H(s) = \frac{s - \bar{\Omega}}{s(s^2 - \Omega^2)} \to H(s) \quad \text{as} \quad \bar{\Omega} \to \Omega \tag{5D.4}$$

A block diagram representation of this system is shown in Fig. 5.7, and the state-space equations corresponding to this representation are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \Omega^2 x_1 - x_3 + u \\ \dot{x}_3 &= \bar{\Omega} u \end{aligned} \tag{5D.5}$$

where x_3 is the state of the integrator in the compensator. The matrices of the process (5D.5) are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \Omega^2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \\ \bar{\Omega} \end{bmatrix}$$

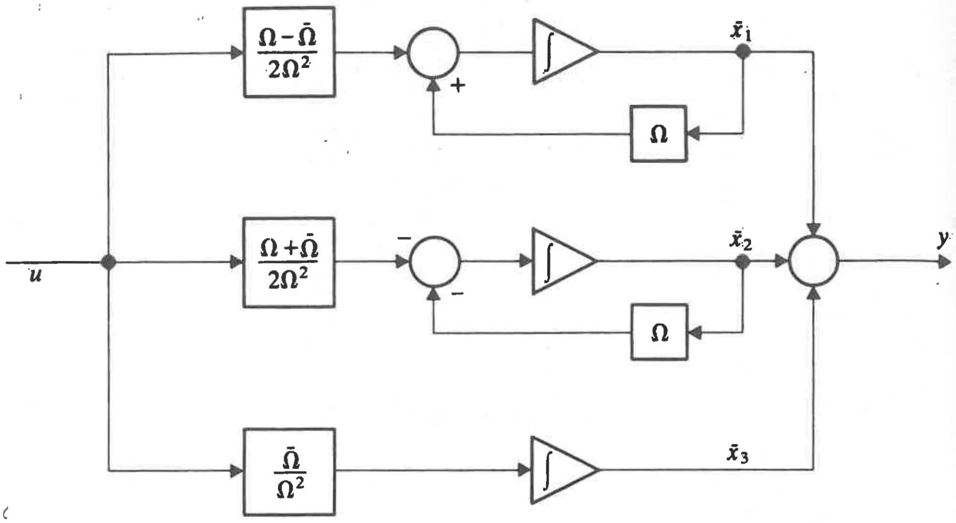

Figure 5.8 Partial fraction representation of Fig. 5.7.

The A matrix can be transformed to diagonal form by the transformation matrix

$$T = \frac{1}{2\Omega^2}\begin{bmatrix} \Omega^2 & \Omega & -1 \\ \Omega^2 & -\Omega & -1 \\ 0 & 0 & 2 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ \Omega & -\Omega & 0 \\ 0 & 0 & \Omega^2 \end{bmatrix}$$

We find that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \Omega & 0 & 0 \\ 0 & -\Omega & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{B} = TB = \frac{1}{2\Omega^2}\begin{bmatrix} \Omega - \bar{\Omega} \\ -(\Omega + \bar{\Omega}) \\ 2\bar{\Omega} \end{bmatrix}$$

The state-space representation of the transformed system is as shown in Fig. 5.8. This block-diagram corresponds directly to the partial-fraction expansion of (5D.4):

$$H_c(s) = \frac{\bar{\Omega}/\Omega^2}{s} + \frac{(\Omega - \bar{\Omega})/2\Omega^2}{s - \Omega} + \frac{-(\Omega + \bar{\Omega})/2\Omega^2}{s + \Omega} \tag{5D.6}$$

Note carefully what happens when $\bar{\Omega} \to \Omega$. In the block-diagram the connection between the control input u and the *unstable* state x_1 is broken, rendering the system uncontrollable and *unstabilizable*. In (5D.6) the residue at the unstable pole vanishes. But now we understand that the vanishing of a residue at a pole of a transfer function does not imply that the subsystem giving rise to the pole disappears, but rather that it becomes "invisible."

If the original inverted pendulum could have arbitrary initial conditions, the transformed system (5D.5) could also have arbitrary initial conditions and hence the inverted pendulum would most assuredly not remain upright, regardless of how the loop were closed between the measurement y and the control input u.

More reasons for unobservability The foregoing examples were instances of uncontrollable systems. Instances of unobservable systems are even more abundant. An unobservable system results any time a state variable is not measured

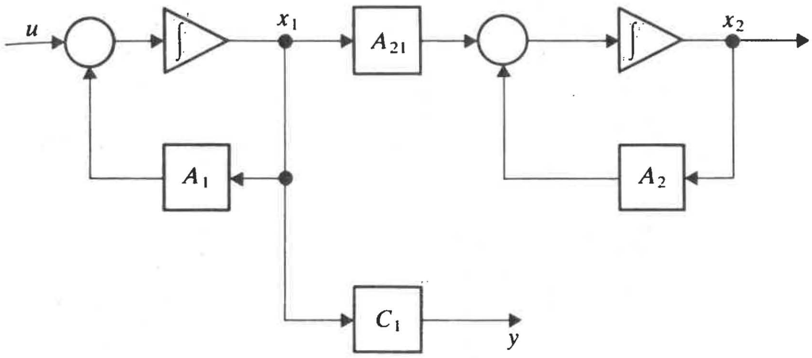

Figure 5.9 Systems in tandem that are unobservable.

directly and is not fed back to those state variables that are measured. Thus, any system comprising two subsystems in tandem (as shown in Fig. 5.9, in which none of the states of the right-hand subsystem can be measured) is unobservable. The transfer function from the inputs to the outputs obviously depends only on the left-hand subsystem.

Physical processes which have the structure shown in Fig. 5.9 are not uncommon. A mass m acted upon by a control force f is unobservable if only its velocity, and not its position, can be measured. This means that no method of velocity feedback can serve as a means of controlling position. In this regard it is noted that the integral of the measured velocity is not the same as the actual position. A control system shown in Fig. 5.10 will not be effective in controlling the position x of the mass, no matter how well it controls the velocity $\dot{x}$; any initial position error will remain in the system indefinitely.

In addition to the obvious reasons for unobservability there are also some of the more subtle reasons such as symmetry, as was illustrated by Example 5C.

