CHAPTER 2

Linear, Discrete, Dynamic-Systems Analysis: The z-Transform

2.1 INTRODUCTION

The unique element in the structure of Fig. 1.1 is the digital computer. The fundamental character of the digital computer is that it takes a finite time to compute answers, and it does so with only finite precision. The purpose of this chapter is to develop tools of analysis necessary to understand and to guide the design of programs for a computer acting as a linear, dynamic control component. Needless to say, digital computers can do many things other than control linear, dynamic systems; it is our purpose in this chapter to examine their characteristics when doing this elementary control task and to develop the basic analysis tools needed to write programs for a real-time control computer.

2.2 LINEAR DIFFERENCE EQUATIONS

We assume that the analog-to-digital converter (A/D) in Fig. 1.1 takes samples of the signal y at discrete times and passes them to the computer so that $\hat{y}(kT) = y(kT)$. The job of the computer is to take these sample values and compute in some fashion the signals to be put out through the digital-to-analog converter (D/A). The characteristics of the A/D and D/A converters will be discussed later. Here we consider the treatment of the data inside the computer. Suppose we call the input signals up to the kth sample $e_0, e_1, e_2, \ldots, e_k$, and the output signals prior to that time $u_0, u_1, u_2, \ldots, u_{k-1}$. Then, to get the next output, we have the machine compute some function, which we can express in symbolic form as

$$u_k = f(e_0, \ldots, e_k; u_0, \ldots, u_{k-1}). \tag{2.1}$$

Because we plan to emphasize the elementary and the dynamic possibilities, we assume that the function f in (2.1) is *linear* and depends on only a *finite* number of past e's and u's. Thus we write

$$u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \cdots - a_n u_{k-n} + b_0 e_k + b_1 e_{k-1} + \cdots + b_m e_{k-m}. \tag{2.2}$$

Equation (2.2) is called a linear recurrence equation or difference equation and, as we shall see, has many similarities with a linear differential equation. The name "difference equation" derives from the fact that we could write (2.2) using u_k plus the differences in u_k, which are defined as

$$\begin{aligned}
\nabla u_k &= u_k - u_{k-1} && \text{(first difference)},\\
\nabla^2 u_k &= \nabla u_k - \nabla u_{k-1} && \text{(second difference)},\\
\nabla^n u_k &= \nabla^{n-1} u_k - \nabla^{n-1} u_{k-1} && (n\text{th difference}).
\end{aligned} \tag{2.3}$$

If we solve (2.3) for the values of u_k, u_{k-1}, and u_{k-2} in terms of differences, we find

$$\begin{aligned}
u_k &= u_k,\\
u_{k-1} &= u_k - \nabla u_k,\\
u_{k-2} &= u_k - 2\nabla u_k + \nabla^2 u_k.
\end{aligned}$$

Thus, for a second-order equation with coefficients a_1, a_2, and b_0 (we let $b_1 = b_2 = 0$ for simplicity), we find the equivalent difference equation to be

$$a_2 \nabla^2 u_k - (a_1 + 2a_2)\nabla u_k + (a_2 + a_1 + 1)u_k = b_0 e_k.$$

Although the two forms are equivalent, the recurrence form of (2.2) is more convenient for computer implementation; we will drop the form using differences. We will continue, however, to refer to our equations as "difference equations." If the a's and b's in (2.2) are constant, then the computer is solving a *constant-coefficient difference equation* (CCDE). We plan to demonstrate later that with such equations the computer can control linear constant dynamic systems and approximate most of the other tasks of linear, constant, dynamic systems, including performing the functions of electronic filters. To do so, it is necessary first to examine methods of obtaining solutions to (2.2) and to study the general properties of these solutions.

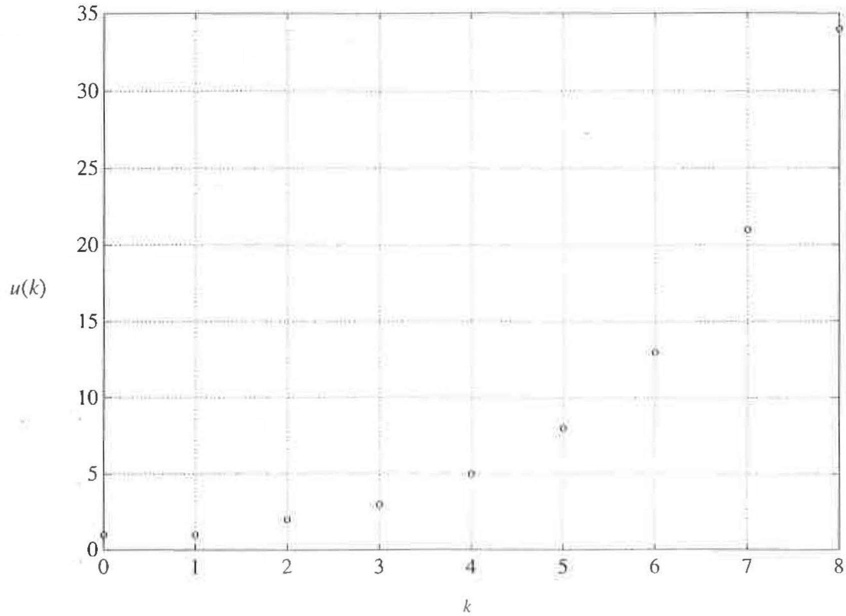


Figure 2.1 The Fibonacci numbers.

To solve a specific CCDE is an elementary matter. We need a starting time (k-value) and some initial conditions to characterize the contents of the computer memory at this time. For example, suppose we take the case

$$u_k = u_{k-1} + u_{k-2} \tag{2.4}$$

and start at $k = 2$. Here there are no input values, and to compute u_2 we need to know the (initial) values for u_0 and u_1. Let us take them to be $u_0 = u_1 = 1$. The first nine values are $1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots$. A plot of the values of u_k versus k is shown in Fig. 2.1.

The results, the Fibonacci numbers, are named after the thirteenth-century mathematician[1] who studied them. For example, (2.4) has been used to model the growth of rabbits in a protected environment.[2] However that may be, the output of the system represented by (2.4) would seem to be

[1] Leonardo Fibonacci of Pisa, who introduced Arabic notation to the Latin world about 1200 A.D.

[2] Wilde (1964). Assume that u_k represents pairs of rabbits and that babies are born in pairs. Assume that no rabbits die and that a new pair begin reproduction after one period. Thus at time k, we have all the old rabbits, u_{k-1}, plus the newborn pairs born to the mature rabbits, which are u_{k-2}.

growing, to say the least. If the response of a dynamic system to any finite initial conditions can grow without bound, we call the system *unstable*. We would like to be able to examine equations like (2.2) and, without having to solve them explicitly, see if they are stable or unstable and even understand the general shape of the solution.

One approach to solving this problem is to assume a form for the solution with unknown constants and to solve for the constants to match the given initial conditions. For continuous, ordinary, differential equations that are constant and linear, exponential solutions of the form e^{st} are used. In the case of linear, constant, difference equations, it turns out that solutions of the form z^k will do where z has the role of s and k is the discrete independent variable replacing time, t. Consider (2.4). If we assume that $u(k) = Az^k$, we get the equation

$$Az^k = Az^{k-1} + Az^{k-2}.$$

Now if we assume $z \neq 0$ and $A \neq 0$, we can divide by A and multiply by z^{-k}, with the result

$$1 = z^{-1} + z^{-2}$$

or

$$z^2 = z + 1.$$

This polynomial of second degree has two solutions, $z = 1/2 \pm \sqrt{5}/2$. Let's call these z_1 and z_2. Since our equation is linear, a sum of the individual solutions will also be a solution. Thus, we have found that a solution to (2.4) is of the form

$$u(k) = A_1 {z_1}^k + A_2 {z_2}^k.$$

We can solve for the unknown constants by requiring that this general solution satisfy the specific initial conditions given. If we substitute $k = 0$ and $k = 1$, we obtain the simultaneous equations

$$1 = A_1 + A_2,$$
$$1 = A_1 z_1 + A_2 z_2.$$

These equations are easily solved[3] to give

$$A_1 = \frac{1+\sqrt{5}}{2\sqrt{5}},$$

$$A_2 = \frac{\sqrt{5}-1}{2\sqrt{5}}.$$

And now we have the complete solution of (2.4) in a closed form. Furthermore, we can see that since $z_1 = (1+\sqrt{5})/2$ is greater than 1, the term in ${z_1}^k$ will grow without bound as k grows, which confirms our suspicion that the equation represents an unstable system. We can generalize this result. The equation in z that we obtain after we substitute $u = z^k$ is a polynomial in z known as the *characteristic equation* of the difference equation. If any solution of this equation is outside the unit circle (has a magnitude greater than one), the corresponding difference equation is unstable in the specific sense that for some finite initial conditions the solution will grow without bound as time goes to infinity. If *all* the roots of the characteristic equation are *inside* the unit circle, the corresponding difference equation is stable.

Example 2.1: Is the equation

$$u(k) = 0.9u(k-1) - 0.2u(k-2)$$

stable? The characteristic equation is

$$z^2 - 0.9z + 0.2 = 0,$$

and the characteristic roots are $z = 0.5$ and $z = 0.4$. Since both these roots are inside the unit circle, the equation is stable.

As an example of the origins of a difference equation with an external input, we consider the discrete approximation to integration. Suppose we have a continuous signal, $e(t)$, of which a segment is sketched in Fig. 2.2, and we wish to compute an approximation to the integral

$$\mathcal{J} = \int_0^t e(t)\,dt, \tag{2.5}$$

[3] Do it!

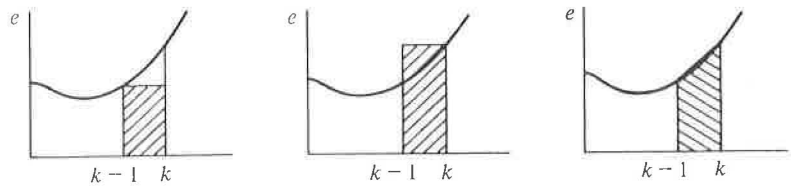


Figure 2.2 Plot of a function and alternative approximations to the area under the curve over a single time interval.

using only the discrete values $e(0), \ldots, e(t_{k-1}), e(t_k)$. We assume that we have an approximation for the integral from zero to the time t_{k-1} and we call it u_{k-1}. The problem is to obtain u_k from this information. Taking the view of the integral as the area under the curve $e(t)$, we see that this problem reduces to finding an approximation to the area under the curve between t_{k-1} and t_k. Three alternatives are sketched in Fig. 2.2. We can use the rectangle of height e_{k-1}, or the rectangle of height e_k, or the trapezoid formed by connecting e_{k-1} to e_k by a straight line. If we take the third choice, the area of the trapezoid is

$$A = \frac{t_k - t_{k-1}}{2}(e_k + e_{k-1}). \tag{2.6}$$

Finally, if we assume that the sampling period, $t_k - t_{k-1}$, is a constant, T, we are led to a simple formula for discrete (trapezoid rule) integration:

$$u_k = u_{k-1} + \frac{T}{2}(e_k + e_{k-1}). \tag{2.7}$$

If $e(t) = t$, then $e_k = kT$ and substitution of $u_k = (T^2/2)k^2$ satisfies (2.7) and is exactly the integral of e. [It should be, because if $e(t)$ is a straight line, the trapezoid is the *exact* area.] If we approximate the area under the curve by the rectangle of height e_{k-1}, the result is called the Forward Rectangular Rule and is described by

$$u_k = u_{k-1} + Te_{k-1}.$$

A third possibility is the Backward Rectangular Rule, given by

$$u_k = u_{k-1} + Te_k.$$

Each of these integration rules is a special case of our general difference equation (2.2). We will examine the properties of these rules later, in Chapter 4, while discussing means to obtain a difference equation that will be equivalent to a given differential equation.

Thus we see that difference equations can be evaluated directly by a digital computer and that they can represent models of physical processes and approximations to integration. It turns out that if the difference equations are linear with coefficients that are constant, we can describe the relation between u and e by a transfer function, and thereby gain a great aid to analysis and also to the design of linear, constant, discrete controls.

2.3 THE DISCRETE TRANSFER FUNCTION

We will obtain the transfer function of linear, constant, discrete systems by the method of z-transform analysis. A logical alternative viewpoint that requires a bit more mathematics but has some appeal is given in Section 2.7.2. The results are the same. We also show how these same results can be expressed in the state space form in Section 2.3.3.

2.3.1 The z-Transform

If a signal has discrete values $e_0, e_1, \ldots, e_k, \ldots$ we define the z-transform of the signal as the function[4,5]

$$\begin{aligned} E(z) &\triangleq \mathcal{Z}\{e_k\} \\ &\triangleq \sum_{k=-\infty}^{\infty} e_k z^{-k}, \qquad r_0 < |z| < R_0, \end{aligned} \tag{2.8}$$

[4] We use the notation $\triangleq$ to mean "is defined as."

[5] In (2.8) the lower limit is $-\infty$ so that values of e_k on both sides of $k = 0$ are included. The transform so defined is sometimes called the two-sided z-transform to distinguish it from the one-sided definition, which would be $\sum_0^\infty e_k z^{-k}$. For signals that are zero for $k < 0$, the transforms obviously give identical results. To take the one-sided transform of u_{k-1}, however, we must handle the value of u_{-1}, and thus are initial conditions introduced by the one-sided transform. Examination of this property and other features of the one-sided transform are invited by the problems. We select the two-sided transform because we need to consider signals that extend into negative time when we study random signals in Chapter 8.

and we assume we can find values of r_0 and R_0 as bounds on the magnitude of the complex variable z for which the series (2.8) converges. A discussion of convergence is deferred until Section 2.7.

Example 2.2: As an example to illustrate (2.8), consider that the data e_k are taken as samples from the time signal $e^{-at}1(t)$ at sampling period T where $1(t)$ is the unit step function, zero for negative t, and one for positive t. Then $e_k = e^{-akT}1(kT)$. The z-transform of this is

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} e_k z^{-k} &= \sum_{0}^{\infty} e^{-akT} z^{-k} \\
&= \sum_{0}^{\infty} (e^{-aT} z^{-1})^k \\
&= \frac{1}{1 - e^{-aT} z^{-1}} \qquad e^{-aT} < |z| < \infty \\
&= \frac{z}{z - e^{-aT}} \qquad |z| > e^{-aT}.
\end{aligned}$$

We will return to the analysis of signals and development of a table of useful z-transforms in Section 2.5; we first examine the use of the transform to reduce difference equations to algebraic equations and techniques for representing these as block diagrams.

2.3.2 The Transfer Function

The z-transform has the same role in discrete systems that the Laplace transform has in analysis of continuous systems. For example, the z-transforms for e_k and u_k in the difference equation (2.2) or in the trapezoid integration (2.7) are related in a simple way that permits the rapid solution of linear, constant, difference equations of this kind. To find the relation, we proceed by direct substitution. We take the definition given by (2.8) and, in the same way, we define the z-transform of the sequence $\{u_k\}$ as

$$U(z) \triangleq \sum_{k=-\infty}^{\infty} u_k z^{-k}. \tag{2.9}$$

Now we multiply (2.7) by z^{-k} and sum over k. We get

$$\sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} u_{k-1} z^{-k} + \frac{T}{2}\left(\sum_{k=-\infty}^{\infty} e_k z^{-k} + \sum_{k=-\infty}^{\infty} e_{k-1} z^{-k}\right). \quad (2.10)$$

From (2.9), we recognize the left-hand side as $U(z)$. In the first term on the right, we let $k-1=j$ to obtain

$$\sum_{k=-\infty}^{\infty} u_{k-1} z^{-k} = \sum_{j=-\infty}^{\infty} u_j z^{-(j+1)} = z^{-1}U(z). \quad (2.11)$$

By similar operations on the third and fourth terms we can reduce (2.10) to

$$U(z) = z^{-1}U(z) + \frac{T}{2}[E(z) + z^{-1}E(z)]. \quad (2.12)$$

Equation (2.12) is now simply an algebraic equation in z and the functions U and E. Solving it we obtain

$$U(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}E(z). \quad (2.13)$$

We *define* the ratio of the transform of the output to the transform of the input as the *transfer function*, $H(z)$. Thus, in this case, the transfer function for trapezoid-rule integration is

$$\frac{U(z)}{E(z)} \triangleq H(z) = \frac{T}{2}\frac{z+1}{z-1}. \quad (2.14)$$

For the more general relation given by (2.2), it is readily verified by the same techniques that

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}},$$

and if $n \geq m$, we can write this as a ratio of polynomials in z as

$$\begin{aligned} H(z) &= \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} \\ &= \frac{b(z)}{a(z)}. \end{aligned} \quad (2.15)$$

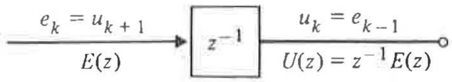


Figure 2.3 The unit delay.

The general input–output relation between transforms with linear, constant, difference equations is

$$U(z) = H(z)E(z). \tag{2.16}$$

Although we have developed the transfer function with the z-transform, it is also true that the transfer function is the ratio of the output to the input when both vary as z^k.

Because $H(z)$ is a rational function of a complex variable, we use the terminology of that subject. Suppose we call the numerator polynomial $b(z)$ and the denominator $a(z)$. The places in z where $b(z) = 0$ are *zeros* of the transfer function, and the places in z where $a(z) = 0$ are the *poles* of $H(z)$. If z_0 is a pole and $(z - z_0)^p H(z)$ has neither pole nor zero at z_0, we say that $H(z)$ has a pole of order p at z_0. If $p = 1$, the pole is simple. The transfer function (2.14) has a simple pole at $z = 1$ and a simple zero at $z = -1$.

We can now give a physical meaning to the variable z. Suppose we let all coefficients in (2.15) be zero except b_1 and we take b_1 to be 1. Then $H(z) = z^{-1}$. But $H(z)$ represents the transform of (2.2), and with these coefficient values the difference equation reduces to

$$u_k = e_{k-1}. \tag{2.17}$$

The present value of the output, u_k, equals the input *delayed by one period.* Thus we see that a transfer function of z^{-1} is a *delay* of one time unit. We can picture the situation as in Fig. 2.3, where both time and transform relations are shown.

Since the relations of (2.7), (2.14), and (2.15) are all composed of delays, they can be expressed in terms of z^{-1}! Consider (2.7). In Fig. 2.4 we illustrate the difference equation (2.7) using the transfer function z^{-1} as the symbol for a unit delay.

We can follow the operations of the discrete integrator by tracing the signals through Fig. 2.4. For example, the present value of e_k is passed to the first summer, where it is added to the previous value e_{k-1}, and the sum is multiplied by $T/2$ to compute the area of the trapezoid between e_{k-1} and e_k. This is the signal marked a_k in Fig. 2.4. After this, there is another sum, where the previous output, u_{k-1}, is added to the new area to form the next

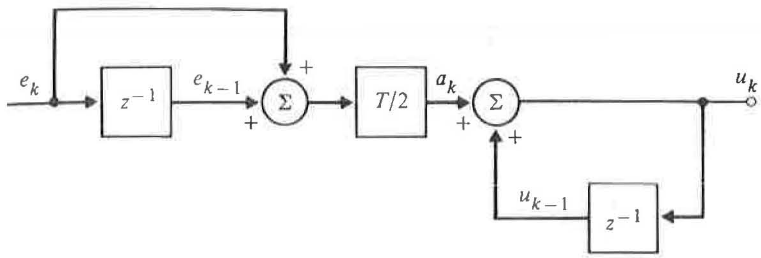


Figure 2.4 A block diagram of trapezoid integration as represented by (2.7).

value of the integral estimate, u_k. The discrete integration occurs in the loop with one delay, z^{-1}, and unity gain.

2.3.3 Block Diagrams and State-Variable Descriptions

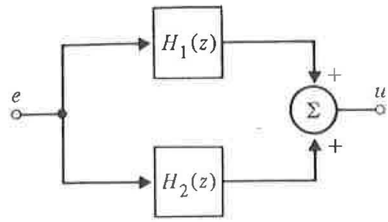
Because (2.16) is a linear algebraic relationship, a system of such relations is described by a system of linear equations. These can be solved by the methods of linear algebra or by the graphical methods of block diagrams. To use block-diagram analysis to manipulate these discrete-transfer-function relationships, there are only four primitive cases:

1. The transfer function of paths in parallel is the sum of the single-path transfer functions (Fig. 2.5).
2. The transfer function of paths in series is the *product* of the path transfer functions (Fig. 2.6).
3. The transfer function of a single loop of paths is the transfer function of the forward path divided by one minus the loop transfer function (Fig. 2.7).
4. The transfer function of an arbitrary multipath diagram is given by combinations of these cases. Mason's rule[6] can also be used.

For the general difference equation of (2.2), we already have the transfer function in (2.15). It is interesting to connect this case with a block diagram *using only simple delay forms for z* in order to see several "canonical" block diagrams and to introduce the description of discrete systems using equations of state.

There are many ways to reduce the difference equation (2.2) to a block diagram involving z only as the delay operator, z^{-1}. The first one we will

[6]Mason (1956). See Franklin, Powell, and Emami-Naeini(1986) for a discussion.



$H(z) = H_1(z) + H_2(z)$

Figure 2.5 Block diagram of parallel blocks.

consider leads to the "control" canonical form. We begin with the transfer function as a ratio of polynomials

$$U(z) = H(z)E(z) = \frac{b(z)}{a(z)}E(z) = b(z)\xi,$$

where

$$\xi = \frac{E(z)}{a(z)}$$

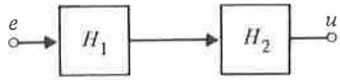
and thus

$$a(z)\xi = E(z).$$

At this point we need to get specific; and rather than carry through with a system of arbitrary order, we will work out the details for the third-order case and leave it to the reader to extend the results in the obvious way to whatever order is desired. In the development that follows, we will consider the variables u, e, and ξ as *time* variables and z as an advance operator such that $zu(k) = u(k+1)$ or $z^{-1}u = u(k-1)$. With this convention (which is simply using the property of z derived earlier), consider the equations

$$(z^3 + a_1z^2 + a_2z + a_3)\xi = e, \tag{2.18}$$

$$(b_0z^3 + b_1z^2 + b_2z + b_3)\xi = u. \tag{2.19}$$



$H(z) = H_1(z)H_2(z)$

Figure 2.6 Block diagram of cascade blocks.

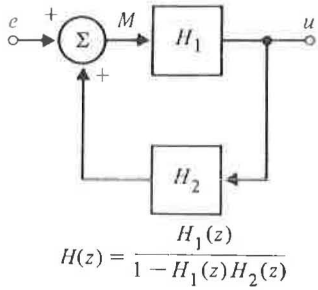


Figure 2.7 Feedback transfer function.

We can write (2.18) as

$$z^3\xi = e - a_1 z^2 \xi - a_2 z\xi - a_3\xi$$

$$\xi(k+3) = e(k) - a_1\xi(k+2) - a_2\xi(k+1) - a_3\xi(k). \tag{2.20}$$

Now assume we have $z^3\xi$, which is to say that we have $\xi(k+3)$ because z^3 is an advance operator of three steps. If we operate on this with z^{-1} three times in a row, we will get back to $\xi(k)$, as shown in Fig. 2.8(a). From (2.20), we can now compute $z^3\xi$ from e and the lower powers of z and ξ given in the block diagram; the picture is now as given in Fig. 2.8(b). To complete the representation of (2.18) and (2.19), we need only add the formation of the output u as a weighted sum of the variables $z^3\xi$, $z^2\xi$, $z\xi$, and ξ according to (2.19). The completed picture is shown in Fig. 2.8(c).

In Fig 2.8(c), the internal variables have been named x_1, x_2, and x_3. These variables comprise the *state* of this dynamic system in this form. Having the block diagram shown in Fig. 2.8(c), we can write down, almost by inspection, the difference equations that describe the evolution of the state, again using the fact that the transfer function z^{-1} corresponds to a one-unit delay. For example, we see that $x_3(k+1) = x_2(k)$ and $x_2(k+1) = x_1(k)$. Finally, expressing the sum at the far left of the figure, we have

$$x_1(k+1) = -a_1x_1(k) - a_2x_2(k) - a_3x_3(k) + e(k).$$

We collect these three equations together in proper order, and we have

$$\begin{aligned} x_1(k+1) &= -a_1x_1(k) - a_2x_2(k) - a_3x_3(k) + e(k), \\ x_2(k+1) &= x_1(k), \\ x_3(k+1) &= x_2(k). \end{aligned} \tag{2.21}$$

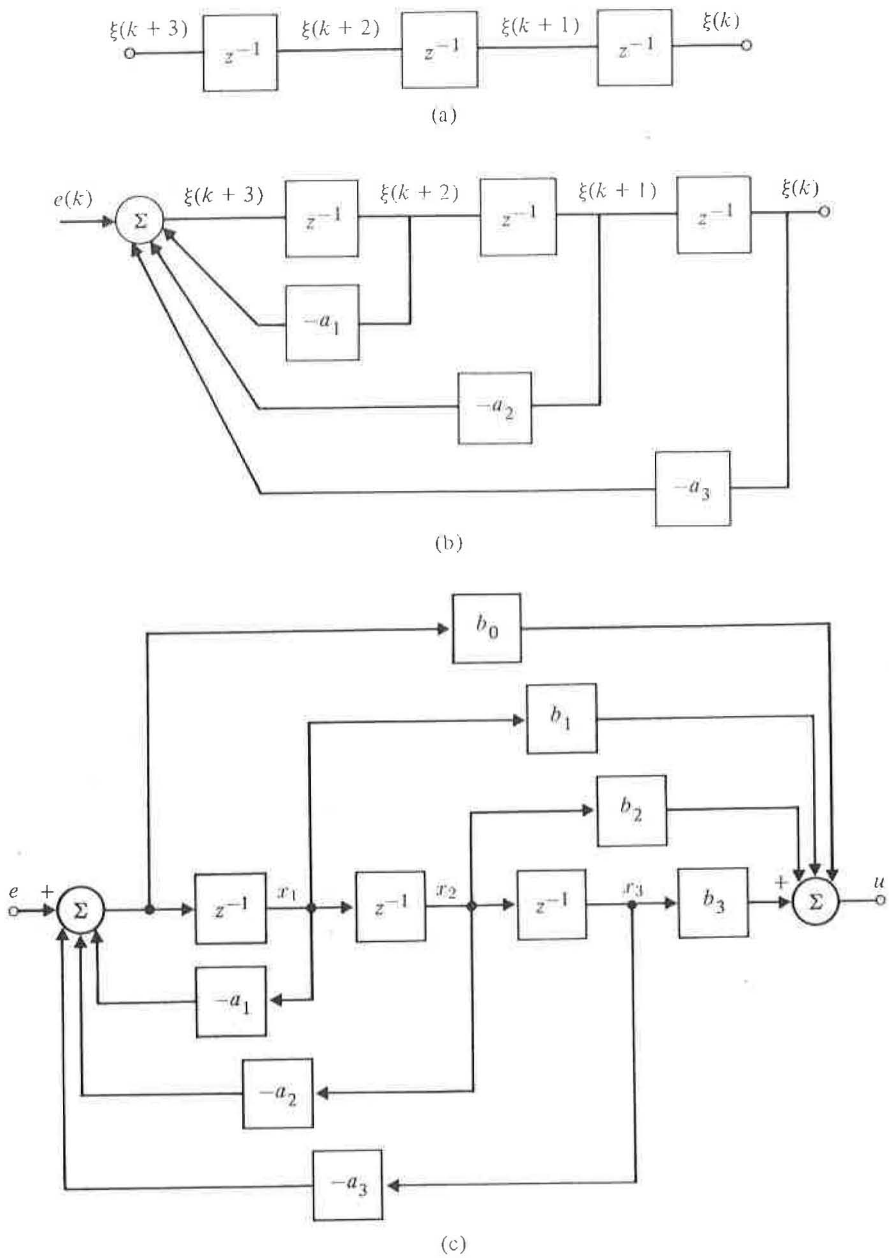


Figure 2.8 Block diagram development of control canonical form. (a) Solving for $\xi(k)$; (b) solving for $\xi(k+3)$ from $e(k)$ and past ξ's; (c) solving for $U(k)$ from ξ's.

Using vector-matrix notation,[7] we can write this in the compact form

$$\mathbf{x}(k+1) = \mathbf{A}_c\mathbf{x}(k) + \mathbf{B}_c e(k),$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},$$

$$\mathbf{A}_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{2.22a}$$

and

$$\mathbf{B}_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2.22b}$$

The output equation is also immediate except that we must watch to catch *all* paths by which the state variables combine in the output. The problem is caused by the b_0 term. If $b_0 = 0$, then $u = b_1x_1 + b_2x_2 + b_3x_3$, and the corresponding matrix form is immediate. However, if b_0 is not 0, x_1 for example not only reaches the output through b_1 but also by the parallel path with gain $-b_0a_1$. The complete equation is

$$u = (b_1 - a_1b_0)x_1 + (b_2 - a_2b_0)x_2 + (b_3 - a_3b_0)x_3) + b_0e.$$

In vector/matrix notation, we have

$$u = \mathbf{C}_c x + \mathbf{D}_c e,$$

where

$$\mathbf{C}_c = [\, b_1 - a_1b_0 \quad b_2 - a_2b_0 \quad b_3 - a_3b_0 \,], \tag{2.23a}$$

$$\mathbf{D}_c = [\, b_0 \,]. \tag{2.23b}$$

[7] We assume the reader has some knowledge of matrices. The results we require and references to study material are given in Appendix C. To distinguish vectors and matrices from scalar variables, we will use bold-face type.

We can combine the equations for the state evolution and the output to give the very useful and most compact equations for the dynamic system,

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}_c\mathbf{x}(k) + \mathbf{B}_c e(k),\\ u(k) &= \mathbf{C}_c x(k) + \mathbf{D}_c e(k),\end{aligned} \tag{2.24}$$

where $\mathbf{A}_c$ and $\mathbf{B}_c$ for this control canonical form are given by (2.22), and $\mathbf{C}_c$ and $\mathbf{D}_c$ are given by (2.23).

The other canonical form we want to illustrate is called the observer canonical form and is found by starting with the difference equations in operator/transform form as

$$z^3u + a_1z^2u + a_2zu + a_3u = b_0z^3e + b_1z^2e + b_2ze + b_3e.$$

In this equation, the external input is $e(k)$, and the response is $u(k)$, which is the solution of this equation. The terms with factors of z are time-shifted toward the future with respect to k and must be eliminated in some way. To do this, we assume at the start that we have the $u(k)$, and of course the $e(k)$, and we rewrite the equation as

$$b_3e - a_3u = z^3u + a_1z^2u + a_2zu - b_0z^3e - b_1z^2e - b_2ze.$$

Here, *every* term on the right is multiplied by at least one power of z, and thus we can operate on the lot by z^{-1} as shown in the partial block diagram drawn in Fig. 2.9(a).

Now in this internal result there appear a_2u and $-b_2e$, which can be cancelled by adding proper multiples of u and e, as shown in Fig. 2.9(b), and once they have been removed, the remainder can again be operated on by z^{-1}.

If we continue this process of subtracting out the terms at k and operating on the rest by z^{-1}, we finally arrive at the place where all that is left is u alone! But that is just what we assumed we had in the first place, so connecting this term back to the start finishes the block diagram, which is drawn in Fig. 2.9(c).

A preferred choice of numbering for the state components is also shown in the figure. Following the technique used for the control form, we find that the matrix equations are given by

$$\begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}_0\mathbf{x}(k) + \mathbf{B}_0 e(k),\\ u(k) &= \mathbf{C}_0 x(k) + \mathbf{D}_0 e(k),\end{aligned} \tag{2.25}$$

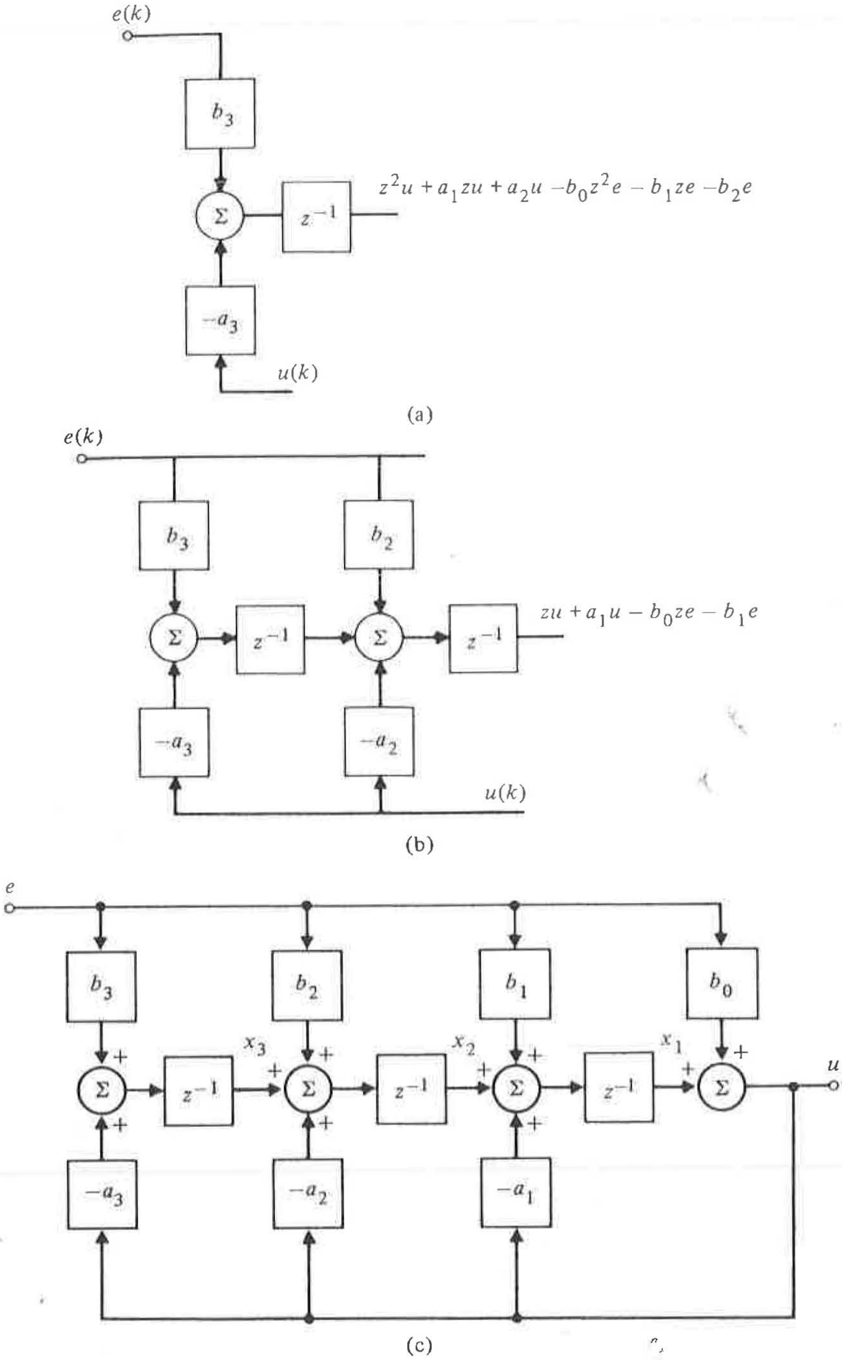


Figure 2.9 Block diagram development of observer canonical form. (a) The first partial sum and delay; (b) the second partial sum and delay; (c) the completion with the solution for $u(k)$.

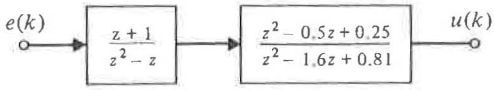


Figure 2.10 Block diagram of a cascade realization.

where

$$\mathbf{A}_0 = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix},$$

$$\mathbf{B}_0 = \begin{bmatrix} b_1 - b_0 a_1 \\ b_2 - b_0 a_2 \\ b_3 - b_0 a_3 \end{bmatrix},$$

$$\mathbf{C}_0 = [\,1 \quad 0 \quad 0\,],$$

$$\mathbf{D}_0 = [\,b_0\,].$$

The block diagrams of Figs. 2.8 and 2.9 are called *direct canonical* realizations of the transfer function $H(z)$ because the gains of the realizations are coefficients in the transfer-function polynomials. Another useful form is obtained if we realize a transfer function by placing several first- or second-order direct forms in series with each other, a *cascade canonical* form. In this case, the $H(z)$ is represented as a product of factors, and the poles and zeros of the transfer function are clearly represented in the coefficients.

For example, suppose we have a transfer function

$$H(z) = \frac{z^3 + 0.5z^2 - 0.25z + 0.25}{z^4 - 2.6z^3 + 2.4z^2 - 0.8z} = \frac{(z+1)(z^2 - 0.5z + 0.25)}{(z^2 - z)(z^2 - 1.6z + 0.8)}.$$

The zero factor $z+1$ can be associated with the pole factor $z^2 - z$ to form one second-order system, and the zero factor $z^2 - 0.5z + 0.25$ can be associated with the second-order pole factor $z^2 - 1.6z + 0.8$ to form another. The cascade factors, which could be realized in a direct form such as control or observer form, make a cascade form as shown in Fig. 2.10.

2.3.4 Relation of Transfer Function to Pulse Response

We have shown that a transfer function of z^{-1} is a unit delay in the time domain. We can also give a time-domain meaning to an arbitrary transfer func-

Table 2.1 Step-by-step construction of the unit pulse response for Fig. 2.4.

k	e_{k-1}	e_k	a_k	u_{k-1}	$u_k \equiv h_k$
0	0	1	$T/2$	0	$T/2$
1	1	0	$T/2$	$T/2$	T
2	0	0	0	T	T
3	0	0	0	T	T

tion. Recall that the z-transform is defined by (2.8) to be $E(z) = \Sigma e_k z^{-k}$, and the transfer function is defined from (2.16) as $H(z)$ when the input and output are related by $U(z) = H(z)E(z)$. Now suppose we deliberately select $e(k)$ to be the unit discrete pulse defined by

$$\begin{aligned} e_k &= \begin{cases} 1, & (k = 0), \\ 0, & (k \neq 0), \end{cases} \\ &\triangleq \delta_k. \end{aligned} \tag{2.26}$$

Then it follows that $E(z) = 1$ and therefore that

$$U(z) = H(z). \tag{2.27}$$

Thus the transfer function $H(z)$ is seen to be the *transform* of the response to a unit-pulse input. For example, let us look at the system of Fig. 2.4 and put a unit pulse in at the e_k-node (with no signals in the system beforehand).[8] We can readily follow the pulse through the block and build Table 2.1.

Thus the unit-pulse response is zero for negative k, is $T/2$ at $k = 0$, and equals T thereafter. The z-transform of this sequence is

$$H(z) = \sum_{-\infty}^{\infty} u_k z^{-k} \triangleq \sum_{-\infty}^{\infty} h_k z^{-k}.$$

[8]In this development we assume that (2.7) is intended to be used as a formula for computing values of u_k as k *increases*. There is no reason why we could not also solve for u_k as k takes on negative values. The direction of time comes from the application and not from the recurrence equation.