5.3 DEFINITIONS AND CONDITIONS FOR CONTROLLABILITY AND OBSERVABILITY

In Secs. 5.1 and 5.2 we found that uncontrollable and/or unobservable systems were characterized by the property that the transfer function from the input to

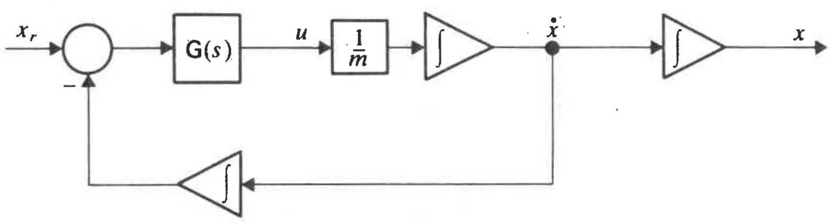

Figure 5.10 Position of mass cannot be observed and cannot be controlled using only velocity feedback.

the output is of lower degree than the order of the dynamic system. We were able to trace this to the fact that some combinations of state variables are not capable of being affected by the input or not being visible in the output.

It is useful to give these concepts more precision with the aid of more precise definitions.

We start with the following basic:

Definition of controllability A system is said to be controllable if and only if it is possible, by means of the input, to transfer the system from *any* initial state $x(t) = x_t$ to *any* other state $x_T = x(T)$ in a *finite* time $T - t \geqq 0$.

The emphasized words "any" and "finite" are essential to the definition. If it is only possible to make the system go from *some* states to *some* other states, then the system is not controllable. Moreover, if it takes an infinite amount of time to go from the arbitrary initial state to the arbitrary final state the system is likewise not controllable.

(In some texts, a system is called *completely controllable* when it is possible to transfer it from *any* state to *any* other state in finite time.[2] A system is not *completely* controllable when it is possible only to transfer it from some states.)

In the definition of controllability the initial time t is not specified and the final time is not fixed. This is done to accommodate time-varying systems, in which it may happen that the possibility of reaching x_T from x_t depends on the initial time t. (See Note 5.2.) In a time-invariant system, however, no generality is lost in taking the initial time t to be zero.

The terminal time T must be finite in order for the system to be controllable. In time-varying systems it may be necessary to restrict T to be greater than some fixed time, say $\bar{T}$. But in time-invariant systems, as we shall see, the only restriction on T is that it be greater than zero. (In fact, if the use of impulsive inputs is permitted, it is possible in a controllable system to go from any state to any other state in zero time, i.e., instantaneously. (See Note 5.2.) As a practical matter, it is possible in a controllable system to go from any state to any other state in an arbitrarily short time if we are willing to use a sufficiently large input.)

Controllability theorem A system is controllable if and only if the matrix

$$P(T, t) = \int_t^T \Phi(T, \lambda)B(\lambda)B'(\lambda)\Phi'(T, \lambda)\, d\lambda \tag{5.6}$$

is nonsingular for some $T > t$, where $\Phi(T, t)$ is the state-transition matrix of the system.

It is not at all obvious what this strange matrix integral, often called the *controllability grammian*, has to do with controllability. The integral appears to have been fetched from out of the sky. Later in the book, we will encounter integrals of this type quite often.

Since there is no obvious connection between the controllability grammian and getting from the state x_t to the state x_T, we should not be surprised that the proof of the controllability theorem is not entirely obvious. And it isn't. The nonobvious part of the theorem is the necessary condition, namely that if the integral $P(T, t)$ is singular for all finite $T > t$, then there are some states that can't be reached by any input.

To prove that the existence of an inverse of the controllability grammian guarantees the ability of going from any state x_t to any other state x_T, we recall from Chap. 3 (Eq. (3.21)) that

$$x_T = \Phi(T, t)x_t + \int_t^T \Phi(T, \lambda)B(\lambda)u(\lambda)\, d\lambda \tag{5.7}$$

Now suppose that $P(T, t)$ is nonsingular (i.e., has an inverse) on the interval $[t, T]$ for some finite T. Then an input that forces the process from x_t to x_T is given by

$$u(\lambda) = B'(\lambda)\Phi'(T, \lambda)P^{-1}(T, t)[x_T - \Phi(T, t)x_t] \qquad \text{for } t \leqq \lambda \leqq T \tag{5.8}$$

To verify this just substitute (5.8) into (5.7):

$$x_T = \Phi(T, t)x_t + \int_t^T \Phi(T, \lambda)B(\lambda)B'(\lambda)\Phi'(T, \lambda)\, d\lambda\, P^{-1}(T, t)\{x_T - \Phi(T, t)x_t\} \tag{5.9}$$

By (5.6) the integral in (5.9) is $P(T, t)$, so (5.9) becomes

$$x_T = \Phi(T, t)x_t + x_T - \Phi(T, t)x_t$$

which is an identity. This verifies that the input (5.8) does indeed transfer the system from x_t at time t to x_T at time T.

Note that the input given by (5.8) requires the inverse $P^{-1}(T, t)$ which exists only if $P(T, t)$ is nonsingular for some t and $T > t$. If $P(T, t)$ is singular for all $T > t$, then the input (5.8) cannot be used to transfer x_t to x_T. But perhaps we can use some other input. The answer is no, and demonstrating this constitutes the second part of the proof. We want to show that there are some states that can't be reached if $P(T, t)$ is singular for all finite $T > t$. Consider some time T for which $P(T, t)$ is singular. Then there must be some *nonzero* vector v such that

$$v'P(T, t)v = 0$$

Thus, by the definition (5.6)

$$\int_t^T v'\Phi(T, \lambda)B(\lambda)B'(\lambda)\Phi'(T, \lambda)v\, d\lambda = 0 \tag{5.10}$$

The integrand can be written

$$\int_t^T z'(\lambda)z(\lambda)\, d\lambda = 0 \tag{5.11}$$

where

$$z(\lambda) = B'(\lambda)\Phi'(T, \lambda)v \tag{5.12}$$

Since the integrand is a sum of squares: $z_1^2(\lambda) + \cdots + z_n^2(\lambda)$ it must be nonnegative. The only way that an integral, over a positive interval $T > t$, of a nonnegative quantity can be zero, is when the integrand itself is *identically zero* over the entire interval. Thus, for a singular grammian we have found a vector v for which

$$z(\lambda) = B'(\lambda)\Phi'(T, \lambda)v \equiv 0 \qquad \text{for all } \lambda \text{ in the interval } [t, T] \tag{5.13}$$