$e_0 \quad +e_1 z^{-1}$	$+e_2 z^{-2}$	$+e_3 2^{-3}$	$+\cdots$
$h_0 \quad +h_1 z^{-1}$	$+h_2 z^{-3}$	$+h_3 2^{-3}$	$+\cdots$
$e_0 h_0 + e_1 h_0 z^{-1}$	$+e_2 h_0 z^{-2}$	$+e_3 h_0 z^{-3}$	
$+e_0 h_1 z^{-1}$	$+e_1 h_1 z^{-2}$	$+e_2 h_1 z^{-3}$	
	$+e_0 h_2 z^{-2}$	$+e_1 h_2 z^{-3}$	
		$+e_0 h_3 z^{-3}$	
$e_0 h_0 + (e_0 h_1 + e_1 h_0) z^{-1}$	$+(e_0 h_2 + e_1 h_1 + e_2 h_0) z^{-2}$	$+(e_0 h_3 + e_1 h_2 + e_2 h_1 + e_3 h_0) z^{-3}$	$+\cdots$

Figure 2.11 Representation of the product $E(z)H(z)$ as a product of polynomials.

If we add $T/2$ to the z^0-term and subtract $T/2$ from the whole series, we have a simpler sum, as follows:

$$
\begin{aligned}
H(z) &= \sum_{k=0}^{\infty} T z^{-k} - \frac{T}{2} \\
&= \frac{T}{1 - z^{-1}} - \frac{T}{2} \qquad (1 < |z|) \\
&= \frac{2T - T(1 - z^{-1})}{2(1 - z^{-1})} \\
&= \frac{T + T z^{-1}}{2(1 - z^{-1})} \\
&= \frac{T}{2}\frac{z + 1}{z - 1} \qquad (1 < |z|). \qquad (2.28)
\end{aligned}
$$

Of course, this is the transfer function we obtained in (2.13) from direct analysis of the difference equation.

A final point of view useful in the interpretation of the discrete transfer function is obtained by multiplying the infinite polynomials of $E(z)$ and $H(z)$ as suggested in (2.16). For purposes of illustration, we will assume that the unit-pulse response, h_k, is zero for $k < 0$. Likewise, we will take $k = 0$ to be the starting time for e_k. Then the product that produces $U(z)$ is the polynomial product given in Fig. 2.11.

Since this product has been shown to be $U(z) = \Sigma u_k z^{-k}$, it must therefore follow that the coefficient of z^{-k} in the product is u_k. Listing these coefficients, we have the relations

$$
\begin{aligned}
u_0 &= e_0 h_0, \\
u_1 &= e_0 h_1 + e_1 h_0, \\
u_2 &= e_0 h_2 + e_1 h_1 + e_2 h_0, \\
u_3 &= e_0 h_3 + e_1 h_2 + e_2 h_1 + e_3 h_0.
\end{aligned}
$$

The extrapolation of this simple pattern gives the result

$$u_k = \sum_{j=0}^{k} e_j h_{k-j}.$$

By extension, we let the lower limit of the sum be $-\infty$ and the upper limit be $+\infty$:

$$u_k = \sum_{j=-\infty}^{\infty} e_j h_{k-j}. \tag{2.29}$$

Negative values of j in the sum correspond to inputs applied before time equals zero. Values for j greater than k occur if the unit-pulse response is nonzero for negative arguments. By definition, such a system, which responds *before* the input that causes it occurs, is called *noncausal*. This is the discrete convolution sum and is the analog of the convolution integral that relates input and impulse response to output in linear, constant, continuous systems.

To verify (2.29) we can take the z-transform of both sides:

$$\sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} z^{-k} \sum_{j=-\infty}^{\infty} e_j h_{k-j}.$$

Interchanging the sum on j with the sum on k leads to

$$U(z) = \sum_{j=-\infty}^{\infty} e_j \sum_{k=-\infty}^{\infty} z^{-k} h_{k-j}.$$

Now let $k - j = l$ in the second sum:

$$U(z) = \sum_{j=-\infty}^{\infty} e_j \sum_{l=-\infty}^{\infty} h_l z^{-(l+j)},$$

but $z^{-(l+j)} = z^{-l}z^{-j}$, which leads to

$$U(z) = \sum_{j=-\infty}^{\infty} e_j z^{-j} \sum_{l=-\infty}^{\infty} h_l z^{-l},$$

and we recognize these two separate sums as

$$U(z) = E(z)H(z).$$

We can also derive the convolution sum from the properties of linearity and stationarity. First we need more formal definitions of "linear" and "stationary."

1. *Linearity:* A system with input e and output u is *linear* if superposition applies, which is to say, if $u_1(k)$ is the response to $e_1(k)$ and $u_2(k)$ is the response to $e_2(k)$, then the system is linear if and only if, for every scalar α and β, the response to $\alpha e_1 + \beta e_2$ is $\alpha u_1 + \beta u_2$.
2. *Stationarity:* A system is *stationary*, or time invariant, if a time shift in the input results in only a time shift in the output. For example, if we take the system at rest (no internal energy in the system) and apply a certain signal $e(k)$, suppose we observe a response $u(k)$. If we repeat this experiment at any later time when the system is again at rest and we apply the shifted input, $e(k-N)$, if we see $u(k-N)$, then the system is stationary.

These properties can be used to derive the convolution in (2.29) as follows. If response to a unit pulse at $k = 0$ is $h(k)$, then response to a pulse of intensity e_0 is $e_0 h(k)$ if the system is linear. Furthermore, if the system is *constant,* then a delay of the input will delay the response. Thus, if

$$e = \begin{cases} e_l, & k = l, \\ 0, & k \neq l, \end{cases}$$

then the response will be $e_l h_{k-l}$.

Finally, by linearity again, the total response at time k to a sequence of these pulses is the *sum* of the responses, namely,

$$u_k = e_0 h_k + e_1 h_{k-1} + \cdots + e_l h_{k-l} + \cdots + e_k h_0,$$

or

$$u_k = \sum_{l=0}^{k} e_l h_{k-l}.$$

Now note that if the input sequence began in the distant past, we must include terms for $l < 0$, perhaps back to $l = -\infty$. Similarly, if the system should be noncausal, future values of e where $l > k$ may also come in. The general case is thus (again)

$$u_k = \sum_{l=-\infty}^{\infty} e_l h_{k-l}. \tag{2.30}$$

2.3.5 External Stability and Jury's Test

A very important qualitative property of a dynamic system is stability, and we can consider internal or external stability. Internal stability is concerned with the responses at all the internal variables such as those that appear at the delay elements in a canonical block diagram as in Fig. 2.8 or Fig. 2.9 (the state). Otherwise we can be satisfied to consider only the *external stability* as given by the study of the input–output relation described for the linear stationary case by the convolution (2.30). These differ in that some internal modes might not be connected to both the input and the output of a given system.

For external stability, the most common definition of *appropriate response* is that for every Bounded Input, we should have a Bounded Output. If this is true we say the system is BIBO stable. A test for BIBO stability can be given directly in terms of the unit-pulse response, h_k. First we consider a sufficient condition. Suppose the input e_k is bounded, that is, there is an M such that

$$|e_l| \le M < \infty \qquad \text{for all } l. \tag{2.31}$$

If we consider the magnitude of the response given by (2.30), it is easy to see that

$$|u_k| \le \left|\sum e_l h_{k-l}\right|,$$

which is surely less than the sum of the magnitudes as given by

$$\le \sum_{-\infty}^{\infty} |e_l|\,|h_{k-l}|.$$

But, because we assume (2.31), this result is in turn bounded by

$$\le M \sum_{-\infty}^{\infty} |h_{k-l}|. \tag{2.32}$$

Thus the output will be bounded for every bounded input if

$$\sum_{l=-\infty}^{\infty} |h_{k-l}| < \infty. \tag{2.33}$$

This condition is also necessary, for if we consider the bounded (by 1!) input

$$\begin{aligned} e_l &= \frac{h_{-l}}{|h_{-l}|} \qquad && (h_{-l} \neq 0) \\ &= 0 && (h_{-l} = 0) \end{aligned}$$

and apply it to (2.30), the output at $k = 0$ is

$$\begin{aligned} u_0 &= \sum_{l=-\infty}^{\infty} e_l h_{-l} \\ &= \sum_{l=-\infty}^{\infty} \frac{(h_{-l})^2}{|h_{-l}|} \\ &= \sum_{l=-\infty}^{\infty} |h_{-l}|. \end{aligned} \tag{2.34}$$

Thus, unless the condition given by (2.34) is true, the system is not BIBO stable.

Example 2.3: The test given by (2.34) can be applied to the unit pulse response used to compute (2.13) and given as the u_k-column in Table 2.1 on page 31:

$$\begin{aligned} h_0 &= T/2, \\ h_k &= T, \qquad k > 0, \end{aligned}$$

$$\sum |h_k| = T/2 + \sum_{1}^{\infty} T = \text{unbounded}. \tag{2.35}$$

Thus this discrete approximation to integration is not (BIBO) stable!

Example 2.4: As a second example, we consider the difference equation (2.2) with all coefficients except a_1 and b_0 equal to zero:

$$u_k = a_1 u_{k-1} + b_0 e_k. \tag{2.36}$$

The unit-pulse response is easily developed from the first few terms to be

$$\begin{aligned} u_0 &= b_0, \qquad u_1 = a_1 b_0, \qquad u_2 = a_1^2 b_0, \ldots \\ u_k &= h_k = b_0 a^k, \qquad k \geq 0. \end{aligned} \tag{2.37}$$

Applying the test, we have

$$\begin{aligned} \sum_{-\infty}^{\infty} |h_l| = \sum_{\infty=0}^{\infty} b_0 |a^l| &= b_0 \frac{1}{1-|a|} \qquad (|a| < 1) \\ &= \text{unbounded} \qquad (|a| \geq 1). \end{aligned}$$

Thus we conclude that the system described by this equation is BIBO stable if $|a| < 1$, and unstable otherwise.

For a more general rational transfer function with many simple poles, we can expand the function in partial fractions about its poles, and the corresponding pulse response will be a sum of respective terms. As we saw earlier, if a pole is inside the unit circle, the corresponding pulse response decays with time geometrically and is stable. Thus, if all poles are inside the unit circle, the system with rational transfer function is stable; if at least one pole is on or outside the unit circle, the corresponding system is not BIBO stable. With modern computer programs available, finding the poles of a particular transfer function is no big deal. Sometimes, however, we wish to test for stability of an entire class of systems; or, as in an adaptive control system, the potential poles are constantly changing and we wish to have a quick test for stability in terms of the literal polynomial coefficients. In the continuous case, such a test was provided by Routh; in the discrete case, the most convenient such test was worked out by Jury and Blanchard(1961).

The Jury test is in the same spirit as the Routh test (see Franklin, Powell, and Emami-Naeini for a discussion) in that we form two rows from the coefficients of length n, and from these, by a series of two-by-two determinants, we compute a successor row of length $n-1$. With this reduced length row we form another successor of length $n-2$ and so on until we have a row of length 1. The test consists of examining the sign of the first entries in selected rows. As with the Routh test, the Jury test is much more difficult to derive than to use; here we illustrate only the use.

If we have a transfer function $H(z) = b(z)/a(z)$, then this system will be stable if and only if all roots of $a(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ are inside the unit circle. To test for this condition by the Jury test, multiply $a(z)$ by -1 if necessary to make the sign of a_0 positive. Then form rows of the coefficients, the even rows being in reversed order, as follows:

$$\begin{array}{cccc} a_0 & a_1 & \cdots & a_n \\ a_n & a_{n-1} & \cdots & a_0 \\ b_0 & b_1 & \cdots & \cdot \\ b_{n-1} & b_{n-2} & \cdots & \cdot \end{array}$$

The entries in the third row are formed from the second-order determinants using the first column of the first two rows with each of the other columns from these rows *starting from the right* and dividing by a_0. The result can be expressed by the formulas:

$$b_0 = a_0 - \frac{a_n}{a_0} a_n,$$

$$b_1 = a_1 - \frac{a_n}{a_0} a_{n-1},$$

$$b_k = a_k - \frac{a_n}{a_0} a_{n-k},$$

The elements in the third row are reversed to form the fourth row and the process is repeated. For example, the elements of the fifth row are given by

$$c_k = b_k - \frac{b_{n-1}}{b_0} b_{n-1-k}.$$

The original polynomial is stable (has all roots inside the unit circle) if all the terms in the first columns of the odd rows are positive, that is, if $a_0 > 0, b_0 > 0, c_0 > 0, \ldots$.

This test is readily implemented in a computer program.[9]

[9] See STABLE in Table E.1 in Appendix E.

Example 2.5: To illustrate the use of Jury's test, we consider first the simple second-order polynomial

$$a(z) = z^2 + a_1 z + a_2.$$

The Jury array is

$$\begin{array}{ccc}
1 & a_1 & a_2 \\
a_2 & a_1 & 1 \\
1 - a_2^2 & a_1 - a_1 a_2 & . \\
a_1 - a_1 a_2 & 1 - a_2^2 & . \\
\dfrac{(1 - a_2^2)^2 - a_1^2(1 - a_2)^2}{1 - a_2^2} & . & .
\end{array}$$

From row three, we have the condition that $1 - a_2{}^2 > 0$, and from this we conclude that

$$-1 < a_2 < 1.$$

From row five, we can factor out $(1 - a_2)^2$ to conclude that

$$(1 + a_2)^2 > a_1{}^2$$

and thus,

$$a_2 + 1 > a_1 \qquad \text{and} \qquad a_2 + 1 > -a_1.$$

From these inequalities, we can draw the *stability triangle* shown in Fig 2.12.

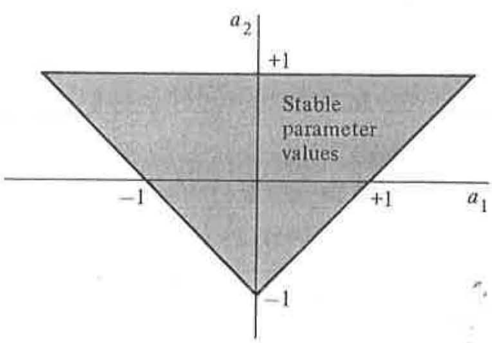


Figure 2.12 Stability triangle for the general, real, second-order polynomial.

Example 2.6: For the next example, consider the polynomial

$$a(z) = z^3 - 2.1z^2 + 1.6z - 0.4.$$

The Jury array is

$$\begin{array}{llll}
1 & -2.1 & 1.6 & -0.4 \\
-0.4 & 1.6 & -2.1 & 1 \\
0.84 & -1.46 & 0.76 & . \\
0.76 & -1.46 & 0.84 & . \\
0.1524 & -0.139 & . & . \\
-0.139 & 0.1524 & . & . \\
0.0256 & . & . & .
\end{array}$$

The test is from the odd rows: $1 > 0, 0.84 > 0, 0.1524 > 0$, $0.0256 > 0$, and we conclude that a system with this polynomial as its denominator would be stable. As a matter of fact, the poles are at $z = 0.5$ and $z = 0.8 \pm 0.4j$.

Example 2.7: As a third example, consider the polynomial

$$z^3 - 2.6z^2 + 2.4z - 0.8$$

The odd rows only of the Jury array are

$$\begin{array}{llll}
1 & -2.6 & 2.4 & -0.8 \\
0.36 & -0.68 & 0.32 & . \\
0.0756 & -0.0756 & . & . \\
0 & . & . & .
\end{array}$$

From these computations we conclude that the polynomial does *not* have all its roots inside the unit circle; and, because the last term is zero and a small perturbation would send it either way, inside or outside, there must be at least one root exactly on the unit circle. In fact, the roots are $z = 1$ and $z = 0.8 \pm 0.4j$.

As an aid to testing stability, it can be shown that it is *necessary* for a stable polynomial (with positive first term) that the polynomial evaluated at $z = 1$ and $z = -1$ must both be positive. The first value is just the sum of the coefficients, and the second is the sum with alternating sign changes. These two tests are quickly done and can save time if the only purpose is to be sure that the system is stable; many unstable systems will be rejected by these simple tests without going through the entire Jury array. In the previous case, for instance, the sum of coefficients is zero and we need go no further; the polynomial cannot be stable.

2.4 DISCRETE MODELS OF SAMPLED-DATA SYSTEMS

The systems and signals we have studied thus far have been defined in discrete time only. Most of the dynamic systems to be controlled, however, are continuous systems and, if linear, are described by continuous transfer functions in the Laplace variable s. The interface between the continuous and discrete domains are the A/D and the D/A converters as shown in Fig. 1.1. In this section we develop the analysis needed to compute the discrete transfer function between the samples that come from the digital computer to the D/A converter and the samples that are picked up by the A/D converter.[10] The situation is drawn in Fig. 2.13.

2.4.1 Using the z-Transform

We wish to find the discrete transfer function from the input samples $u(kT)$ (which probably come from a computer of some kind) to the output samples, $y(kT)$ picked up by the A/D converter. Although it is possibly confusing at first, we follow convention and call the discrete transfer function $G(z)$ when the continuous transfer function is $G(s)$. Although $G(z)$ and $G(s)$ are entirely different functions, they do describe the *same* plant, and the use of s for the continuous transform and z for the discrete transform is always maintained. To find $G(z)$ we need only observe that the $y(kT)$ are samples of the plant output when the input is from the D/A converter. As for the D/A converter, we assume that this device, commonly called a zero-order hold or ZOH, accepts a sample $u(kT)$ at $t = kT$ and holds its output constant

[10]In Chapter 3, a comprehensive frequency analysis of sampled data systems is presented. Here we undertake only the special problem of finding the sample-to-sample discrete transfer function of a continuous system between a D/A and an A/D.

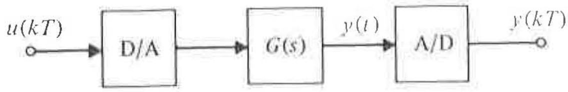


Figure 2.13 The prototype sampled-data system.

at this value until the next sample is sent at $t = kT + T$. The piecewis constant output of the D/A is the signal, $u(t)$, that is applied to the plant

Our problem is now really quite simple because we have just seen tha the discrete transfer function is the z-transform of the samples of the outpu when the input samples are the unit pulse at $k = 0$. If $u(kT) = 1$ for $k =$ and $u(kT) = 0$ for $k \neq 0$, the output of the D/A converter is a pulse of widt T seconds and height 1, as sketched in Fig. 2.14. Mathematically, this puls is given by $1(t) - 1(t - T)$. Let us call the particular output in response t the pulse shown in Fig. 2.14 $y_1(t)$. This response is the difference betwee the step response [to $1(t)$] and the delayed step response [to $1(t - T)$]. Th Laplace transform of the step response is $G(s)/s$. Thus in the transfor domain the unit pulse response of the plant is

$$Y_1(s) = (1 - e^{-Ts})\frac{G(s)}{s}, \qquad (2.38$$

and the required transfer function is the z-transform of the samples of th inverse of $Y_1(s)$, which can be expressed as

$$\begin{aligned} G(z) &= \mathcal{Z}\{y_1(kT)\} \\ &= \mathcal{Z}\{\mathcal{L}^{-1}\{Y_1(s)\}\} \triangleq \mathcal{Z}\{Y_1(s)\} \\ &= \mathcal{Z}\left\{(1 - e^{-Ts})\frac{G(s)}{s}\right\}. \end{aligned}$$

This is the sum of two parts. The first is $\mathcal{Z}\{G(s)/s\}$, and the second is

$$\mathcal{Z}\{e^{-Ts}G(s)/s\} = z^{-1}\mathcal{Z}\{G(s)/s\}$$

because e^{-Ts} is exactly a delay of one period. Thus the transfer function

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \qquad (2.3$$

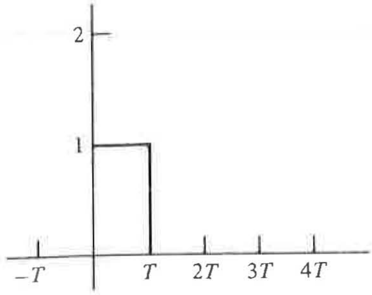


Figure 2.14 D/A output for unit-pulse input.

Example 2.8: As a first example of computing such discrete transfer functions, suppose $G(s) = a/(s+a)$. Then

$$\frac{G(s)}{s} = \frac{a}{s(s+a)} = \frac{1}{s} - \frac{1}{s+a},$$

and the corresponding time function is

$$\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} = 1(t) - e^{-at}1(t).$$

The samples of this signal are $1(kT) - e^{-akT}1(kT)$, and the z-transform of these samples is

$$\begin{aligned}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} &= \frac{z}{z-1} - \frac{z}{z-e^{-aT}} \\ &= \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}.\end{aligned}$$

We could have gone to the tables in Appendix B and found this result directly as entry 12. Now we can compute the desired transform by applying (2.39)

$$\begin{aligned}G(z) &= \frac{z-1}{z}\,\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})} \\ &= \frac{1-e^{-aT}}{z-e^{-aT}}.\end{aligned} \tag{2.40}$$

Example 2.9: We consider the double integrator characteristic of a single mass, such as the satellite, for which the transfer function is $G(s) = 1/s^2$. We have

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s^3}\right\}$$

This time we refer to the tables in Appendix B and find that the transform of $1/s^3$ is

$$\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3},$$

and therefore

$$G(z) = \frac{T^2(z+1)}{2(z-1)^2}. \tag{2.41}$$

For more complex systems than those in Examples 2.8 and 2.9, use of a CAD package is recommended.[11]

2.4.2 Continuous Time Delay

We now consider computing the discrete transfer function of a continuous system with pure time delay. The responses of many chemical process-control plants exhibit pure time delay because there is a finite time of transport of fluids or materials between the process and the controls and/or the sensors. Also, we must often consider finite computation time in the digital controller, and this is exactly the same as if the process had a pure time delay. With the techniques we have developed here, it is possible to obtain the discrete transfer function of such processes exactly, as Example 2.10 illustrates.

Example 2.10: We consider the example suggested by the fluid mixer problem described in Appendix A.3, for which

$$G(s) = e^{-\lambda s}H(s).$$

[11] See X-C2D in Table E.1.

The term $e^{-\lambda s}$ represents the delay of λ seconds, which includes both the process delay and the computation delay, if any. We assume that $H(s)$ is a rational transfer function. To prepare this function for computation of the z-transform, we first define an integer ℓ and a positive number m less than 1.0 such that $\lambda = \ell T - mT$. With these definitions we can write

$$\frac{G(s)}{s} = e^{-\ell Ts} \frac{e^{mTs} H(s)}{s}.$$

Because ℓ is an integer, this term reduces to $z^{-\ell}$ when we take the z-transform. Because $m < 1$, the transform of the other term is quite direct. We select $H(s) = a/(s+a)$ and, after the partial fraction expansion of $H(s)/s$, we have

$$G(z) = \frac{z-1}{z^{\ell+1}} \mathcal{Z} \left\{ \frac{e^{mTs}}{s} - \frac{e^{mTs}}{s+a} \right\}$$

To complete the transfer function, we need the z-transforms of the inverses of the terms in the braces. The first term is a unit step shifted left by mT seconds, and the second term is an exponential shifted left by the same amount. Because $m < 1$, these shifts are less than one full period, and no sample is picked up in negative time. The signals are sketched in Fig. 2.15.

The samples are given by $1(kT)$ and $e^{-aT(k+m)}1(kT)$. The corresponding z-transforms are $z/(z-1)$ and $ze^{-amT}/(z - e^{-aT})$. Consequently the final transfer function is

$$\begin{aligned} G(z) &= \frac{z-1}{z} \frac{1}{z^{\ell}} \left\{ \frac{z}{z-1} - \frac{ze^{-amT}}{z - e^{-aT}} \right\} \\ &= \frac{z-1}{z^{\ell}} \left\{ \frac{z[z - e^{-aT} - (z-1)e^{-amT}]}{(z-1)(z - e^{-aT})} \right\} \\ &= (1 - e^{-amT}) \frac{z + \alpha}{z^{\ell}(z - e^{-aT})}, \end{aligned}$$

where the zero position is at $-\alpha = -(e^{-amT} - e^{-aT})/(1 - e^{-amT})$. Notice that this zero is near the origin of the z-plane when m is near 1 and moves outside the unit circle to near $-\infty$ when m approaches 0. For the specific values of the mixer, we take $a = 1$, $T = 1$, and

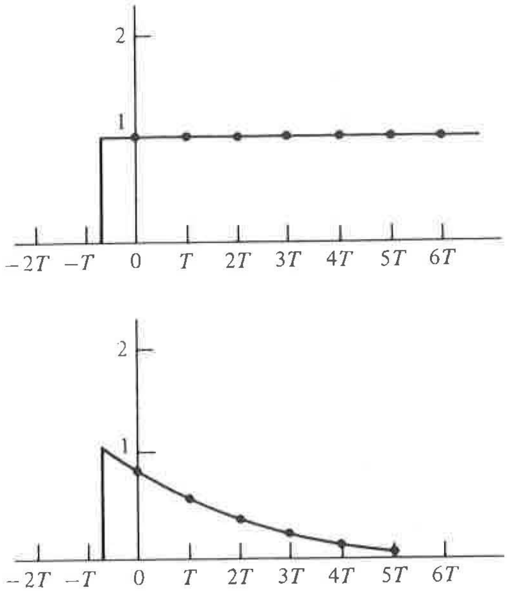


Figure 2.15 Sketch of the shifted signals showing sample points.

$\lambda = 1.5$. Then we can compute that $\ell = 2$ and $m = 0.5$. For these values, we get

$$G(z) = \frac{z + 0.6065}{z^2(z - 0.3679)}. \tag{2.42}$$

2.4.3 State-Space Form

Computing the z-transform using the Laplace transform as in (2.39) is a very tedious business that is unnecessary with the availability of computers. We will next develop a formula using state descriptions that will remove most of the calculations to the computer, where it is better done. A continuous, linear, constant-coefficient system of differential equations can always be expressed as a set of first-order matrix differential equations:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u + \mathbf{G}_1 w, \tag{2.43}$$

where u is the control input to the system and w is a disturbance input. The output can be expressed as a linear combination of the state, $\mathbf{x}$, and the input as

$$y = \mathbf{H}\mathbf{x} + Ju. \tag{2.44}$$

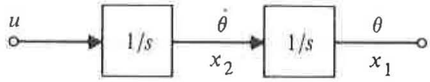


Figure 2.16 Satellite attitude control in classical representation.

Often the sampled-data system being described is the plant of a control problem, and the parameter J in (2.44) is zero and will frequently be omitted.

Example 2.11: Application of state representation to the equations of the satellite attitude-control example shown in Fig. 2.16 and described in Appendix A yields

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\mathbf{F}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{G}} u,$$
$$\theta = y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad (2.45)$$

which, in this case, turns out to be a rather involved way of writing

$$\ddot{\theta} = u.$$

The representations (2.43) and (2.44) are not unique. Given one state representation, any nonsingular linear transformation of that state such as $\boldsymbol{\xi} = \mathbf{Tx}$ is also an allowable alternative realization of the same system.

If we let $\boldsymbol{\xi} = \mathbf{Tx}$ in (2.43) and (2.44), we find

$$\begin{aligned} \dot{\boldsymbol{\xi}} = \mathbf{T}\dot{\mathbf{x}} &= \mathbf{T}(\mathbf{Fx} + \mathbf{G}u + \mathbf{G}_1 w) \\ &= \mathbf{TFx} + \mathbf{TG}u + \mathbf{TG}_1 w, \\ \dot{\boldsymbol{\xi}} &= \mathbf{TFT}^{-1}\boldsymbol{\xi} + \mathbf{TG}u + \mathbf{TG}_1 w, \\ y &= \mathbf{HT}^{-1}\boldsymbol{\xi} + Ju. \end{aligned}$$

If we designate the system matrices for the new state $\boldsymbol{\xi}$ as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and$\mathbf{D}$, then

$$\dot{\boldsymbol{\xi}} = \mathbf{A}\boldsymbol{\xi} + \mathbf{B}u + \mathbf{B}_1 w, \qquad y = \mathbf{C}\boldsymbol{\xi} + \mathbf{D}u,$$

Figure 2.17 System definition with sampling operations shown.

where

$$\mathbf{A} = \mathbf{TFT}^{-1}, \quad \mathbf{B} = \mathbf{TG}, \quad \mathbf{B}_1 = \mathbf{TG}_1, \quad \mathbf{C} = \mathbf{HT}^{-1}, \quad D = J.$$

Example 2.12: As an illustration, we can let $\xi_1 = x_2$ and $\xi_2 = x_1$ in (2.45); or, in matrix notation, the transformation to interchange the states is

$$\mathbf{T} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case $\mathbf{T}^{-1} = \mathbf{T}$, and application of the transformation equations to the system matrices of (2.45) gives

$$\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{C} = [0 \quad 1].$$

Most often, a change of state is made to bring the description matrices into a useful canonical form. We saw earlier how a single high-order difference equation could be represented by a state description in control or in observer canonical form. Also, there is a very useful state description corresponding to the partial-fraction expansion of a transfer function. State transformations can take a general description for either a continuous or a discrete system and, subject to some technical restrictions, convert it into a description in one or the other of these forms, as needed.

We wish to use the state description to establish a general method for obtaining the difference equations that represent the behavior of the continuous plant. Fig. 2.17 again depicts the portion of our system under consideration. Ultimately, the digital controller will take the samples $y(k)$, operate on that sequence by means of a difference equation, and put out a sequence of numbers, $u(k)$, which are the inputs to the plant. The loop will, therefore, be closed. To analyze the result, we must be able to relate the samples of the output $y(k)$ to the samples of the control $u(k)$. To do this, we must solve (2.43).

We will solve the general equation in two steps. We begin by solving the equation with only initial conditions and no external input. This is the homogeneous equation

$$\dot{\mathbf{x}}_h = \mathbf{F}\mathbf{x}_h(t), \qquad \mathbf{x}_h(t_0) = \mathbf{x}_0 \tag{2.46}$$

To solve this, we assume the solution is sufficiently smooth that a series expansion of the solution is possible:

$$\mathbf{x}_h(t) = \mathbf{A}_0 + \mathbf{A}_1(t - t_0) + \mathbf{A}_2(t - t_0)^2 + \cdots. \tag{2.47}$$

If we let $t = t_0$, we find immediately that $\mathbf{A}_0 = \mathbf{x}_0$. If we differentiate (2.47) and substitute into (2.46), we have

$$\mathbf{A}_1 + 2\mathbf{A}_2(t - t_0) + 3\mathbf{A}_3(t - t_0)^2 + \cdots = \mathbf{F}\mathbf{x}_h$$

and, at $t = t_0$, $\mathbf{A}_1 = \mathbf{F}\mathbf{x}_0$. Now we continue to differentiate the series and the differential equation and equate them at t_0 to arrive at the series

$$\mathbf{x}_h(t) = \left[\mathbf{I} + \mathbf{F}(t - t_0) + \frac{\mathbf{F}^2(t - t_0)^2}{2} + \frac{\mathbf{F}^3(t - t_0)^3}{6} + \cdots\right]\mathbf{x}_0.$$

This series is defined as the matrix exponential and written

$$\mathbf{x}_h(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0), \tag{2.48}$$

where, by definition, the matrix exponential is

$$\begin{aligned} e^{\mathbf{F}(t-t_0)} &= \mathbf{I} + \mathbf{F}(t - t_0) + \mathbf{F}^2\frac{(t - t_0)^2}{2!} + \mathbf{F}^3\frac{(t - t_0)^3}{3!} + \cdots \\ &= \sum_{k=0}^{\infty} \mathbf{F}^k \frac{(t - t_0)^k}{k!}. \end{aligned} \tag{2.49}$$

It can be shown that the solution given by (2.48) is unique, which leads to very interesting properties of the matrix exponential. For example, consider two values of t : t_1 and t_2. We have

$$\mathbf{x}(t_1) = e^{\mathbf{F}(t_1-t_0)}\mathbf{x}(t_0)$$

and

$$\mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_0)}\mathbf{x}(t_0).$$

Because t_0 is arbitrary also, we can express $\mathbf{x}(t_2)$ as if the equation solution began at t_1, for which

$$\mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_1)}x(t_1).$$

Substituting for $\mathbf{x}(t_1)$ gives

$$\mathbf{x}(t_2) = e^{\mathbf{F}(t_2-t_1)}e^{\mathbf{F}(t_1-t_0)}\mathbf{x}(t_0).$$

We now have two separate expressions for $\mathbf{x}(t_2)$, and, if the solution is unique, these must be the same. Hence we conclude that

$$e^{\mathbf{F}(t_2-t_0)} = e^{\mathbf{F}(t_2-t_1)}e^{\mathbf{F}(t_1-t_0)} \tag{2.50}$$

for all t_2, t_1, t_0. Note especially that if $t_2 = t_0$, then

$$\mathbf{I} = e^{-\mathbf{F}(t_1-t_0)}e^{\mathbf{F}(t_1-t_0)}.$$

Thus we can obtain the inverse of $e^{\mathbf{F}t}$ by merely changing the sign of t! We will use this result in computing the particular solution to (2.43).

The particular solution when u is not zero is obtained by using the method of *variation of parameters.*[12] We guess the solution to be in the form

$$\mathbf{x}_p(t) = e^{\mathbf{F}(t-t_0)}\mathbf{v}(t), \tag{2.51}$$

where $\mathbf{v}(t)$ is a vector of variable parameters to be determined [as contrasted to the constant parameters $\mathbf{x}(t_0)$ in (2.48)]. Substituting (2.51) into (2.43), we obtain

$$\mathbf{F}e^{\mathbf{F}(t-t_0)}\mathbf{v} + e^{\mathbf{F}(t-t_0)}\dot{\mathbf{v}} = \mathbf{F}e^{\mathbf{F}(t-t_0)}\mathbf{v} + \mathbf{G}u,$$

and, using the fact that the inverse is found by changing the sign of the exponent, we can solve for $\dot{\mathbf{v}}$ as

$$\dot{\mathbf{v}}(t) = e^{-\mathbf{F}(t-t_0)}\mathbf{G}u(t).$$

[12]Due to Joseph Louis Lagrange, French mathematician (1736–1813). We assume $w = 0$, but because the equations are linear, the effect of w can be added later.

Assuming that the control $u(t)$ is zero for $t < t_0$, we can integrate $\dot{\mathbf{v}}$ from t_0 to t to obtain

$$\mathbf{v}(t) = \int_{t_0}^{t} e^{-\mathbf{F}(\tau - t_0)} \mathbf{G} u(\tau) d\tau.$$

Hence, from (2.51), we get

$$\mathbf{x}_p(t) = e^{\mathbf{F}(t-t_0)} \int_{t_0}^{t} e^{-\mathbf{F}(\tau - t_0)} \mathbf{G} u(\tau) d\tau,$$

and simplifying, using the results of (2.50), we obtain the particular solution (convolution)

$$\mathbf{x}_p(t) = \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)} \mathbf{G} u(\tau) d\tau. \tag{2.52}$$

The total solution for $w = 0$ and $u \neq 0$ is the sum of (2.48) and (2.52):

$$\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)} \mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)} \mathbf{G} u(\tau) d\tau. \tag{2.53}$$

We wish to use this solution over one sample period to obtain a difference equation: hence we juggle the notation a bit (let $t = kT + T$ and t_0 equal kT) and arrive at a particular version of (2.53):

$$\mathbf{x}(kT + T) = e^{\mathbf{F}T} \mathbf{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)} \mathbf{G} u(\tau) d\tau. \tag{2.54}$$

This result is not dependent on the type of hold because u is specified in terms of its continuous time history, $u(t)$, over the sample interval. A common and typically valid assumption is that of a zero-order hold (ZOH) with no delay, that is,

$$u(\tau) = u(kT), \qquad kT \leq \tau < kT + T.$$

If some other hold is implemented or if there is a delay between the application of the control from the ZOH and the sample point, this fact can be accounted for in the evaluation of the integral in (2.54). The equations for a delayed ZOH will be given in the next subsection. To facilitate the solution of (2.54) for a ZOH with no delay, we change variables in the integral from τ to η such that

$$\eta = kT + T - \tau.$$

Then we have

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G} u(kT). \tag{2.55}$$

If we define

$$\mathbf{\Phi} = e^{\mathbf{F}T}, \tag{2.56a}$$

$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G}, \tag{2.56b}$$

Eqs. (2.55) and (2.44) reduce to difference equations in standard form:

$$\begin{aligned} \mathbf{x}(k+1) &= \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma} u(k) + \mathbf{\Gamma}_1 w(k), \\ y(k) &= \mathbf{H}\mathbf{x}(k), \end{aligned} \tag{2.57}$$

where we include the effect of an impulsive or piecewise constant disturbance, w, and assume that $J = 0$ in this case. If w is a constant, then $\mathbf{\Gamma}_1$ is given by (2.56b) with $\mathbf{G}$ replaced by $\mathbf{G}_1$. If w is an impulse, then $\mathbf{\Gamma}_1 = \mathbf{G}_1$.[13] The $\mathbf{\Phi}$ series expansion

$$\mathbf{\Phi} = e^{\mathbf{F}T} = \mathbf{I} + \mathbf{F}T + \frac{\mathbf{F}^2T^2}{2!} + \frac{\mathbf{F}^3T^3}{3!} + \cdots,$$

can also be written

$$\mathbf{\Phi} = \mathbf{I} + \mathbf{F}T\mathbf{\Psi}, \tag{2.58}$$

where

$$\mathbf{\Psi} = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2T^2}{3!} + \cdots$$

[13] If $w(t)$ is *not* a function of only its sample values, then an integral like that of (2.54) is required to describe its influence on $\mathbf{x}(k+1)$. Random disturbances are treated in Chapter 9.

1. Select sampling period T and description matrices $\mathbf{F}$ and $\mathbf{G}$.
2. Matrix $\mathbf{I} \leftarrow$ Identity
3. Matrix $\mathbf{\Psi} \leftarrow I$
4. $k \leftarrow 11$ [We are using $N = 11$ in (2.60).]
5. If $k = 1$, go to step 9.
6. Matrix $\mathbf{\Psi} \leftarrow \mathbf{I} + \dfrac{\mathbf{F}T}{k}\mathbf{\Psi}$
7. $k \leftarrow k - 1$
8. Go to step 5.
9. Matrix $\mathbf{\Gamma} \leftarrow T\mathbf{\Psi}\mathbf{g}$
10. Matrix $\mathbf{\Phi} \leftarrow \mathbf{I} + \mathbf{F}T\mathbf{\Psi}$

Figure 2.18 Program logic to compute $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ from $\mathbf{F}$, $\mathbf{G}$, and T for simple cases. (The left arrow, $\leftarrow$, is to be read "is replaced by.")

The $\mathbf{\Gamma}$ integral in (2.56) can be evaluated term by term to give

$$\begin{aligned} \mathbf{\Gamma} &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^{k+1}}{(k+1)!}\mathbf{G} \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^{k}}{(k+1)!} T\mathbf{G} \\ &= \mathbf{\Psi} T\mathbf{G}. \end{aligned} \tag{2.59}$$

We evaluate $\mathbf{\Psi}$ by a series in the form

$$\mathbf{\Psi} \approx \mathbf{I} + \frac{\mathbf{F}T}{2}\left(\mathbf{I} + \frac{\mathbf{F}T}{3}\left(\cdots \frac{\mathbf{F}T}{N-1}\left(\mathbf{I} + \frac{\mathbf{F}T}{N}\right)\right)\cdots\right), \tag{2.60}$$

which has better numerical properties than the direct series of powers. We then find $\mathbf{\Gamma}$ from (2.59) and $\mathbf{\Phi}$ from (2.58). A discussion of the selection of N and a technique to compute $\mathbf{\Psi}$ for comparatively large T is given by Källström (1973), and a review of various methods is found in a classic paper by Moler and Van Loan (1978). The program logic for computation of $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ for simple cases is given in Fig. 2.18. All control design packages that we know of contain logic to compute $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ from the continuous matrices $\mathbf{F}$, $\mathbf{G}$, and the sample period T.[14]

[14] See X-C2D in Table E.1 in Appendix E.