Then, as we shall see, it is impossible in the time interval $[t, T]$ to get from the origin to any state in the direction of v. Suppose the contrary: that it is possible to go to a state cv (where c is a scalar) in the direction of v. Then, with $x_t = 0$, by (5.7) we must have

$$cv = \int_t^T \Phi(T, \lambda)B(\lambda)u(\lambda)\, d\lambda \tag{5.14}$$

On premultiplying both sides of (5.14) by v' we find

$$cv'v = \int_t^T v'\Phi(T, \lambda)B(\lambda)u(\lambda)\, d\lambda \tag{5.15}$$

The left-hand side of (5.15) is clearly nonzero. But the integrand of the right-hand side is $z'(\lambda)u(\lambda)$ and in (5.13) we have found that $z(\lambda) \equiv 0$ in the entire interval $[t, T]$. Thus, the right-hand side of (5.15) is zero, independent of the input. This is a contradiction. We are forced to conclude that no input can transfer the system from the origin to a state cv in the interval $[t, T]$. If the controllability grammian is singular for all t and T there will always be some states that we will not be able to reach in any finite time. Hence the controllability grammian must be nonsingular for some t and $T > t$, in order that the system be controllable. This completes the proof.

The controllability theorem and its proof are the contributions of R. E. Kalman. See Note 5.1.

For a time-invariant system, the controllability grammian is given by

$$P(T-t) = \int_t^T e^{A(T-\lambda)}BB'\, e^{A'(T-\lambda)}\, d\lambda = \int_0^{T-t} e^{At}BB'\, e^{A't}\, dt$$

or simply

$$P(T) = \int_0^T e^{At}BB'\, e^{A't}\, dt \tag{5.16}$$

The matrix used for the controllability test is sometimes written

$$\tilde{P}(T) = \int_0^T e^{-At}BB'\, e^{-A't}\, dt \tag{5.17}$$

which is not the same matrix as $P(T)$ but which can easily be shown to have the same rank as $P(T)$.

Matrices having the form of the controllability grammian (5.6) in the general case, or in the form of (5.16) in the time-invariant case, sometimes need to be evaluated for optimum control and estimation problems, as will be discussed from Chap. 9 onward. But to evaluate the integrals merely for the purpose of testing controllability of a system seems a great deal of effort to achieve a simple objective. A simpler test would be most welcome. For time-varying systems there does not seem to be a simpler alternative. But for time-invariant systems, several simpler alternatives are available. We have already used one of the alternatives in the examples: transform the matrix to diagonal form (or block-diagonal form) and find whether or not any subsystem cannot be reached by the input. It may not always be easy to do this, however. A still simpler criterion, based on the rank of the matrix $[B, AB, \ldots, A^{k-1}B]$, will be given in the next section. We postpone a discussion of this condition, however, until addressing the topic of observability. We shall see that a close similarity exists between the concept of controllability and the concept of observability, which make it desirable to treat the two concepts together.

Just as the output y is not considered in the definition of controllability, the input u is generally not considered in defining observability. Thus, we deal with the unforced system

$$\dot{x} = A(t)x$$

with the observation given by

$$y(t) = C(t)x(t)$$

We use the following:

Definition of observability An unforced system is said to be observable if and only if it is possible to determine *any* (arbitrary initial) state $x(t) = x_t$ by using only a finite record, $y(\tau)$ for $t \leqq \tau \leqq T$, of the output.

This definition seems to square with our intuitive concept of what ought to constitute an observable system. Note that the definition requires ability to determine the initial state no matter where that state might be in the state-space. If only *some*, but not all, initial states can be determined, then the system is not observable.

The general condition for observability is given by the following:

Observability theorem A system is observable if and only if the matrix

$$M(T, t) = \int_t^T \Phi'(\lambda, t)C'(\lambda)C(\lambda)\Phi(\lambda, t)\, d\lambda \qquad (5.18)$$

is nonsingular for some $T > t$, where $\Phi(T, t)$ is the state-transition matrix of the system.

The matrix $M(T, t)$ for testing observability is often called the ***observability grammian***, and bears a strong resemblance to the controllability grammian (5.6): in place of the transition matrix $\Phi(T, \lambda)$ in (5.6), its transpose appears in (5.18); in place of the control matrix B in (5.6), the transpose of the observation matrix C appears in (5.18). Because of the close resemblance between controllability and observability, these are frequently referred to as *dual* concepts.

To prove the observability theorem we use the fact that the output y is given by

$$y(\lambda) = C(\lambda)\Phi(\lambda, t)x_t \qquad (\lambda \geqq t) \tag{5.19}$$

when the system starts in the state x_t. Multiply both sides of (5.19) by $\Phi'(\lambda, t)C'(\lambda)$ and integrate over the interval $[t, T]$ to obtain

$$\int_t^T \Phi'(\lambda, t)C'(\lambda)y(\lambda)\, d\lambda = \left(\int_t^T \Phi'(\lambda, t)C'(\lambda)C(\lambda)\Phi(\lambda, t)\, d\lambda\right)x_t \tag{5.20}$$

The integral on the right-hand side of (5.20) is recognized as the observability grammian $M(T, t)$ of (5.18). Thus, if the observability grammian is nonsingular, we can solve (5.20) for x_t:

$$x_t = M^{-1}(T, t)\int_t^T \Phi'(\lambda, t)C'(\lambda)y(\lambda)\, d\lambda \tag{5.21}$$

This formula furnishes an actual procedure for finding the initial state x_t, given $y(t)$ over the interval of the integral. Of course, it may not be the only way to determine x_t. Perhaps another way can be found to determine x_t that does not entail the inverse of the observability grammian. The answer, as we already suspect, is no. The reason why the answer is no is a consequence of an argument like that used for establishing the dual result for controllability: if the observability grammian $M(T, t)$ is singular then there exists a vector w for which the function

$$q(\lambda) = C(\lambda)\Phi(\lambda, t)w \equiv 0 \qquad \text{for all } \lambda \text{ in the interval } [t, T]$$

This function $q(\lambda)$, which is identically zero over the interval $[t, T]$ is precisely the output of the system when the initial state is w. It thus follows that if the initial state is w or anywhere on the line cw it will yield an output of zero and there will be no way of determining that initial state. If the observability grammian is singular for every t and T, there will always be some initial state which will produce zero outputs for intervals of any length, and hence the system is not observable.