To compare this method of representing the plant with the discrete transfer functions, we can take the z-transform of (2.57) with $w = 0$ and obtain

$$[z\mathbf{I} - \mathbf{\Phi}]\mathbf{X}(z) = \mathbf{\Gamma}U(z), \tag{2.61a}$$

$$Y(z) = \mathbf{H}\mathbf{X}(z); \tag{2.61b}$$

therefore

$$\frac{Y(z)}{U(z)} = \mathbf{H}[z\mathbf{I} - \mathbf{\Phi}]^{-1}\mathbf{\Gamma}. \tag{2.62}$$

Example 2.13: For the satellite attitude-control example, the $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ matrices are easy to calculate using (2.58) and (2.59) and the values for $\mathbf{F}$ and $\mathbf{G}$ defined in (2.45). Since $\mathbf{F}^2 = 0$ in this case, we have

$$\begin{aligned}
\mathbf{\Phi} &= \mathbf{I} + \mathbf{F}T + \frac{\mathbf{F}^2T^2}{2!} + \cdots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} T = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \\
\mathbf{\Gamma} &= \left[\mathbf{I}T + \mathbf{F}\frac{T^2}{2!} + \frac{\mathbf{F}^2T^3}{3!}\right]\mathbf{G} \\
&= \left\{\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\frac{T^2}{2}\right\}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix};
\end{aligned}$$

hence, using (2.61), we obtain

$$\begin{aligned}
\frac{Y(z)}{U(z)} &= [1 \quad 0]\left\{z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\right\}^{-1}\begin{bmatrix} T^2/2 \\ T \end{bmatrix} \\
&= \frac{T^2}{2}\frac{(z+1)}{(z-1)^2},
\end{aligned}$$

which is the same result that would be obtained using (2.39) and the z-transform tables.

Note that to compute Y/U we find that the denominator is the determinant $\det(z\mathbf{I} - \mathbf{\Phi})$, which comes from the matrix inverse in (2.62). This determinant is the characteristic polynomial of the transfer function, and the zeros of the determinant are the poles of the plant. We have two poles at $z = 1$ in this case, corresponding to the two integrations in this plant's equations of motion.

We can explore further the question of poles and zeros and the state-space description by considering again the transform equations (2.61). An interpretation of transfer-function poles from the perspective of the corresponding difference equation is that a pole is a value of z such that the equation has a nontrivial solution when the forcing input is zero. From (2.61a), this implies that the linear eigenvalue equations

$$[z\mathbf{I} - \mathbf{\Phi}]\mathbf{X}(z) = [0]$$

have a nontrivial solution. From matrix algebra the well-known requirement for this is that $\det(z\mathbf{I} - \mathbf{\Phi}) = 0$. In the present case, we have

$$\begin{aligned}\det[z\mathbf{I} - \mathbf{\Phi}] &= \det\left[\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}\right] \\ &= \det\begin{bmatrix} z-1 & T \\ 0 & z-1 \end{bmatrix} \\ &= (z-1)^2 = 0,\end{aligned}$$

which is the characteristic equation, as we have seen. To compute the poles numerically when the matrices are given, one would use an eigenvalue routine.[15]

Along the same line of reasoning, a system zero is a value of z such that the system output is zero even with a nonzero state-and-input combination. Thus if we are able to find a nontrivial solution for $\mathbf{X}(z_0)$ and $U(z_0)$ such that $Y(z_0)$ is zero, then z_0 is a zero of the system. Combining the two parts of (2.57), we must satisfy the requirement

$$\begin{bmatrix} z\mathbf{I} - \mathbf{\Phi} & -\mathbf{\Gamma} \\ \mathbf{H} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{X}(z) \\ U(z) \end{bmatrix} = [0]. \tag{2.63}$$

Once more the condition for the existence of nontrivial solutions is that the determinant of the square coefficient system matrix be zero.[16] For the

[15] See EIGENV in Table E.1.

[16] We do not consider here the case of different numbers of inputs and outputs.

satellite example, we have

$$\det \begin{bmatrix} z-1 & -T & -T^2/2 \\ 0 & z-1 & -T \\ 1 & 0 & 0 \end{bmatrix} = 1 \cdot \det \begin{bmatrix} -T & -T^2/2 \\ z-1 & -T \end{bmatrix}$$

$$= +T^2 + \left(\frac{T^2}{2}\right)(z-1)$$

$$= +\frac{T^2}{2}z + \frac{T^2}{2}$$

$$= +\frac{T^2}{2}(z+1).$$

Thus we have a single zero at $z = -1$, as we have seen from the transfer function. Again, to compute the values of the zeros, called transmission zeros, good algorithms exist in matrix algebra.[17]

2.4.4 State-Space Models for Systems with Delay

Thus far we have discussed the calculation of discrete state models from continuous, ordinary differential equations of motion. Now we present the formulas for including a time delay in the model and also a time prediction up to one period which corresponds to the modified z-transform as defined by Jury. We begin with a state-variable model that includes a delay in control action. The state equations are

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{F}\mathbf{x}(t) + \mathbf{G}u(t-\lambda), \\ y &= \mathbf{H}\mathbf{x}. \end{aligned} \tag{2.64}$$

The general solution to (2.64) is given by (2.53); it is

$$\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{F}(t-\tau)}\mathbf{G}u(\tau-\lambda)\,d\tau.$$

If we let $t_0 = kT$ and $t = kT + T$, then

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau-\lambda)\,d\tau.$$

[17] See ZEROS in Table E.1. In using this function, one must be careful to account properly for the zeros that are at infinity; the function might return them as very large numbers that the user must remove to uncover the finite zeros.

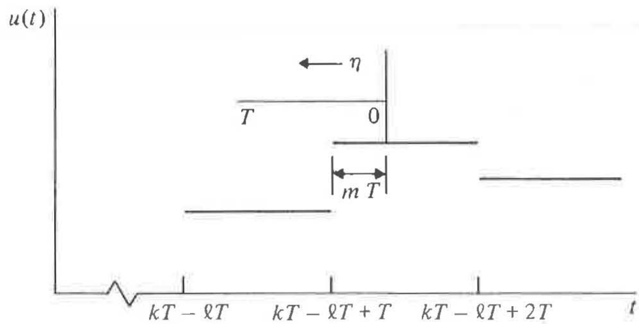


Figure 2.19 Sketch of a piecewise input and time axis for a system with time delay.

If we substitute $\eta = kT + T - \tau$ for τ in the integral, we find a modification of (2.55):

$$\begin{aligned}\mathbf{x}(kT+T) &= e^{\mathbf{F}T}\mathbf{x}(kT) + \int_T^0 e^{\mathbf{F}\eta}\mathbf{G}u(kT+T-\lambda-\eta)(-d\eta)\\ &= e^{\mathbf{F}T}\mathbf{x}(kT) + \int_0^T e^{\mathbf{F}\eta}\mathbf{G}u(kT+T-\lambda-\eta)\,d\eta.\end{aligned}$$

If we now separate the system delay λ into an integral number of sampling periods plus a fraction, we can define an integer ℓ and a positive number m less than one such that

$$\lambda = \ell T - mT, \tag{2.65}$$

and

$$\begin{aligned}&\ell \geq 0,\\ &0 \leq m < 1.\end{aligned}$$

With this substitution, we find that the discrete system is described by

$$\mathbf{x}(kT+T) = e^{\mathbf{F}T}\mathbf{x}(kT) + \int_0^T e^{\mathbf{F}\eta}\mathbf{G}u(kT+T-\ell T+mT-\eta)\,d\eta. \tag{2.66}$$

If we sketch a segment of the time axis near $t = kT - \ell T$ (Fig. 2.19), the nature of the integral in (2.66) with respect to the variable η will become clear. The integral runs for η from 0 to T, which corresponds to t from $kT - \ell T + T + mT$ backward to $kT - \ell T + mT$. Over this period, the control, which we assume is piecewise constant, takes on first the value $u(kT - \ell T + T)$

and then the value $u(kT-\ell T)$. Therefore, we can break the integral in (2.66) into two parts as follows:

$$
\begin{aligned}
\mathbf{x}(kT+T) &= e^{\mathbf{F}T}\mathbf{x}(kT) + \int_0^{mT} e^{\mathbf{F}\eta}\mathbf{G}\, d\eta\, u(kT-\ell T+T) \\
&\quad + \int_{mT}^{T} e^{\mathbf{F}\eta}\mathbf{G}\, d\eta\, u(kT-\ell T) \\
&= \mathbf{\Phi}\mathbf{x}(kT) + \mathbf{\Gamma}_1 u(kT-\ell T) + \mathbf{\Gamma}_2 u(kT-\ell T+T). \qquad (2.67)
\end{aligned}
$$

In (2.67) we defined

$$
\mathbf{\Phi} = e^{\mathbf{F}T}, \qquad \mathbf{\Gamma}_1 = \int_{mT}^{T} e^{\mathbf{F}\eta}\mathbf{G}\, d\eta, \qquad \text{and} \qquad \mathbf{\Gamma}_2 = \int_0^{mT} e^{\mathbf{F}\eta}\mathbf{G}\, d\eta. \qquad (2.68)
$$

To complete our analysis it is necessary to express (2.67) in standard state-space form. To do this we must consider separately the cases of $\ell = 0$, $\ell = 1$, and $\ell > 1$.

For $\ell = 0$, $\lambda = -mT$ according to (2.65), which implies not delay but prediction. Because mT is restricted to be less than T, however, the output will not show a sample before $k = 0$, and the discrete system will be causal. The result is that the discrete system computed with $\ell = 0$, $m \neq 0$ will show the response at $t = 0$, which the same system with $\ell = 0$, $m = 0$ would show at $t = mT$. In other words, by taking $\ell = 0$ and $m \neq 0$ we pick up the response values *between* the normal sampling instants. In z-transform theory, the transform of the system with $\ell = 0$, $m \neq 0$ is called the *modified z-transform.*[18] The state-variable form requires that we evaluate the integrals in (2.68). To do so we first convert $\mathbf{\Gamma}_1$ to a form similar to the integral for $\mathbf{\Gamma}_2$. From (2.68) we factor out the constant matrix G to obtain

$$
\mathbf{\Gamma}_1 = \int_{mT}^{T} e^{\mathbf{F}\eta}\, d\eta\, \mathbf{G}.
$$

If we set $\sigma = \eta - mT$ in this integral, we have

$$
\begin{aligned}
\mathbf{\Gamma}_1 &= \int_0^{T-mT} e^{\mathbf{F}(mT+\sigma)} d\sigma \mathbf{G} \\
&= e^{\mathbf{F}m} \int_0^{T-mT} e^{\mathbf{F}\sigma} d\sigma \mathbf{G}. \qquad (2.69)
\end{aligned}
$$

[18] See Jury (1964) or Ogata (1987).

For notational purposes we will define, for any positive nonzero scalar number, a, the two matrices

$$\mathbf{\Phi}(a) = e^{\mathbf{F}a}, \quad \mathbf{\Psi}(a) = \frac{1}{a}\int_0^a e^{\mathbf{F}\sigma}\,d\sigma. \tag{2.70}$$

In terms of these matrices, we have

$$\begin{aligned} \mathbf{\Gamma}_1 &= (T - mT)\mathbf{\Phi}(mT)\mathbf{\Psi}(T - mT)\mathbf{G}, \\ \mathbf{\Gamma}_2 &= mT\mathbf{\Psi}(mT)\mathbf{G}. \end{aligned} \tag{2.71}$$

The definition (2.70) is also useful from a computational point of view. If we recall the series definition of the matrix exponential,

$$\mathbf{\Phi}(a) = e^{\mathbf{F}a} = \sum_{k=0}^{\infty} \frac{\mathbf{F}^k a^k}{k!},$$

then we get

$$\begin{aligned} \mathbf{\Psi}(a) &= \frac{1}{a}\int_0^a \sum_{k=0}^{\infty} \frac{\mathbf{F}^k \sigma^k}{k!}\,d\sigma \\ &= \frac{1}{a}\sum_{k=0}^{\infty} \frac{\mathbf{F}^k}{k!}\,\frac{a^{k+1}}{k+1} \\ &= \sum_{k=0}^{\infty} \frac{\mathbf{F}^k a^k}{(k+1)!}. \end{aligned} \tag{2.72}$$

But now we note that the series for $\mathbf{\Phi}(a)$ can be written as

$$\mathbf{\Phi}(a) = \mathbf{I} + \sum_{k=1}^{\infty} \frac{\mathbf{F}^k a^k}{k!}.$$

If we let $k = j + 1$ in the sum, then, as in (2.58), we have

$$\begin{aligned} \mathbf{\Phi}(a) &= I + \sum_{j=0}^{\infty} \frac{\mathbf{F}^{j+1} a^{j+1}}{(j+1)!} \\ &= I + \sum_{j=0}^{\infty} \frac{\mathbf{F}^j a^j}{(j+1)!}\,aF \\ &= I + a\mathbf{\Psi}(a)\mathbf{F}. \end{aligned} \tag{2.73}$$

The point of (2.73) is that only the series for $\boldsymbol{\Psi}(a)$ needs to be computed and from this single sum we can compute $\boldsymbol{\Phi}$ and $\boldsymbol{\Gamma}$.

If we return to the case $\ell = 0$, $m \neq 0$, the discrete state equations are

$$\mathbf{x}(k+1) = \boldsymbol{\Phi}\mathbf{x}(k) + \boldsymbol{\Gamma}_1 u(k) + \boldsymbol{\Gamma}_2 u(k+1),$$

where $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ are given by (2.71). In order to put these equations in state-variable form, we must eliminate the term in $u(k+1)$. To do this, we define a new state, $\boldsymbol{\xi}(k) = \mathbf{x}(k) - \boldsymbol{\Gamma}_2 u(k)$. Then the equations are

$$\begin{aligned}
\boldsymbol{\xi}(k+1) &= \mathbf{x}(k+1) - \boldsymbol{\Gamma}_2 u(k+1) \\
&= \boldsymbol{\Phi}\mathbf{x}(k) + \boldsymbol{\Gamma}_1 u(k) + \boldsymbol{\Gamma}_2 u(k+1) - \boldsymbol{\Gamma}_2 u(k+1), \\
\boldsymbol{\xi}(k+1) &= \boldsymbol{\Phi}[\boldsymbol{\xi}(k) + \boldsymbol{\Gamma}_2 u(k)] + \boldsymbol{\Gamma}_1 u(k) \\
&= \boldsymbol{\Phi}\boldsymbol{\xi}(k) + (\boldsymbol{\Phi}\boldsymbol{\Gamma}_2 + \boldsymbol{\Gamma}_1) u(k) \\
&= \boldsymbol{\Phi}\boldsymbol{\xi}(k) + \boldsymbol{\Gamma} u(k). \qquad (2.74)
\end{aligned}$$

The output equation is

$$\begin{aligned}
y(k) &= \mathbf{H}\mathbf{x}(k) \\
&= \mathbf{H}[\boldsymbol{\xi}(k) + \boldsymbol{\Gamma}_2 u(k)] \\
&= \mathbf{H}\boldsymbol{\xi}(k) + \mathbf{H}\boldsymbol{\Gamma}_2 u(k) \\
&= \mathbf{H}_d\boldsymbol{\xi}(k) + J_d u(k). \qquad (2.75)
\end{aligned}$$

Thus for $\ell = 0$, the state equations are given by (2.71), (2.74), and (2.75). Note especially that if $m = 0$, then $\boldsymbol{\Gamma}_2 = 0$, and these equations reduce to the previous model with no delay.

Our next case is $\ell = 1$. From (2.67), the equations are given by

$$\mathbf{x}(k+1) = \boldsymbol{\Phi}\mathbf{x}(k) + \boldsymbol{\Gamma}_1 u(k-1) + \boldsymbol{\Gamma}_2 u(k).$$

In this case, we must eliminate $u(k-1)$ from the right-hand side, which we do by defining a new state $x_{n+1}(k) = u(k-1)$. We have thus an increased dimension of the state, and the equations are

$$\begin{aligned}
\begin{bmatrix} \mathbf{x}(k+1) \\ x_{n+1}(k+1) \end{bmatrix} &= \begin{bmatrix} \boldsymbol{\Phi} & \boldsymbol{\Gamma}_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ x_{n+1}(k) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Gamma}_2 \\ 1 \end{bmatrix} u(k), \\
y(k) &= [\mathbf{H} \quad 0] \begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix}. \qquad (2.76)
\end{aligned}$$

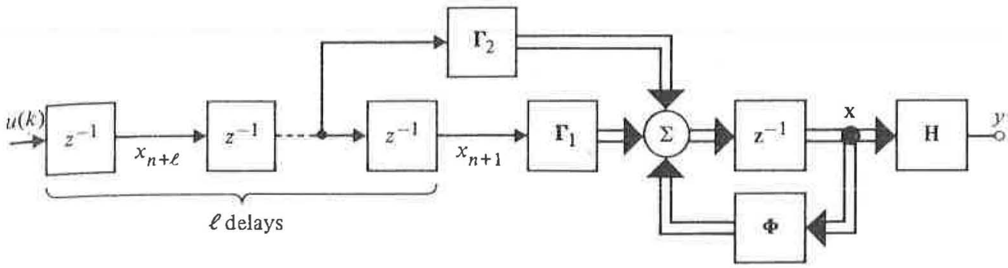


Figure 2.20 Block diagram of system with delay of more than one period. Double line indicates vector valued variables.

For our final case, we consider $\ell > 1$. In this case, the equations are

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}_1 u(k-\ell) + \mathbf{\Gamma}_2 u(k-\ell+1),$$

and we must eliminate the past controls up to $u(k)$. To do this we introduce ℓ new variables such that

$$x_{n+1}(k) = u(k-\ell), \qquad x_{n+2}(k) = u(k-\ell+1), \dots, x_{n+\ell}(k) = u(k-1).$$

The structure of the equations is

$$\begin{bmatrix} \mathbf{x}(k+1) \\ x_{n+1}(k+1) \\ x_{n+2}(k+1) \\ \vdots \\ x_{n+\ell}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Gamma}_1 & \mathbf{\Gamma}_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ & & & & & 1 \\ 0 & 0 & 0 & \vdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ x_{n+1}(k) \\ x_{n+2}(k) \\ \vdots \\ x_{n+\ell}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(k),$$

$$y(k) = [\mathbf{H} \quad 0] \begin{bmatrix} \mathbf{x} \\ x_{n+1} \\ \vdots \\ x_{n+\ell} \end{bmatrix}. \tag{2.77}$$

This final situation is easily visualized in terms of a block diagram, as shown in Fig. 2.20.

2.4.5 Numerical Considerations in Computing $\mathbf{\Phi}$ and $\mathbf{\Gamma}$

The numerical considerations of these computations are centered in the approximation to the infinite sum for $\mathbf{\Psi}$ given by (2.72) or, for $a = T$, by (2.58).

The problem is that if $\mathbf{F}T$ is large, then $(\mathbf{F}T)^N/N!$ becomes extremely large before it becomes small, and before acceptable accuracy is realized most computer number representations will overflow, destroying the value of the computation. Källström (1973) has analyzed a technique used by Kalman and Englar (1966), which has been found effective by Moler and Van Loan (1978). The basic idea comes from (2.50) with $t_2 - t_0 = 2T$ and $t_1 - t_0 = T$, namely,

$$(e^{\mathbf{F}T})^2 = e^{\mathbf{F}T}e^{\mathbf{F}T} = e^{\mathbf{F}2T}. \tag{2.78}$$

Thus, if T is too large, we can compute the series for $T/2$ and square the result. If $T/2$ is too large, we compute the series for $T/4$, and so on, until we find a k such that $T/2^k$ is *not* too large. We need a test for deciding on the value of k. We propose to approximate the series for $\mathbf{\Psi}$, which can be written

$$\mathbf{\Psi}\left(\frac{T}{2^k}\right) = \sum_{j=0}^{N-1} \frac{[\mathbf{F}(T/2^k)]^j}{(j+1)!} + \sum_{j=N}^{\infty} \frac{(\mathbf{F}T/2^k)^j}{(j+1)!} = \hat{\mathbf{\Psi}} + \mathbf{R}.$$

We will select k, the factor that decides how much the sample period is divided down, to yield a small remainder term $\mathbf{R}$. Källström suggests that we estimate the size of $\mathbf{R}$ by the size of the first term ignored in $\hat{\mathbf{\Psi}}$, namely,

$$\hat{\mathbf{R}} \cong (\mathbf{F}T)^N/(N+1)!2^{Nk}.$$

A simpler method is to select k such that the size of $\mathbf{F}T$ divided by 2^k is less than 1. In this case, the series for $\mathbf{F}T/2^k$ will surely converge. The rule is to select k such that

$$2^k > ||\,\mathbf{F}T\,|| = \max_j \sum_{i=1}^{n} |\,F_{ij}\,|T.$$

Taking the log of both sides, we find

$$k > \log_2 ||\,\mathbf{F}T\,||,$$

from which we select

$$k = \max(\lceil \log_2 ||\,\mathbf{F}T\,||, 0), \tag{2.79}$$

where the symbol $\lceil x$ means the smallest integer greater than x. The maximum of this integer and zero is taken because it is possible that $||\,\mathbf{F}T\,||$ is

1. Select $\mathbf{F}$ and T.
2. Comment: Compute $\| \mathbf{F}T \|$.
3. $V \leftarrow \max_j\{\Sigma_i \mid F_{ij} \mid\} \times T$
4. $k \leftarrow$ smallest nonnegative integer greater than $\log_2 V$.
5. Comment: compute $\mathbf{\Psi}(T/2^k)$.
6. $T_1 \leftarrow T/2^k$
7. $\mathbf{I} \leftarrow$ Identity
8. $\mathbf{\Psi} \leftarrow \mathbf{I}$
9. $j \leftarrow 11$
10. If $j = 1$,
11. $\mathbf{\Psi} \leftarrow I + \dfrac{\mathbf{F}T_1}{j}\mathbf{\Psi}$
12. $j \leftarrow j - 1$
13. Go to step 10.
14. Comment: Now double $\mathbf{\Psi}$ k times.
15. If $k = 0$, stop.
16. $\mathbf{\Psi} \leftarrow \left(\mathbf{I} + \dfrac{\mathbf{F}T}{2^{k+1}}\mathbf{\Psi}\right)\mathbf{\Psi}$
17. $k \leftarrow k - 1$
    go to step 14.
18. Go to step 15.

Figure 2.21 Logic for a program to compute $\mathbf{\Psi}$ using automatic time scaling.

already so small that its log is negative, in which case we want to select $k = 0$.

Having selected k, we now have the problem of computing $\hat{\mathbf{\Psi}}(T)$ from $\hat{\mathbf{\Psi}}(T/2^k)$. Our original concept was based on the series for $\mathbf{\Phi}$, which satisfied (2.78). To obtain the suitable formula for $\mathbf{\Psi}$, we use the relation between $\mathbf{\Phi}$ and $\mathbf{\Psi}$ given by (2.58) as follows to obtain the "doubling" formula for $\mathbf{\Psi}$:

$$\begin{aligned}
\mathbf{\Phi}(2T) &= \mathbf{\Phi}(T)\mathbf{\Phi}(T), \\
\mathbf{I} + 2T\mathbf{F}\mathbf{\Psi}(2T) &= [\mathbf{I} + T\mathbf{F}\mathbf{\Psi}(T)][\mathbf{I} + T\mathbf{F}\mathbf{\Psi}(T)] \\
&= \mathbf{I} + 2T\mathbf{F}\mathbf{\Psi}(T) + T^2\mathbf{F}^2\mathbf{\Psi}^2(T);
\end{aligned}$$

therefore

$$2T\mathbf{F}\mathbf{\Psi}(2T) = 2T\mathbf{F}\mathbf{\Psi}(T) + T^2\mathbf{F}^2\mathbf{\Psi}^2(T).$$

This is equivalent to

$$\mathbf{\Psi}(2T) = \left(\mathbf{I} + \frac{T\mathbf{F}}{2}\mathbf{\Psi}(T)\right)\mathbf{\Psi}(T),$$

which is the form to be used. The program logic for computing $\mathbf{\Psi}$ is shown in Fig. 2.21.[19] This algorithm does not include the delay discussed in Section 2.4.4. For that, we must implement the logic shown in Fig. 2.20.[20]

2.5 SIGNAL ANALYSIS AND DYNAMIC RESPONSE

In Section 2.3 we demonstrated that if two variables are related by a linear constant difference equation, then the ratio of the z-transform of the output signal to that of the input is a function of the system equation alone, and the ratio is called the transfer function. A method for study of linear constant discrete systems is thereby indicated, consisting of the following steps:

1. Compute the transfer function of the system $H(z)$.
2. Compute the transform of the input signal, $E(z)$.
3. Form the product, $E(z)H(z)$, which is the transform of the output signal, U.
4. Invert the transform to obtain $u(kT)$.

If the system description is available in difference-equation form, and if the input signal is elementary, then the first three steps of this process require very little effort or computation. The final step, however, is tedious if done by hand, and, because we will later be preoccupied with design of transfer functions to give desirable responses, we attach great benefit to gaining intuition for the kind of response to be expected from a transform without actually inverting it. Our approach to this problem is to present a repertoire of elementary signals with known features and to learn their representation in the transform or z-domain. Thus, when given an unknown transform, we will be able, by reference to these known solutions, to infer the major features of the time-domain signal and thus to determine whether the unknown is of sufficient interest to warrant the effort of detailed time-response computation. To begin this process of attaching a connection between the time domain and the z-transform domain, we compute the transforms of a few elementary signals.

[19]See X-C2D in Table E.1

[20]See DELAY in Table E.1.

2.5.1 The Unit Pulse

We have already seen that the unit pulse is defined by[21]

$$\begin{aligned} e_1(k) &= 1 \qquad (k = 0) \\ &= 0 \qquad (k \neq 0) \\ &= \delta_k; \end{aligned}$$

therefore we have

$$E_1(z) = \sum_{-\infty}^{\infty} \delta_k z^{-k} = z^0 = 1. \tag{2.80}$$

This result is much like the continuous case, wherein the Laplace transform of the unit impulse is the constant 1.0.

The quantity $E_1(z)$ gives us an instantaneous method to relate signals to systems: To characterize the system $H(z)$, consider the signal $u(k)$, which is the unit pulse response; then $U(z) = H(z)$.

2.5.2 The Unit Step

Consider the unit step function defined by

$$\begin{aligned} e_2(k) &= 1 \qquad (k \geq 0) \\ &= 0 \qquad (k < 0) \\ &\triangleq 1(k). \end{aligned}$$

In this case, the z-transform is

$$\begin{aligned} E_2(z) &= \sum_{k=-\infty}^{\infty} e_2(k) z^{-k} = \sum_{k=0}^{\infty} z^{-k} \\ &= \frac{1}{1 - z^{-1}} \qquad (|z^{-1}| < 1) \\ &= \frac{z}{z - 1} \qquad (|z| > 1). \end{aligned} \tag{2.81}$$

[21]We have shifted notation here to use $e(k)$ rather than e_k for the kth sample. We use subscripts to identify different signals.

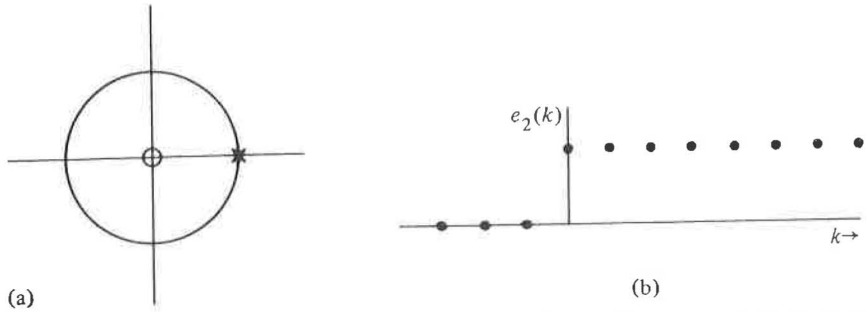


Figure 2.22 (a) Pole and zero of $E_2(z)$ in the z-plane. The unit circle is shown for reference. (b) Plot of $e_2(k)$.

Here the transform is characterized by a zero at $z = 0$ and a pole at $z = 1$. The significance of the convergence being restricted to $|z| > 1$ will be explored later when we consider the inverse transform operation. The Laplace transform of the unit step is $1/s$; we may thus keep in mind that a pole at $s = 0$ for a continuous signal corresponds in some way to a pole at $z = 1$ for discrete signals. We will explore this further later. In any event, we record that a pole at $z = 1$ with convergence outside the unit circle, $|z| = 1$, will correspond to a constant for positive time and zero for negative time.

To emphasize the connection between the time domain and the z-plane, we sketch in Fig. 2.22 the z-plane with the unit circle shown and the pole of $E_2(z)$ marked $\times$ and the zero marked $\circ$. Beside the z-plane, we sketch the time plot of $e_2(k)$.

2.5.3 Exponential

The one-sided exponential in time is

$$\begin{aligned} e_3(k) &= r^k \qquad (k \geq 0) \\ &= 0 \qquad (k < 0), \end{aligned} \tag{2.82}$$

which is the same as $r^k 1(k)$, using the symbol $1(k)$ for the unit step function.

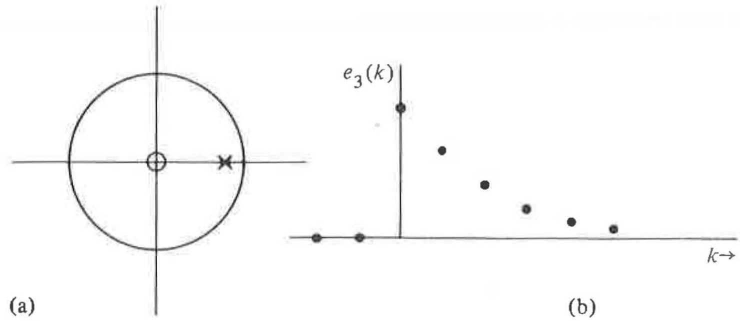


Figure 2.23 (a) Pole and zero of $E_3(z)$ in the z-plane. (b) Plot of $e_3(k)$.

Now we get

$$\begin{aligned} E_3(z) &= \sum_{k=0}^{\infty} r^k z^{-k} \\ &= \sum_{k=0}^{\infty} (rz^{-1})^k \\ &= \frac{1}{1 - rz^{-1}} \qquad (|rz^{-1}| < 1) \\ &= \frac{z}{z-r} \qquad (|z| > |r|). \end{aligned} \tag{2.83}$$

The pole of $E_3(z)$ is at $z = r$. From (2.82) we know that $e_3(k)$ grows without bound if $|r| > 1$. From (2.83) we conclude that a z-transform that converges for large z and has a real pole *outside* the circle $|z| = 1$ corresponds to a growing signal. If such a signal were the unit-pulse response of our system, such as our digital control program, we would say the program was *unstable* as we saw in (2.37). We plot in Fig. 2.23 the z-plane and the corresponding time history of $E_3(z)$ as $e_3(k)$ for the stable value, $r = 0.6$.

2.5.4 General Sinusoid

Our next example considers the modulated sinusoid $e_4(k) = [r^k \cos k\theta]1(k)$, where we assume $r > 0$. Actually, we can decompose $e_4(k)$ into the sum of two complex exponentials as

$$e_4(k) = r^k \left(\frac{e^{jk\theta} + e^{-jk\theta}}{2} \right) 1(k),$$

and because the z-transform is linear,[22] we need only compute the transform of each single complex exponential and add the results later. We thus take first

$$e_5(k) = r^k e^{jk\theta} 1(k) \tag{2.84}$$

and compute

$$\begin{aligned} E_5(z) &= \sum_{k=0}^{\infty} r^k e^{j\theta k} z^{-k} \\ &= \sum_{k=0}^{\infty} (re^{j\theta} z^{-1})^k \\ &= \frac{1}{1 - re^{j\theta} z^{-1}} \\ &= \frac{z}{z - re^{j\theta}} \qquad (|z| > r). \end{aligned} \tag{2.85}$$

The signal $e_5(k)$ grows without bound as k gets large if and only if $r > 1$, and a system with this pulse response is BIBO stable if and only if $|r| < 1$. The boundary of stability is the unit circle. To complete the argument given above for $e_4(k) = r^k \cos k\theta 1(k)$, we see immediately that the other half is found by replacing θ by $-\theta$ in (2.85),

$$\mathcal{Z}\{r^k e^{-j\theta k} 1(k)\} = \frac{z}{z - re^{-j\theta}} \qquad (|z| > r), \tag{2.86}$$

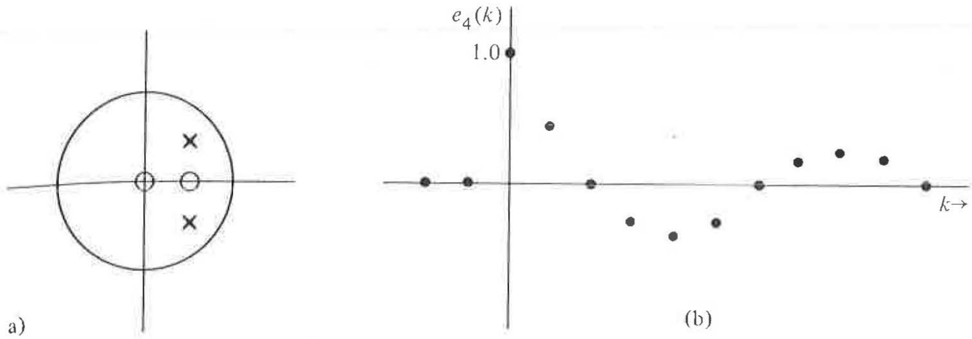
and thus that

$$\begin{aligned} E_4(z) &= \frac{1}{2}\left\{\frac{z}{z - re^{j\theta}} + \frac{z}{z - re^{-j\theta}}\right\} \\ &= \frac{z(z - r\cos\theta)}{z^2 - 2r(\cos\theta)z + r^2} \qquad (|z| > r). \end{aligned} \tag{2.87}$$

The z-plane pole-zero pattern of $E_4(z)$ and the time plot of $e_4(k)$ are shown in Fig. 2.24 for $r = 0.7$ and $\theta = 45°$.

We note in passing that if $\theta = 0$, then e_4 reduces to e_3 and, with $r = 1$, to e_2, so that three of our signals are special cases of e_4. By exploiting the

[22] We have not shown this formally. The demonstration, using the definition of linearity given above, is simple and is given in Section 2.7.



igure 2.24 (a) Poles and zeros of $E_4(z)$ for $\theta = 45°$, $r = 0.7$ in the z-plane. (b) lot of $e_4(k)$.

atures of $E_4(z)$, we can draw a number of conclusions about the relation etween pole locations in the z-plane and the time-domain signals to which e poles correspond. We collect these for later reference.

The settling time of a transient, defined as the time required for the signal to decay to one percent of its maximum value, is set mainly by the value of the radius, r, of the poles.

a) $r > 1$ corresponds to a growing signal that will not decay at all.

b) $r = 1$ corresponds to a signal with constant amplitude (which is *not* BIBO stable as a pulse response).

c) For $r < 1$, the closer r is to 0 the shorter the settling time. The corresponding system is BIBO stable. We can compute the settling time in samples, N, in terms of the pole radius, r.

pole radius, r	**response duration,** N
0.9	43
0.8	21
0.6	9
0.4	5

d) A pole at $r = 0$ corresponds to a transient of finite duration.

The number of samples per oscillation of a sinusoidal signal is determined by θ. If we require $\cos\theta k = \cos(\theta(k + N))$, we find that a period

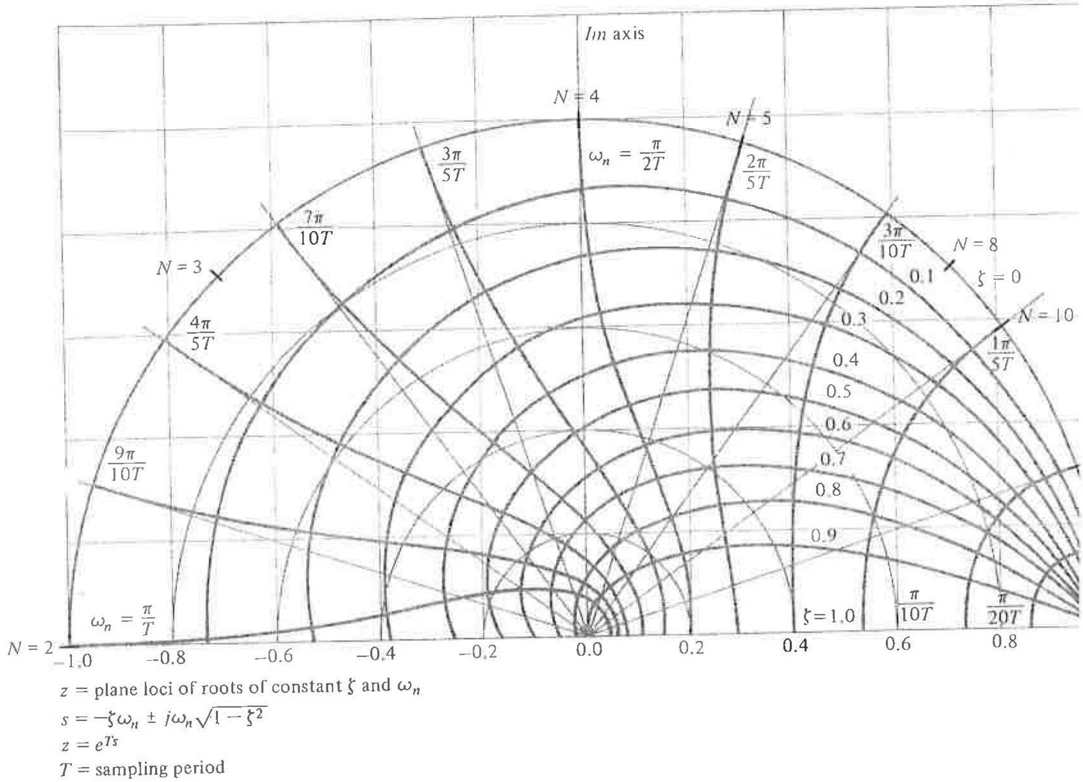


Figure 2.25 Sketch of the unit circle with angle θ marked in numbers of samples per cycle.

of 2π rad contains N samples, where

$$N = \left.\frac{2\pi}{\theta}\right|_{\text{rad}} = \left.\frac{360}{\theta}\right|_{\text{deg}} \text{ samples/cycle.}$$

For $\theta = 45°$, we have $N = 8$, and the plot of $e_4(k)$ given in Fig. 2.24(b) shows the eight samples in the first cycle very clearly. A sketch of the unit circle with several points corresponding to various numbers of samples per cycle marked is drawn in Fig. 2.25. The sampling frequency in Hertz is $1/T$, and the signal frequency is $f = 1/NT$ so that $N = f_s/f$ and $1/N$ is a *normalized* signal frequency. Since $\theta = (2\pi)/N$, θ is the normalized signal frequency in radians/sample. θ/T is the frequency in radians/second.