For time-invariant systems the observability grammian of (5.18) may be written

$$M(T - t) = \int_t^T e^{A'(\lambda - t)}C'C\,e^{A(\lambda - t)}\, d\lambda = \int_0^{T-t} e^{A'\tau}C'C\,e^{A\tau}\, d\tau$$

or simply

$$M(T) = \int_0^T e^{A'\tau} C' C e^{A\tau} \, d\tau \qquad (5.22)$$

Other forms of the observability grammian are also used, such as

$$\tilde{M}(T) = \int_0^T e^{-A't} C' C e^{-At} \, dt$$

Again, as with controllability, these matrices are not equal to $M(t)$ but have the same rank.

Also, as is the case with controllability, it is not necessary to evaluate the observability grammian to test for observability. There is a simpler algebraic test which is the subject of the next section.

5.4 ALGEBRAIC CONDITIONS FOR CONTROLLABILITY AND OBSERVABILITY

In the previous section we have seen that the necessary and sufficient condition for controllability of a time-invariant system is that the controllability grammian $P(T)$, given by (5.16), be nonsingular for some finite time T.

The algebraic criterion equivalent to this is expressed by the following:

Algebraic controllability theorem The time-invariant system $\dot{x} = Ax + Bu$ is controllable if and only if the rank $r(Q)$ of the *controllability test matrix*

$$Q = [B \quad AB \quad \cdots \quad A^{k-1}B] \qquad (5.23)$$

is equal to k, the order of the system.

Note that Q is a matrix having k rows and kl columns, where l is the number of inputs. The rank of Q thus cannot be greater than k. But the rank of Q can be smaller than k. If so, the system is not controllable.

To prove the algebraic controllability theorem we note that if $P(T)$ is singular, then, by (5.13), there is a nonzero vector v such that the function

$$z(t) = B' e^{A't} v \equiv 0 \qquad \text{for} \qquad 0 \leqq t \leqq T \qquad (5.24)$$

Since the function $z(t)$ is *identically* zero (flat), all its derivatives must also be identically zero. Thus we must have

$$\begin{aligned} \dot{z}(t) &= B'A' e^{A't} v \equiv 0 \\ \ddot{z}(t) &= B'(A')^2 e^{A't} v \equiv 0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ z^{(k-1)}(t) &= B'(A')^{k-1} e^{A't} v \equiv 0 \end{aligned} \qquad (5.25)$$

We can keep going with this process but there is no need to do so.

We can arrange (5.24) and (5.25) in the following form:

$$\underbrace{\begin{bmatrix} B' \\ B'A' \\ \vdots \\ B'(A')^{k-1} \end{bmatrix}}_{Q'} e^{A't}v \equiv 0 \qquad (5.26)$$

(k columns; kl rows)

The long matrix in (5.26), is Q'. Let its columns be denoted by $q_1, q_2, \ldots, q_k$:

$$Q' = [q_1 \quad q_2 \quad \cdots \quad q_k]$$

Also let

$$e^{A't}v = \begin{bmatrix} \alpha_1(t) \\ \vdots \\ \alpha_k(t) \end{bmatrix} \qquad \text{a } k\text{-dimensional vector}$$

Thus (5.26) becomes

$$\alpha_1(t)q_1 + \alpha_2(t)q_2 + \cdots + \alpha_k(t)q_k \equiv 0 \qquad (5.27)$$

In other words, the columns of Q' are linearly *dependent*, which implies that the rank of the matrix Q' must be less than the order k of the system. We have thus established that if the controllability grammian is singular the rank of the matrix Q' is less than k. Since the rank of any matrix is equal to the rank of its transpose, we also can say that the singularity of the controllability grammian implies that the rank of Q is less than k.

To prove the converse, we expand e^{At} in a power series in t:

$$e^{At} = I + At + A^2t^2/2! + \cdots + A^{k-1}t^{k-1}/(k-1)! + A^kt^k/k! + \cdots \qquad (5.28)$$

By the Cayley–Hamilton theorem, however,

$$A^k = -a_1A^{k-1} - a_2A^{k-2} - \cdots - a_kI \qquad (5.29)$$

where $a_1, a_2, \ldots, a_k$ are the coefficients of the characteristic polynomial of A. Thus, by repeated use of (5.29), any power of A greater than $k-1$ can be expressed as a polynomial of degree $k-1$ in A. Thus (5.28) can be written as

$$e^{At} = If_1(t) + Af_2(t) + \cdots + A^{k-1}f_k(t)$$

where $f_1(t), f_2(t), \ldots, f_k(t)$ are the time functions obtained by substituting the expressions for powers of A higher than $k-1$ into (5.28) and collecting terms.

(The proof doesn't depend on what these functions are.) Thus

$$e^{At}B = [Bf_1(t) + ABf_2(t) + \cdots + A^{k-1}Bf_k(t)]$$

$$= Q\begin{bmatrix} f_1(t) \\ \vdots \\ f_k(t) \end{bmatrix}$$

Thus the controllability grammian can be expressed as

$$P(T) = Q\int_0^T \begin{bmatrix} f_1^2(t) & \cdots & f_1(t)f_k(t) \\ \cdots & \cdots & \cdots \\ f_1(t)f_k(t) & \cdots & f_k^2(t) \end{bmatrix} dt\, Q' \qquad (5.30)$$

$$= QGQ'$$

If we knew the functions $f_1, \ldots, f_k$ we would have an expression for $P(T)$ in terms of the grammian matrix G appearing in (5.30) between Q and Q'. But no matter what this matrix is, the rank of $P(T)$ cannot be greater than the rank of Q, since the rank of the product of matrices cannot exceed the rank of any of its factors. Thus, if the rank of Q is less than k, then the rank of $P(T)$ must surely be less than k, which means, of course, that $P(T)$ must be singular.

This completes the proof of the algebraic controllability theorem.