2.5.5 Correspondence with Continuous Signals

From the calculation of these few z-transforms, we have established that the duration of a time signal is related to the radius of the pole locations and the number of samples per cycle is related to the angle, θ. Another set of very useful relationships can be established by considering the signals to be samples from a continuous signal, $e(t)$, with Laplace transform $E(s)$. With this device we can exploit our knowledge of s-plane features by transferring them to equivalent z-plane properties. For the specific numbers represented in the illustration of e_4, we take the continuous signal

$$y(t) = e^{-at} \cos bt\, 1(t) \tag{2.88}$$

with

$$\begin{aligned} aT &= 0.3567, \\ bT &= \pi/4. \end{aligned}$$

And, taking samples one second apart ($T = 1$), we have

$$\begin{aligned} y(kT) &= (e^{-0.3567})^k \cos \frac{\pi k}{4} 1(k) \\ &= (0.7)^k \cos \frac{\pi k}{4} 1(k) \\ &= e_4(k). \end{aligned}$$

The poles of the Laplace transform of $y(t)$ (in the s-plane) are at

$$s_{1,2} = -a + jb,\ -a - jb.$$

From (2.87), the z-transform of $E_4(z)$ has poles at

$$z_{1,2} = re^{j\theta},\ re^{-j\theta},$$

but because $y(kT)$ equals $e_4(k)$, it follows that

$$\begin{aligned} r &= e^{-aT}, & \theta &= bT, \\ z_{1,2} &= e^{s_1 T}, & & e^{s_2 T}. \end{aligned}$$

If $E(z)$ is a ratio of polynomials in z, which will be the case if $e(k)$ is generated by a linear difference equation with constant coefficients, then by partial fraction expansion, $E(z)$ can be expressed as a sum of elementary

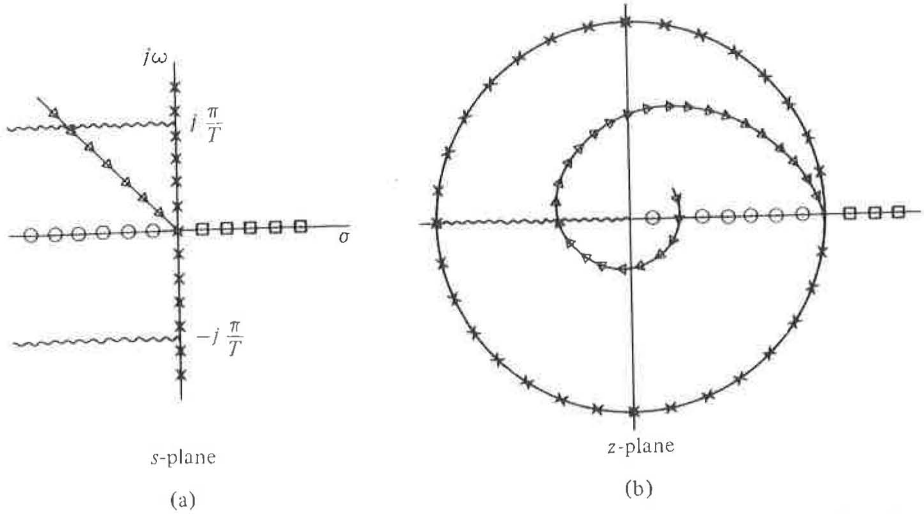


Figure 2.26 Corresponding lines in the s-plane and the z-plane according to $z = e^{sT}$.

terms like E_4 and E_3.[23] In all such cases, the discrete signal can be generated by samples from continuous signals where the relation between the s-plane poles and the corresponding z-plane poles is given by

$$z = e^{sT}. \tag{2.89}$$

If we know what it means to have a pole in a certain place in the s-plane, then (2.89) shows us where to look in the z-plane to find a representation of discrete samples having the *same time features*. It is useful to sketch several major features from the s-plane to the z-plane according to (2.89) to help fix these ideas. Such a sketch is shown in Fig. 2.26.

Each feature should be traced in the mind to obtain a good grasp of the relation. These features are given in Table 2.2. We note in passing that the map $z = e^{sT}$ of (2.89) is many-to-one. There are many values of s for each value of z. In fact, if

$$s_2 = s_1 + j\frac{2\pi}{T}N,$$

then $e^{s_1T} = e^{s_2T}$. The (great) significance of this fact will be explored in Chapter 3.

[23]Unless a pole of $E(z)$ is repeated. We have yet to compute the discrete version of a signal corresponding to a higher-order pole. The result is readily shown to be a polynomial in k multiplying $r^k e^{jk\theta}$.

able 2.2 Descriptions of corresponding lines in s-plane and z-plane.

s-plane	Symbol	z-plane
$s = j\omega$ Real frequency axis	× × ×	$\lvert z \rvert = 1$ Unit circle
$s = \sigma \geq 0$	□□□	$z = r \geq 1$
$s = \sigma \leq 0$	○○○	$z = r,\ 0 \leq r \leq 1$
$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$ $= -a + jb$ Constant damping ratio if ζ is fixed and ω_n varies	△△△	$z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T) = e^{-aT}$, $\theta = \omega_n T\sqrt{1-\zeta^2} = bT$ Logarithmic spiral
$s = \pm j(\pi/T) + \sigma, \sigma \leq 0$	∼∼∼∼	$z = -r$

2.5.6 Step Response

Our eventual purpose, of course, is to design digital controls, and our interest in the relation between z-plane poles and zeros and time-domain response comes from our need to know how a proposed design will respond in a given dynamic situation. The generic dynamic test for controls is the step response, and we will conclude this discussion of discrete system dynamic response with an examination of the relationships between the pole-zero patterns of elementary systems and the corresponding step responses for a discrete transfer function from u to y of a hypothetical plant. Our attention will be restricted to the step responses of the discrete system shown in Fig. 2.27 for a selected set of values of the parameters.

Note that if $z_1 = p_1$, the members of the one pole-zero pair cancel out; and if at the same time $z_2 = r\cos(\theta)$, $a_1 = -2r\cos(\theta)$, and $a_2 = r^2$,

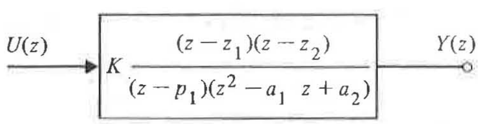


Figure 2.27 Definition of the parameters of the system whose step responses are to be catalogued.

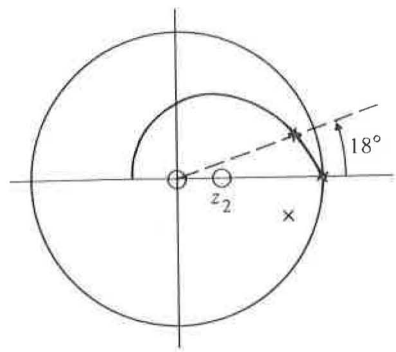


Figure 2.28 Pole-zero pattern of $Y(z)$ for the system of Fig. 2.27, with $z_1 = p_1, U(z) = z/(z-1), a_1$ and a_2 selected for $\theta = 18°$, and $\zeta = 0.5$.

the system response, $Y(z)$, to the input with transform $U(z) = 1$ (a unit pulse) is

$$Y(z) = \frac{z - r\cos\theta}{z^2 - 2r\cos\theta z + r^2}. \tag{2.90}$$

This transform, when compared with the transform $E_4(z)$ given in (2.87), is seen to be

$$Y(z) = z^{-1}E_4(z),$$

and we conclude that under these circumstances the system pulse response is a delayed version of $e_4(k)$, a typical second-order system pulse response.

For our first study we consider the effect of zero location. We let $z_1 = p_1$ and explore the effect of the (remaining) zero location, z_2, on the step-response overshoot for three sets of values of a_1 and a_2. We select a_1 and a_2 so that the poles of the system correspond to a response with damping ratio $\zeta = 0.5$ and consider values of θ of 18, 45, and 72 degrees. In every case, we will take the gain K to be such that the steady-state output value equals the step size. The situation in the z-plane is sketched in Fig. 2.28 for $\theta = 18°$. The curve for $\zeta = 0.5$ is also shown for reference. In addition to the two poles and one zero of $H(z)$, we show the pole at $z = 1$ and the zero at $z = 0$, which come from the transform of the input step, $U(z)$, given by $z/(z-1)$.

The major effect of the zero z_2 on the step response $y(k)$ is to change the percent overshoot, as can be seen from the four step responses for this case plotted in Fig. 2.29. To summarize all these data, we plot the percent overshoot versus zero location in Fig. 2.30 for $\zeta = 0.5$ and in Fig. 2.31

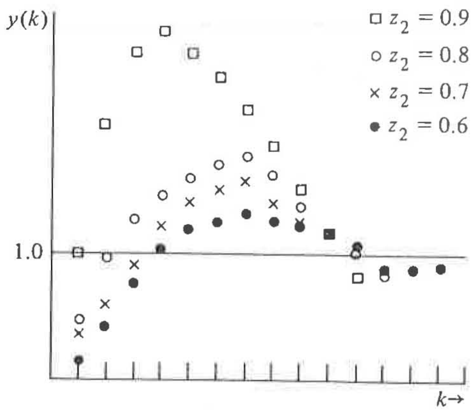


Figure 2.29 Plot of step responses for a discrete plant described by the pole-zero pattern of Fig. 2.28 for various values of z_2.

for $\zeta = 0.707$. The major feature of these plots is that the zero has very little influence when on the negative axis, but its influence is dramatic as it comes near $+1$. Also included on the plots of Fig. 2.30 are overshoot

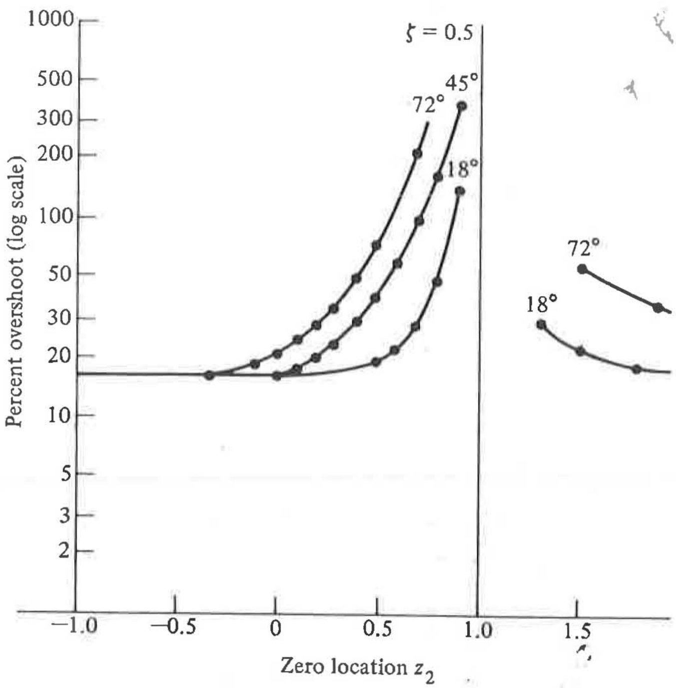


Figure 2.30 Effects of an extra zero on a discrete second-order system, $\zeta = 0.5$; $\theta = 18°$, $45°$, and $72°$.

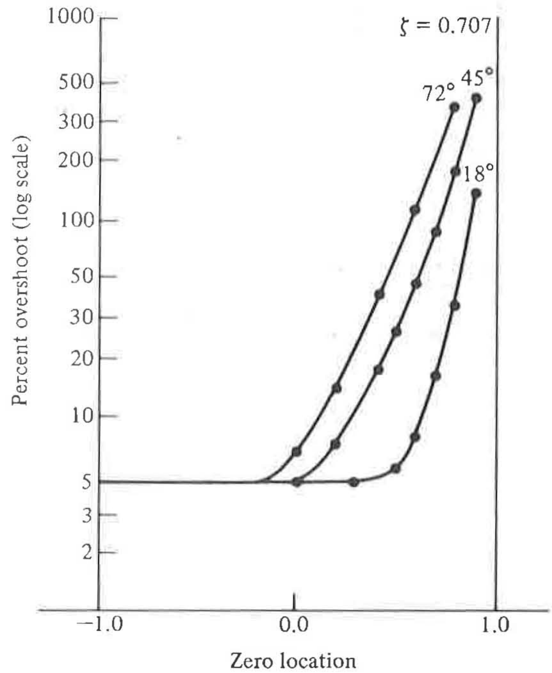


Figure 2.31 Effects of extra zero on second-order system when $\zeta = 0.707$; $\theta = 18°$, $45°$, $72°$. Percent overshoot versus zero location.

figures for a zero in the unstable region on the positive real axis. These responses go in the *negative* direction at first, and for the zero very near $+1$, the negative peak is larger than one![24]

Our second class of step responses corresponds to a study of the influence of a third pole on a basically second-order response. For this case we again consider the system of Fig. 2.27, but this time we fix $z_1 = z_2 = -1$ and let p_1 vary from near -1 to near $+1$. In this case, the major influence of the moving singularity is on the rise time of the step response. We plot this effect for $\theta = 18$, 45, and 72 degrees and $\zeta = 0.5$ on Fig. 2.32. In the figure we defined the rise time as the time required for the response to rise to 0.95, which is to 5% of its final value. We see here that the extra pole causes the rise time to get very much longer as the location of p_1 moves toward $z = +1$ and comes to dominate the response.

[24]Such systems are called nonminimum phase by Bode because the phase shift they impart to a sinusoidal input is greater than the phase of a system whose *magnitude* response is the same but that has a zero in the stable rather than the unstable region.

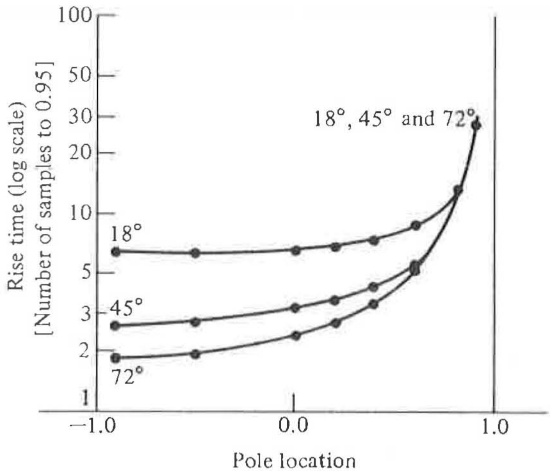


Figure 2.32 Effects of extra pole on system rise time. Two zeros at -1, one zero at ∞; $\zeta = 0.5$; $\theta = 18°, 45°, 72°$.

Our conclusions from these plots are that the addition of a pole or zero to a given system has only a small effect if the added singularities are in the range from 0 to -1. However, a zero moving toward $z = +1$ greatly increases the system overshoot. A pole placed toward $z = +1$ causes the response to slow down and thus primarily affects the rise time, which is progressively increased.

2.6 FREQUENCY RESPONSE

A very important concept in linear systems analysis is the frequency response. If a sinusoid at frequency ω_o is applied to a stable, linear, constant, continuous system, the response is a transient plus a sinusoidal steady state at the *same frequency*, ω_o, *as the input.* If the transfer function is written in gain-phase form as $H(j\omega) = A(\omega)e^{j\psi(\omega)}$, then the steady-state response to a unit-amplitude sinusoidal signal has amplitude $A(\omega_o)$ and phase $\psi(\omega_o)$ relative to the input signal.

We can say almost exactly the same respecting the frequency response of a stable, linear, constant, discrete system. If the system has a transfer function $H(z)$, we define its magnitude and phase for z taking on values around the unit circle by $H(e^{j\omega T}) = A(\omega T)e^{j\psi(\omega T)}$. If a unit-amplitude sinusoid is applied, then in the steady state, the response samples will be on a sinusoid of the same frequency with amplitude $A(\omega_o T)$ and phase $\psi(\omega_o T)$. It is worthwhile going through the calculations to fix ideas on this point.

From (2.16), the discrete response transform is

$$U(z) = H(z)E(z). \tag{2.91}$$

If $e(k) = \cos(\omega_o Tk)1(k)$, then, from (2.87) with $r = 1$ and $\theta = \omega_o T$, we have

$$E(z) = \frac{1}{2}\left\{\frac{z}{z - e^{j\omega_o T}} + \frac{z}{z - e^{-j\omega_o T}}\right\}. \tag{2.92}$$

If we substitute (2.92) into (2.91), we obtain

$$U(z) = \frac{1}{2}\left\{\frac{zH(z)}{z - e^{j\omega_o T}} + \frac{zH(z)}{z - e^{-j\omega_o T}}\right\}. \tag{2.93}$$

The steady state of $u(kT)$ corresponds to the terms in the expansion of (2.93) associated with the two poles on the unit circle. If we expand $U(z)/z$ into partial fractions and multiply back by z, the steady state part can be found as

$$U_{ss}(z) = \frac{1}{2}\frac{H(e^{j\omega_o T})z}{z - e^{j\omega_o T}} + \frac{1}{2}\frac{H(e^{-j\omega_o T})z}{z - e^{-j\omega_o T}}.$$

If $H(e^{j\omega_o T}) = A(\omega_o T)e^{j\psi(\omega_o T)}$, then we have

$$U_{ss}(z) = \frac{A}{2}\frac{e^{j\psi}z}{z - e^{j\omega_o T}} + \frac{A}{2}\frac{e^{-j\psi}z}{z - e^{-j\omega_o T}}, \tag{2.94}$$

and the inverse transform of $U_{ss}(z)$ is

$$\begin{aligned} U_{ss}(kT) &= \frac{A}{2}e^{j\psi}e^{j\omega_o Tk} + \frac{A}{2}e^{-j\psi}e^{-j\omega_o Tk} \\ &= A\cos(\omega_o Tk + \psi), \end{aligned} \tag{2.95}$$

which, of course, are samples at kT instants on a sinusoid of amplitude A, phase ψ, and frequency ω_o.

We will defer the plotting of particular frequency responses until later chapters (see, for example, Figs. 4.3, 4.8, 5.16, and 5.23). However, it should be noticed here that although a sinusoid of frequency ω_o *could* be passed through the samples of (2.95), there are other continuous sinusoids of frequency $\omega_o + \ell 2\pi/T$ for integer ℓ which also pass through these points. This is the phenomenon of aliasing, to which we will return in Chapter 3. Here, we repeat, we define the discrete frequency response of a transfer function $H(z)$ to sinusoids of frequency ω_o as $H(e^{j\omega_o T})$.

The Discrete Fourier Transform (DFT). The analysis developed above based on the z-transform is adequate for considering the theoretical frequency response of a linear, constant system or the corresponding difference equation, but it is not the best for the analysis of real-time signals as they occur in the laboratory or in other experimental situations. For the analysis of real data, we need a transform defined over a finite data record, which can be computed quickly and accurately. The required formula is that of the Discrete Fourier Transform, the DFT, and its numerical cousin, the Fast Fourier Transform, the FFT. Implementation of a version of the FFT algorithm is contained in all signal-processing software and in most computer-aided control-design software.