Since Q is a constant matrix it has constant rank. Thus, if Q is singular, then $P(T)$ is singular for *every* T. Similarly, if $P(T)$ is nonsingular for any $T > 0$, it must be nonsingular for every $T > 0$. This means that if a system is controllable, there is an input that will transfer the system from any starting state to any other state in an arbitrarily short time. The shorter the time, the larger the needed input, of course.

From the manner in which the algebraic controllability theorem was established using the controllability grammian, we can immediately assert the dual:

Algebraic observability theorem The (unforced) time invariant system

$$\dot{x} = Ax$$

with the observation vector

$$y = Cx$$

is observable if and only if the rank $r(N)$ of the *observability test matrix*

$$N = [C' \quad A'C' \quad \cdots \quad (A')^{k-1}C'] \qquad (5.31)$$

is equal to k, the order of the system.

Intuitively, one might conjecture that the ranks of the controllability grammian P and the corresponding matrix Q are related to each other and to the dimension of the subspace of states that can be reached. It turns out that this conjecture is quite correct. In fact, the ranks of $P(T)$, and Q, and the dimension of the "controllable subspace" are all *equal.* Likewise, the ranks of the observability grammian $M(T)$ and N and the dimension of the "observable subspace" are all equal. Complete proofs of all these facts is beyond the scope of this book. (See Note 5.2.) But we can gain insight into why this happens by considering transformations of state variables. In particular, suppose

$$\bar{x} = Tx$$

so that the matrices for the transformed system are

$$\bar{A} = TAT^{-1} \qquad \bar{B} = TB \qquad \bar{C} = CT^{-1} \tag{5.32}$$

Then the controllability test matrix for the transformed system is

$$\bar{Q} = [\bar{B} \quad \bar{A}\bar{B} \quad \cdots \quad \bar{A}^{k-1}\bar{B}]$$

But from (5.32)

$$\bar{A}^l\bar{B} = TA^lT^{-1}TB = TA^lB$$

Thus

$$\bar{Q} = [TB \quad TAB \quad \cdots \quad TA^{k-1}B] = TQ$$

Since the rank of a product cannot exceed the rank of either factor

$$r(\bar{Q}) \leqq r(Q)$$

But

$$Q = T^{-1}\bar{Q}$$

So

$$r(Q) \leqq r(\bar{Q})$$

Thus we conclude that

$$r(Q) = r(\bar{Q})$$

In other words, the rank of the controllability matrix is invariant to a change of state variable. Suppose that the transformed system is block-diagonal, i.e.,

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix} \begin{matrix} \updownarrow k_1 \\ \updownarrow k_2 \end{matrix} \qquad (k_1 + k_2 = k)$$

and, moreover, that the subsystem (of order k_1) corresponding to $\bar{A}_1$ is controllable, but that no input at all goes to the subsystem (of order k_2) corresponding to $\bar{A}_2$. Thus, we have

$$\bar{B} = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$$

The controllability test matrix is

$$Q = \begin{bmatrix} \bar{B}_1 & \bar{A}_1\bar{B}_1 & \cdots & \bar{A}_1^{k_1-1}\bar{B} & \bar{A}_1^{k_1}\bar{B} & \cdots & \bar{A}_1^{k_1+k_2}\bar{B} \\ 0 & 0 & & 0 & 0 & & 0 \end{bmatrix} \begin{matrix} \updownarrow k_1 \\ \updownarrow k_2 \end{matrix}$$

The upper left submatrix $[\bar{B}_1 \quad \bar{A}_1\bar{B}_1 \quad \cdots \quad \bar{A}_1^{k_1-1}\bar{B}_1]$ is the controllability test matrix of the controllable subsystem, so it is of rank k_1. Thus the matrix $\bar{Q}$ is at least of rank k_1. But it cannot be of rank greater than k_1 because it contains k_2 rows of zero elements. Thus the *rank deficiency* $k_2 = k - k_1$ of $\bar{Q}$ is precisely equal to the dimension of the subspace which receives no input. Because the controllability matrix of the original system has the same rank as $\bar{Q}$, the dimension of the subspace that is uncontrollable remains equal to k_2.

The very same transformation concept applies to the relationship between the dimension of the "unobservable" subspace and the rank of the observability test matrix.

The block-diagonal matrix $\bar{A}$ can be the matrix of the Jordan canonical form. In that case, as discussed in Chap. 4, the state variables are the "normal modes" of the system. All the normal modes which can be controlled can be identified with subsystem 1 and all those normal modes which cannot be controlled can be identified with subsystem 2. It thus follows immediately that the rank of the controllability test matrix Q is equal to the number of controllable normal modes. Similarly, the number of observable normal modes is equal to the rank of the observability test matrix N.

These concepts are illustrated by the following examples.

Example 5E Hypothetical system (continued from Example 5A) The test matrix Q for controllability of the hypothetical system of Example 5A is

$$Q = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -2 & 4 & -10 & 28 \\ 2 & -6 & 18 & -54 \\ -1 & 3 & -9 & 27 \end{bmatrix}$$

The sum of the elements in each column of Q is zero so Q is clearly singular. Moreover, the sum of the elements of the first two rows minus the fourth row are also zero. Thus, only two rows of Q are linearly independent and the rank of Q is thus 2.

The test matrix N for observability of the system is

$$N = [C' \quad A'C' \quad (A')^2C' \quad (A')^3C'] = \begin{bmatrix} 7 & -10 & 16 & 28 \\ 6 & -9 & 15 & -27 \\ 4 & -6 & 10 & -18 \\ 2 & -3 & 5 & -9 \end{bmatrix}$$

It is similarly verified that the rank of N is 2.

Thus, there are two observable modes and two controllable modes. This is clear from Fig. 5.2. But since one of the controllable modes is also an observable mode, one mode remains that is neither observable nor controllable.

Example 5F Coupled masses with an internal force The controllability test matrix for the system of Example 5B is

$$Q = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 0 & -1/m_1 & 0 & km/m_1^2m_2 \\ 0 & 1/m_2 & 0 & -km/m_1m_2^2 \\ -1/m_1 & 0 & km/m_1^2m_2 & 0 \\ 1/m_2 & 0 & -km/m_1m_2^2 & 0 \end{bmatrix} \qquad (m = m_1 + m_2)$$

The third and fourth columns of Q are proportional to the first and second, respectively, so the rank of Q is only 2, as expected from Example 5B. The two uncontrollable state variables are the position of the center of mass x_c and its derivative $\dot{x}_c$.

Example 5G Distillation column A schematic diagram corresponding to the simplified model of the distillation column as developed by Gilles and Retzbach[3] (Example 2G) is shown in Fig. 4.2 on page 120. It is observed that there is a path from the input Δu_1 (steam flow rate) to each of the state variables. Nevertheless, the process cannot be controlled by u_1 alone, because x_2 and x_3 are both integrators and thus give the appearance of redundant state variables. It is also evident that the process is not controllable from the input Δu_2 (vapor side stream flow rate): there is not even a path from u_2 to x_1 and to x_2. But, by using both inputs, the process is controllable.

These facts can be verified by use of the algebraic controllability criterion. First, consider the single input u_1. The corresponding control matrix is

$$B_1 = \begin{bmatrix} b_{11} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence, the controllability matrix Q_1 corresponding to u_1 is given by

$$Q_1 = [B_1 \quad AB_1 \quad A^2B_1 \quad A^3B_1]$$

$$= \begin{bmatrix} 1 & a_{11} & a_{11}^2 & a_{11}^3 \\ 0 & a_{21} & a_{21}(a_{11}+a_{22}) & a_{21}[a_{11}^2 + a_{22}(a_{11}+a_{22})] \\ 0 & 0 & a_{21}a_{32} & a_{32}a_{21}(a_{11}+a_{22}) \\ 0 & 0 & a_{21}a_{42} & a_{42}a_{21}(a_{11}+a_{22}) \end{bmatrix} b_{11}$$

The upper left-hand 3-by-3 submatrix is triangular and thus has a nonzero determinant (unless $a_{21} = 0$). Thus the rank of Q_1 is at least 3. But

$$|Q_1| = b_{11}^4 a_{21} \begin{vmatrix} a_{21}a_{32} & a_{32}a_{21}(a_{11}+a_{22}) \\ a_{21}a_{42} & a_{42}a_{21}(a_{11}+a_{22}) \end{vmatrix} = 0$$

Thus, the rank of $Q_1 < 4$. Thus, we conclude that the rank of $Q_1 = 3$, which means that the process is not controllable using only Δu_1. The control matrix for the input Δu_2 is

$$B_2 = \begin{bmatrix} 0 \\ 0 \\ b_{32} \\ b_{42} \end{bmatrix}$$

and the corresponding controllability matrix is

$$Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{32} & 0 & 0 & 0 \\ b_{42} & 0 & 0 & 0 \end{bmatrix}$$

which has a rank of 1.

The controllability matrix using both inputs is a 4×8 matrix whose columns are the columns of Q_1 and Q_2, interlaced. If the system is controllable, the resulting matrix must have four linearly independent columns, for example, the first three columns of Q_1 and the first column of Q_2. The determinant of the matrix formed from these columns is

$$\Delta = \begin{vmatrix} b_{11} & b_{11}a_{11} & b_{11}a_{11}^2 & 0 \\ 0 & b_{11}a_{21} & b_{11}(a_{11}+a_{22}) & 0 \\ 0 & 0 & b_{11}a_{21}a_{32} & b_{32} \\ 0 & 0 & b_{11}a_{21}a_{42} & b_{42} \end{vmatrix}$$

$$= b_{11}^3 a_{21}^2 \begin{vmatrix} a_{32} & b_{32} \\ a_{42} & b_{42} \end{vmatrix}$$

Except for specific values of a_{32}, a_{42}, b_{32}, and b_{42} the determinant $\Delta \neq 0$ and hence, the controllability matrix has a full rank of 4 and the process is, in general, controllable using both inputs. From the numerical data given with Example 2G, it is seen that a_{32} and a_{42} are of the same magnitude, while b_{32} and b_{42} are very much different in magnitude. Thus, Δ is not even approximately zero and the process is easily controllable using both inputs.

It is very important to recognize that the algebraic controllability and observability tests are only valid for time-invariant systems. That they are not generally valid for time-varying systems is vividly illustrated by a simple, but practical example, in which the state vector x is a constant:

$$\dot{x} = 0 \tag{5.33}$$

hence the dynamics matrix A is zero. If the observation matrix C is constant then the observability test matrix

$$N = [C' \quad 0 \quad 0 \quad \cdots \quad 0] \tag{5.34}$$

N has rank k if and only if C has rank k, i.e., that there are as many independent components of the observation vector as there are components in x. If C is time-varying, the observability test matrix is still given by (5.34) which would imply that x is unobservable, unless C is of rank k. But in fact x may be observable even if the observation vector y is a scalar, if C is time varying. Consider the scalar observation

$$y(t) = C(t)x = c'(t)x = c_1(t)x_1 + \cdots + c_k(t)x_k \tag{5.35}$$

At k different time instants $t_1, t_2, \ldots, t_k$ we have

$$\begin{gathered} y(t_1) = c'(t_1)x \\ y(t_2) = c'(t_2)x \\ \cdots\cdots\cdots\cdots \\ y(t_k) = c'(t_k)x \end{gathered}$$

or

$$\begin{bmatrix} y(t_1) \\ \vdots \\ y(t_k) \end{bmatrix} = \begin{bmatrix} c'(t_1) \\ \vdots \\ c'(t_k) \end{bmatrix} x$$

If the time instants t_i are chosen such that the matrix multiplying x is nonsingular, then

$$x = \begin{bmatrix} c'(t_1) \\ \vdots \\ c'(t_k) \end{bmatrix}^{-1} \begin{bmatrix} y(t_1) \\ \vdots \\ y(t_k) \end{bmatrix}$$

Determination of an unknown constant vector by looking at it at different times, or "from different angles" is a standard procedure in the calibration of instruments, and the selection of suitable time instants, depending on the nature of $c(t)$, or the design of a suitable $c(t)$, is an important issue in practical calibration procedures. See Note 5.4.

5.5 DISTURBANCES AND TRACKING SYSTEMS: EXOGENOUS VARIABLES

In order to use state-space methods on design problems in which there are reference inputs and/or disturbances, it is frequently desirable to represent these inputs and disturbances by additional state variables.

The particular dynamic process we might wish to control would be of the form

$$\dot{x} = Ax + Bu + Fx_d \tag{5.36}$$

where x_d is a disturbance vector (which may or may not be subject to direct measurement).

In addition, we might wish to require that the state x track a reference state x_r.

To formulate the problem purely in terms of state variables, it is often expedient to assume that x_d and x_r satisfy known differential equations:

$$\dot{x}_d = A_d x_d \tag{5.37}$$

$$\dot{x}_r = A_r x_r \tag{5.38}$$

These supplementary states are surely not subject to control by the designer, so that these are *unforced* differential equations. The system comprising x, x_d, and x_r is necessarily uncontrollable. (Fig. 5.11.)

In general, we are concerned with the error defined by

$$e = x - x_r \tag{5.39}$$

The differential equation for the error using (5.36) and (5.38) becomes

$$\begin{aligned} \dot{e} = \dot{x} - \dot{x}_r &= A(e + x_r) + Fx_d + Bu - A_r x_r \\ &= Ae + (A - A_r)x_r + Fx_d + Bu = Ae + Ex_0 + Bu \end{aligned} \tag{5.40}$$

where

$$E = [A - A_r \;\vdots\; F] \tag{5.41}$$

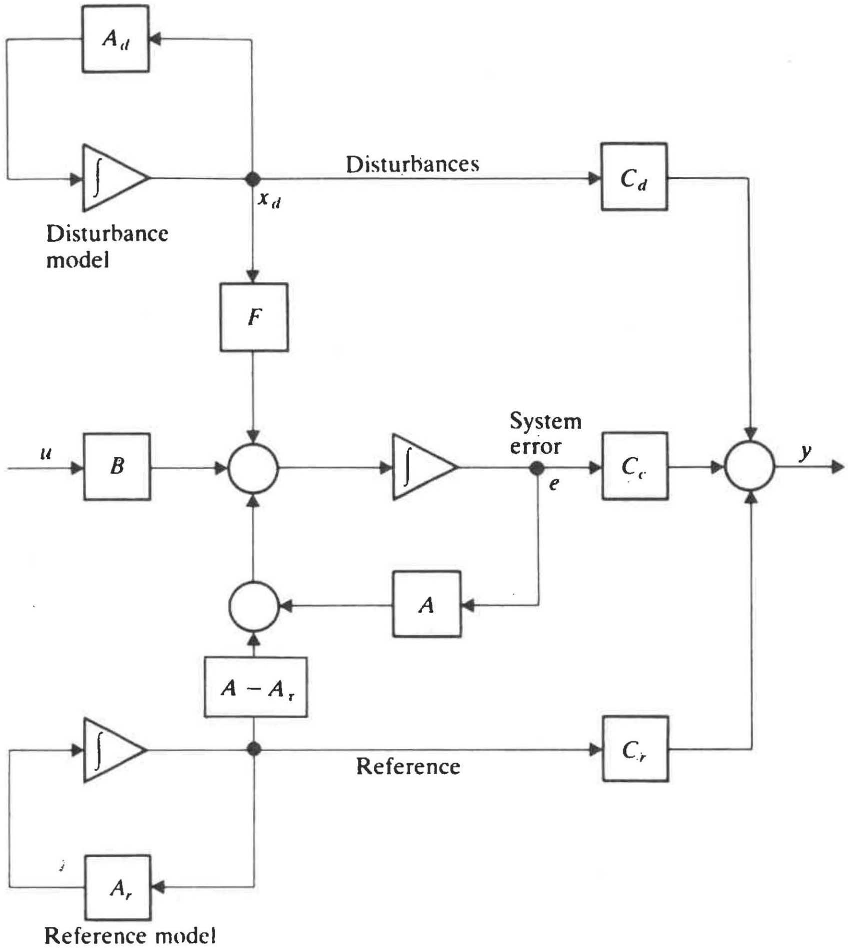

Figure 5.11 State-space representation of system with disturbances and reference input. (Models for disturbances and reference state are uncontrollable.)

and

$$x_0 = \begin{bmatrix} x_r \\ \hline x_d \end{bmatrix} \tag{5.42}$$

The vector x_0 represents the "exogenous" inputs to the system. To the differential equation of the error is adjoined the equations for the reference and disturbance states to produce a system of order $2k+1$ having the "metastate" vector

$$\mathbf{x} = \begin{bmatrix} e \\ \hline x_0 \end{bmatrix} \begin{matrix} \updownarrow\ k \\ \updownarrow\ k+1 \end{matrix} \tag{5.43}$$

and satisfying the "metastate equation"

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{5.44}$$

where

$$\mathbf{A} = \left[\begin{array}{c|c} A & E \\ \hline 0 & \mathbf{A}_0 \end{array}\right] \qquad \mathbf{B} = \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] \tag{5.45}$$

where

$$\mathbf{A}_0 = \left[\begin{array}{c|c} A_r & 0 \\ \hline 0 & A_d \end{array}\right]$$

is the dynamics matrix for the exogenous inputs, now a substate of the metastate vector **x**.

In some cases, only the error can be measured. In that case, the observation equation is

$$y = Ce = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{C} = [C \mid 0 \quad 0]$$

More generally, however, it might be possible to measure the error, the reference state, and the disturbance state. Hence the general form of the observation equation is

$$y = C_e e + C_r x_r + C_d x_d$$

and hence, the general observation matrix is given by

$$\mathbf{C} = [C_e \mid C_r \quad C_d]$$

A schematic representation of the metasystem is shown in Fig. 5.11. The subsystems for the disturbance x_d and the reference x_r are clearly not controllable. With C_d and C_r present, the system is likely to be observable. But even if only C_e is present, the system may be observable because there is a path from x_r to the output through the subsystem that generates the error.

The very natural way in which an uncontrollable system arises when exogenous disturbances and reference inputs are modeled does not alter the fact that such systems *are* uncontrollable and hence that design techniques based on the premise of a controllable system cannot be applied willy-nilly to the metasystem. This doesn't imply that these methods are useless for this type of metasystem (or other types of uncontrollable systems) but rather that it is necessary to be cautious in their use.

PROBLEMS

Problem 5.1 Exogenous variables: controllability and observability

Consider the metasystem (5.44) with **A** and **B** as given by (5.45)

(*a*) Using the algebraic controllability test (5.23), show that the metasystem is *not* controllable. (This result is intuitively obvious.)

(*b*) Assume that only the state x (and not x_0) is measured, i.e., $y = Cx$, and that the original system $\dot{x} = Ax + Bu$ is observable (i.e., $[C', A'C', \ldots, (A')^{k-1}C']$ is of rank k). Discuss the conditions under which the metasystem is observable.

Problem 5.2 Two-car train

Consider the two-car train of Probs. 2.5 and 3.9.
(*a*) Is it controllable using only one motor?
(*b*) Is it controllable using both motors?
(*c*) Is it observable if only the position z_1 of the first car is measurable?
(*d*) Is it observable if the velocity v_1 of the first car is measurable?
(*e*) Is it observable if the velocities of both cars are measurable?

Problem 5.3 Aircraft lateral dynamics: controllability

Consider the lateral aircraft dynamics of Prob. 4.4.
(*a*) Is the dynamic process controllable using only the ailerons?
(*b*) Is the dynamic process controllable using only the rudder?

Problem 5.4 Inverted pendulum on cart: observability

Consider the inverted pendulum on a motor-driven cart described in Probs. 2.1 and 3.6. Determine whether or not it is observable with the following sets of observations:
(*a*) Cart displacement: $y = x_1$;
(*b*) Pendulum angle: $y = x_3$;
(*c*) Cart velocity: $y = x_2$;
(*d*) Cart velocity and pendulum angle: $y_1 = x_2$, $y_2 = x_3$.

Problem 5.5 Double-effect evaporator: controllability

Determine whether or not the evaporator of Example 2H is controllable from each of the following combinations of inputs:
(*a*) u_1 only;
(*b*) u_1 and u_2;
(*c*) u_1 and u_3;
(*d*) u_2 and u_3.
If the system is not controllable for any of the above cases, explain why not and, if possible, identify the states that are not controllable. *Hint:* Refer to Fig. 2.21.

Problem 5.6 Double-effect evaporator: observability

Determine whether or not the evaporator of Example 2H is observable from each of the following combinations of outputs:
(*a*) x_1 and x_4;
(*b*) x_3 and x_5;
(*c*) x_3, x_4, and x_5.
If, in any case, the system is not observable, explain why not and, if possible, identify the unobservable states.

NOTES

Note 5.1 Background of controllability and observability

In 1954 Bergen and Ragazzini[4] presented a method of compensating a sampled-data system by solving for the transfer function of the compensator given the desired closed-loop transfer function. They recognized that this method of compensation entailed cancellation of undesirable poles and zeros of the plant and substitution of more desirable ones. A mathematically exact cancellation would not be possible with real hardware. Thus they developed rules governing the incorporation of "nonminimum phase" (Note 4.7) poles and zeros into the specification of the desired closed-loop transfer function.

Kalman observed that the problem of nonminimum phase pole-zero cancellation would be present even if the cancellation were mathematically perfect, because the resulting system would turn out to be uncontrollable. With J. E. Bertram he presented a state-space design procedure[5] making use of state variable feedback in which the concept of controllability is hinted at. By 1960, Kalman had fully elucidated the concept of controllability and the dual concept of observability.[6]

Note 5.2 Varieties of controllability and observability

In this book we say a system is *controllable* if it is possible to find an input which brings it to the origin (or any other state) from any state in a finite time. (Kalman called such a system *completely controllable*.[1]) If a system is not controllable, it can be divided into two subsystems, one of which (if it exists) is controllable and the other is uncontrollable. If the uncontrollable subsystem is stable, the entire system is said to be *stabilizable*. The set of stabilizable systems thus includes the controllable systems as a subset: every controllable system is stabilizable, but not every stabilizable system is controllable. Similar distinctions apply with regard to observability. A system that is not observable (*completely* observable, in Kalman's terminology) can be divided into two subsystems, one of which (if it exists) is observable and the other is not. If the unobservable subsystem is stable, the entire system is said to be *detectable*. Thus the observable systems are a subset of the detectable systems.

These definitions and concepts are adequate for time-invariant systems, a category that includes most systems considered in this book. When time-varying systems are considered, however, the situation becomes more complicated. In a time-invariant system controllability is independent of the initial time. If this is true in a time-varying system, the system is said to be *uniformly* controllable. The dual of uniform controllability is uniform observability. With regard to the latter, it is noted that our definition of observability requires the ability to determine the present state based on *future* outputs. In a time-invariant system this is equivalent to the ability to determine the present state on the basis of *past* outputs. These are not necessarily equivalent, however, in time-varying systems. Thus we have another concept, namely *reconstructability*, which is the ability to determine the present state from past inputs.

A reasonably comprehensive treatment of observability, controllability, and various derivative concepts can be found in [2].

Note 5.3 Degree of controllability

By the definition of this chapter, a system is either controllable or it is not. In the real world, however, it may not be possible to make such sharp distinctions. An electrical bridge network, for example, is uncontrollable (or unobservable) for one discrete combination of its parameters. Since exact mathematical balancing is not possible, every practical bridge network is controllable and observable. If the balance condition is close to being satisfied, however, it will be very difficult to control or to observe all the state variables of the bridge. The problem with the standard definitions of controllability and observability is that they can lead to discontinuous functions of the system parameters: an arbitrarily small change in a system parameter can cause an abrupt change in the rank of the matrix by which controllability or observability is determined. It would be desirable to have definitions which can vary continuously with the parameters of the system and thus can reflect the degree of controllability of the system. Kalman et al.[6] recognized the need and suggested using the determinant of the corresponding test matrix or grammian as a measure of the degree of controllability or observability. Friedland,[7] noting that basing the degree of controllability or observability on the determinant of the test matrix suffers from sensitivity to the scaling of the state variables, suggested using the ratio of the smallest of the singular values to the largest as a preferable measure. Moore[8] subsequently elaborated upon this suggestion.

Note 5.4 Application to calibration

The development of an analytical technique for the determination of a constant vector b based on time-varying measurement signal $y(t) = C(t)b$ and generalizations of this technique is considered by Friedland.[9]

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