To understand the DFT, it is useful to consider two properties of a signal and its Fourier transform that are complements of each other: the property of being periodic and the property of being discrete. In ordinary Fourier analysis, we have a signal that is neither periodic nor discrete and its Fourier transform is also neither discrete nor periodic. If, however, the time function $f(t)$ is periodic with period T_0, then the appropriate form of the transform is the Fourier series, and the transform is defined only for the discrete frequencies $\omega = 2\pi n/T_0$. In other words, if the function in time is periodic, the function in frequency is discrete. The case where the properties are reversed is the z-transform we have just been studying. In this case, the time functions are discrete, being sampled, and the z-transform is periodic in ω; for if $z = e^{j\omega T}$, corresponding to real frequencies, then replacing $\omega = \omega + 2\pi k/T$ leaves z unchanged. We can summarize these results with the following table:

	Time	**Frequency**
Fourier series	periodic	discrete
z-transform	discrete	periodic

Suppose we now have a time function that is both periodic and discrete. Based on what we have seen, we would expect the transform of this function also to be both periodic and discrete. And this is the case, which leads us to the finite discrete Fourier transform and its finite inverse. Let the time function in question be $f(kT) = f(kT + NT)$. Because the function is periodic, the transform can be defined as the finite sum

$$F\left(\frac{2\pi n}{NT}\right) = \sum_{k=0}^{N-1} f(kT)e^{-j2\pi(nkT)/(NT)}.$$

This is the same as the z-transform over one period evaluated at the discrete frequencies of a Fourier series $\omega = 2\pi n/NT$. It is standard practice to suppress all the arguments except the indices of time and frequency and write

$$F_n = \sum_{k=0}^{N-1} f_k e^{-j2\pi(nk)/N}. \tag{2.96}$$

To complete the DFT, we need the inverse transform, which, by analogy with the standard Fourier transform, we guess to be the sum

$$\sum_{n=0}^{N-1} F_n e^{j2\pi(nk)/N}.$$

If we substitute (2.96) with summing index ℓ into this, we find

$$\sum_{n=0}^{N-1} \left\{ \sum_{\ell=0}^{N-1} f_\ell e^{-j2\pi(n\ell)/N} \right\} e^{j2\pi(nk)/N}.$$

Interchanging the order of the summations gives

$$\sum_{\ell=0}^{N-1} f_\ell \left\{ \sum_{n=0}^{N-1} e^{j2\pi[n(k-\ell)]/N} \right\}.$$

The sum in the braces is a finite geometric series, which we can evaluate as follows:

$$\begin{aligned}
\sum_{n=0}^{N-1} e^{j2\pi[n(k-\ell)]/N} &= \frac{1 - e^{j2\pi(k-\ell)}}{1 - e^{j2\pi(k-\ell)/N}} \\
&= \begin{cases} N & k - \ell = 0 \\ 0 & k - \ell = 1, 2, \ldots, N-1. \end{cases}
\end{aligned}$$

The sum is periodic with period N. With this evaluation, we see that the sum we have been considering is Nf_k, and thus we have the inverse sum,

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{j2\pi(nk)/N}. \tag{2.97}$$

Equations (2.96) and (2.97) comprise the DFT:

$$F_n = \sum_{k=0}^{N-1} f_k e^{-j2\pi(nk)/N},$$

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{j2\pi(nk)/N}.$$

Because there are N terms in the sum in (2.96), it would appear that to compute the DFT for one frequency it will take on the order of N multiply and add operations, and to compute the DFT for all N frequencies, it would take on the order of N^2 multiply and add operations. However, several authors, especially Cooley and Tukey(1965), have showed how to take advantage of the circular nature of the exponential so that all N values of F_n can be computed with on the order of $N\log(N)$ operations if N is a power of 2. For $N = 1024$, this is a saving of a factor of 100, a very large value. Their algorithm and related schemes are called the Fast Fourier Transform, or FFT.

To use the DFT/FFT in evaluating frequency response, we suppose we have a system described by (2.91) and that the input is a sinusoid at frequency $\omega_\ell = 2\pi\ell/NT$ so that $e(kT) = A\sin(2\pi\ell kT/NT)$. We apply this input to the system *and wait until all transients have died away.* At this time, the output is given by $u(kT) = B\sin(2\pi\ell k/N + \psi)$. The DFT of $e(k)$ is

$$\begin{aligned} E_n &= \sum_{k=0}^{N-1} A\sin\left(\frac{2\pi\ell k}{N}\right) e^{-j(2\pi nk)/N} \\ &= \sum_{k=0}^{N-1} \frac{A}{2j}\left[e^{j(2\pi\ell k)/N} - e^{-j(2\pi\ell k)/N}\right] e^{-j(2\pi nk)/N} \\ &= \begin{cases} 0, & \ell \neq n, \\ \dfrac{NA}{2j}, & \ell = n. \end{cases} \end{aligned}$$

The DFT of the output is

$$
\begin{aligned}
U_n &= \sum_{k=0}^{N-1} B \sin\left(\frac{2\pi \ell k}{N} + \psi\right) e^{-j(2\pi nk)/N} \\
&= \sum_{k=0}^{N-1} \frac{B}{2j}\left[e^{j\psi} e^{j(2\pi \ell k)/N} - e^{-j\psi} e^{-j(2\pi \ell k)/N}\right] e^{-j(2\pi nk)/N} \\
&= \begin{cases} 0, & \ell \neq n, \\ \dfrac{NB}{2j} e^{j\psi}, & \ell = n. \end{cases}
\end{aligned}
$$

Dividing these results, we see that with sinusoidal input and output, the frequency response at the frequency $\omega = (2\pi\ell)/NT$ is given by

$$
H(e^{j(2\pi\ell)/N}) = \frac{U_\ell}{E_\ell},
$$

where $U_\ell = FFT(u_k)$ and $E_\ell = FFT(e_k)$, each evaluated at $n = \ell$. We will discuss in Chapter 8 the general problem of estimation of the total frequency response from experimental data using the DFT/FFT as well as other tools.

2.7 PROPERTIES OF THE z-TRANSFORM

We have used the z-transform to show that linear, constant, discrete systems can be described by a transfer function that is the z-transform of the system's unit-pulse response, and we have studied the relationship between the pole-zero patterns of transfer functions in the z-plane and the corresponding time responses. We began a table of z-transforms, and a more extensive table is given in Appendix B. In Section 2.7.1 we turn to consideration of some of the properties of the z-transform that are essential to the effective and correct use of this important tool. In Section 2.7.2 an alternate derivation of the transfer function is given.

2.7.1 z-Transform Properties

In order to make maximum use of a table of z-transforms, one must be able to use a few simple properties of the z-transform which follow directly from the definition. Some of these, such as linearity, we have already used without making a formal statement of it, and others, such as the transform of the convolution, we have previously derived. For reference, we will demonstrate a few properties here and collect them into Appendix B for future reference. In all the properties listed below, we assume that $F_i(z) = \mathcal{Z}\{f_i(kT)\}$.

1. *Linearity:* A function $f(x)$ is linear if $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$. Applying this result to the definition of the z-transform, we find immediately that

$$\begin{aligned} \mathcal{Z}\{\alpha f_1(kT) + \beta f_2(kT)\} &= \sum_{k=-\infty}^{\infty} \{\alpha f_1(k) + \beta f_2(k)\} z^{-k} \\ &= \alpha \mathcal{Z}\{f_1(k)\} + \beta \mathcal{Z}\{f_2(k)\} \\ &= \alpha F_1(z) + \beta F_2(z). \end{aligned}$$

   Thus the z-transform is a linear function. It is the linearity of the transform that makes the partial-fraction technique work.

2. *Convolution of Time Sequences:*

$$\mathcal{Z}\left\{ \sum_{l=-\infty}^{\infty} f_1(l) f_2(k-l) \right\} = F_1(z) F_2(z).$$

   We have already developed this result in connection with (2.30). It is this result with linearity that makes the transform so useful in linear-constant-system analysis because the analysis of a combination of such dynamic systems can be done by linear algebra on the transfer functions.

3. *Time Shift:*

$$\mathcal{Z}\{f(k+n)\} = z^{+n} F(z). \tag{2.98}$$

   We demonstrate this result also by direct calculation:

$$\mathcal{Z}\{f(k+n)\} = \sum_{k=-\infty}^{\infty} f(k+n) z^{-k}.$$

   If we let $k + n = j$, then

$$\begin{aligned} \mathcal{Z}\{f(k+n)\} &= \sum_{j=-\infty}^{\infty} f(j) z^{-(j-n)} \\ &= z^n F(z). \quad \text{QED} \end{aligned}$$

This property is the essential tool in solving linear constant-coefficient difference equations by transforms. We should note here that the transform of the time shift is not the same for the one-sided transform because a shift can introduce terms with negative argument which are not included in the one-sided transform and must be treated separately. This effect causes initial conditions for the difference equation to be introduced when solution is done with the one-sided transform. See Problem 2.13.

4. *Scaling in the z-plane:*

$$\mathcal{Z}\{r^{-k}f(k)\} = F(rz). \tag{2.99}$$

By direct substitution, we obtain

$$\begin{aligned}\mathcal{Z}\{r^{-k}f(k)\} &= \sum_{k=-\infty}^{\infty} r^{-k}f(k)z^{-k} \\ &= \sum_{k=-\infty}^{\infty} f(k)(rz)^{-k} \\ &= F(rz). \qquad \text{QED}\end{aligned}$$

As an illustration of this property, we consider the z-transform of the unit step, $1(k)$, which we have computed before:

$$\mathcal{Z}\{1(k)\} = \sum_{k=0}^{\infty} z^{-k} = \frac{z}{z-1}.$$

By property 4 we have immediately that

$$\mathcal{Z}\{r^{-k}1(k)\} = \frac{rz}{rz-1} = \frac{z}{z-(1/r)}.$$

As a more general example, if we have a polynomial $a(z) = z^2 + a_1 z + a_2$ with roots $re^{\pm j\theta}$, then the scaled polynomial $\alpha^2 z^2 + a_1 \alpha z + a_2$ has roots $(r/\alpha)e^{\pm j\theta}$. This is an example of radial projection whereby the roots of a polynomial can be projected radially simply by changing the coefficients of the polynomial. The technique is sometimes used in pole-placement designs as described in Chapter 6, and sometimes used in adaptive control as described in Chapter 11.

5. *Final-Value Theorem:* If $F(z)$ converges for $|z| > 1$ and all poles of $(z-1)F(z)$ are inside the unit circle, then

$$\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z-1)F(z). \tag{2.100}$$

The conditions on $F(z)$ assure that the only possible pole of $F(z)$ not strictly inside the unit circle is a simple pole at $z = 1$, which is removed in $(z-1)F(z)$. Furthermore, the fact that $F(z)$ converges as the magnitude of z gets arbitrarily large ensures that $f(k)$ is zero for negative k. Therefore, all components of $f(k)$ tend to zero as k gets large, with the possible exception of the constant term due to the pole at $z = 1$. The size of this constant is given by the coefficient of $1/(z-1)$ in the partial-fraction expansion of $F(z)$, namely,

$$C = \lim_{z \to 1} (z-1)F(z).$$

However, because all other terms in $f(k)$ tend to zero, the constant C is the final value of $f(k)$, and (2.100) results. QED

As an illustration of this property, we consider the signal whose transform is given by

$$U(z) = \frac{z}{z-0.5} \frac{T}{2} \frac{z+1}{z-1}, \qquad |z| > 1.$$

Because $U(z)$ satisfies the conditions of (2.100), we have

$$\begin{aligned}
\lim_{k \to \infty} u(k) &= \lim_{z \to 1} (z-1) \frac{z}{z-0.5} \frac{T}{2} \frac{z+1}{z-1} \\
&= \lim_{z \to 1} \frac{z}{z-0.5} \frac{T}{2} (z+1) \\
&= \frac{1}{1-0.5} \frac{T}{2} (1+1) \\
&= 2T.
\end{aligned}$$

This result can be checked against the closed form for $u(k)$ given by (2.106) below.

6. *Inversion:* As with the Laplace transform, the z-transform is actually one of a pair of transforms that connect functions of time to functions of the complex variable z. The z-transform computes a function of z from

a sequence in k. (We identify the sequence number k with time in our analysis of dynamic systems, but there is nothing in the transform *per se* that requires this.) The inverse z-transform is a means to compute a sequence in k from a given function of z. We first examine two elementary schemes for inversion of a given $F(z)$ which can be used if we know beforehand that $F(z)$ is rational in z and converges as z approaches infinity. For a sequence $f(k)$, the z-transform has been defined as

$$F(z) = \sum_{k=-\infty}^{\infty} f(k)z^{-k}, \qquad r_0 < |z| < R_0. \tag{2.101}$$

If any value of $f(k)$ for negative k is nonzero, then there will be a term in (2.101) with a positive power of z. This term will be unbounded if the magnitude of z is unbounded; and thus if $F(z)$ converges as $|z|$ approaches infinity, we know that $f(k)$ is zero for $k < 0$. In this case, (2.101) is one-sided, and we can write

$$F(z) = \sum_{k=0}^{\infty} f(k)z^{-k}, \qquad r_0 < |z|. \tag{2.102}$$

The right-hand side of (2.102) is a series expansion of $F(z)$ about infinity or about $z^{-1} = 0$. Such an expansion is especially easy if $F(z)$ is the ratio of two polynomials in z^{-1}. We need only divide the numerator by the denominator in the correct way, and when the division is done, the coefficient of z^{-k} is automatically the sequence value $f(k)$. An example we have worked out before will illustrate the process. Suppose we take our system to be the trapezoid-rule integration with transfer function given by (2.14):

$$H(z) = \frac{T}{2}\frac{z+1}{z-1}, \qquad |z| > 1.$$

We will take the input to be the geometric series represented by $e_3(k)$ with $r = 0.5$. Then we have

$$E_3(z) = \frac{z}{z-0.5}, \qquad |z| > 0.5,$$

$$\begin{aligned} U(z) &= E_3(z)H(z) \\ &= \frac{z}{z-0.5}\frac{T}{2}\frac{z+1}{z-1}, \qquad |z| > 1. \end{aligned} \tag{2.103}$$

Equation (2.103) represents the transform of the system output, $u(k)$. Keeping out the factor of $T/2$, we write $U(z)$ as a ratio of polynomials in z^{-1},

$$U(z) = \frac{T}{2}\frac{1+z^{-1}}{1-1.5z^{-1}+0.5z^{-2}}, \tag{2.104}$$

and divide as follows:

$$\begin{array}{r l}
 & \frac{T}{2}[1 + 2.5z^{-1} + 3.25z^{-2} + 3.625z^{-3} + \cdots \\
\cline{2-2}
1 - 1.5z^{-1} + 0.5z^{-2} & \big) \, 1 + z^{-1} \\
 & \underline{1 - 1.5z^{-1} + 0.5z^{-2}} \\
 & \qquad 2.5z^{-1} - 0.5z^{-2} \\
 & \qquad \underline{2.5z^{-1} - 3.75z^{-2} + 1.25z^{-3}} \\
 & \qquad\qquad\qquad 3.25z^{-2} - 1.25z^{-3} \\
 & \qquad\qquad\qquad \underline{3.25z^{-2} - 4.875z^{-3} + 1.625z^{-4}} \\
 & \qquad\qquad\qquad\qquad\qquad 3.625z^{-3} - 1.625z^{-4} \\
 & \qquad\qquad\qquad\qquad\qquad \underline{3.625z^{-3} - \cdots}
\end{array}$$

By direct comparison with $U(z) = \sum_0^\infty u(k)z^{-k}$, we conclude that

$$\begin{aligned} u_0 &= T/2, \\ u_1 &= (T/2)2.5, \\ u_2 &= (T/2)3.25, \\ &\vdots \end{aligned} \tag{2.105}$$

Clearly, the use of a computer will greatly aid the speed of this process in all but the simplest of cases. Some may prefer to use synthetic division and omit copying over all the extraneous z's in the division. The process is identical to converting $F(z)$ to the equivalent difference equation and solving for the unit-pulse response.

The second special method for the inversion of z-transforms is to decompose $F(z)$ by partial-fraction expansion and look up the components of the sequence $f(k)$ in a previously prepared table. We consider again (2.103) and expand $U(z)$ as a function of z^{-1} as follows:

$$U(z) = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}\frac{1}{1-0.5z^{-1}} = \frac{A}{1-z^{-1}} + \frac{B}{1-0.5z^{-1}}.$$

We multiply both sides by $1 - z^{-1}$, let $z^{-1} = 1$, and compute

$$A = \frac{T}{2}\frac{2}{0.5} = 2T.$$

Similarly, at $z^{-1} = 2$, we evaluate

$$B = \frac{T}{2}\frac{1+2}{1-2} = -\frac{3T}{2}.$$

Looking back now at e_2 and e_3, which constitute our "table" for the moment, we can copy down that

$$\begin{aligned} u_k &= Ae_2(k) + Be_3(k) \\ &= 2Te_2(k) - \frac{3T}{2}e_3(k) \\ &= \left(2T - \frac{3T}{2}\left(\frac{1}{2}\right)^k\right) 1(k) \\ &= \frac{T}{2}\left[4 - \frac{3}{2^k}\right] 1(k). \end{aligned} \tag{2.106}$$

Evaluation of (2.106) for $k = 0, 1, 2, \ldots$ will, naturally, give the same values for $u(k)$ as we found in (2.105).

We now examine more closely the role of the region of convergence of the z-transform and present the inverse-transform integral. We begin with another example. The sequence

$$f(k) = \begin{cases} -1, & k < 0, \\ 0, & k \geq 0, \end{cases}$$

has the transform

$$F(z) = \sum_{k=-\infty}^{-1} -z^{-k} = -\left[\sum_{0}^{\infty} z^k - 1\right]$$
$$= \frac{z}{z-1}, \qquad |z| < 1.$$

This transform is exactly the same as the transform of the unit step $1(k)$, (2.81), except that this transform converges *inside* the unit circle and the transform of the $1(k)$ converges outside the unit circle. Knowledge of the region of convergence is obviously essential to the proper inversion of the transform to obtain the time sequence. The inverse z-transform is the closed, complex integral[25]

$$f(k) = \frac{1}{2\pi j} \oint F(z) z^k \frac{dz}{z}, \tag{2.107}$$

where the contour is a circle in the region of convergence of $F(z)$. To demonstrate the correctness of the integral and to use it to compute inverses it is useful to apply Cauchy's residue calculus [see Churchill and Brown (1984)]. Cauchy's result is that a closed integral of a function of z which is analytic on and inside a closed contour C except at a finite number of isolated singularities z_i is given by

$$\frac{1}{2\pi j} \oint_C F(z)\, dz = \sum_i \mathrm{Res}(z_i). \tag{2.108}$$

In (2.108), $\mathrm{Res}(z_i)$ means the residue of $F(z)$ at the singularity at z_i. We will be considering only rational functions, and these have only poles as singularities. If $F(z)$ has a pole of order n at z_1, then $(z - z_1)^n F(z)$ is regular at z_1 and can be expanded in a Taylor series near z_1 as

$$\begin{aligned}(z - z_1)^n F(z) = & A_{-n} + A_{-n+1}(z - z_1) + \cdots + A_{-1}(z - z_1)^{n-1} \\ & + A_0 (z - z_1)^n + \cdots . \end{aligned} \tag{2.109}$$

The residue of $F(z)$ at z_1 is A_{-1}.

---

[25]If it is known that $f(k)$ is causal, that is, $f(k) = 0$ for $k < 0$, then the region of convergence is outside the smallest circle that contains all the poles of $F(z)$ for rational transforms. It is this property that permits inversion by partial-fraction expansion and long division.

First we will use Cauchy's formula to verify (2.108). If $F(z)$ is the z-transform of $f(k)$, then we write

$$\mathcal{I} = \frac{1}{2\pi j} \oint \sum_{l=-\infty}^{\infty} f(l) z^{-l} z^k \frac{dz}{z}.$$

We assume that the series for $F(z)$ converges uniformly on the contour of integration, so the series can be integrated term by term. Thus we have

$$\mathcal{I} = \frac{1}{2\pi j} \sum_{l=-\infty}^{\infty} f(l) \oint z^{k-l} \frac{dz}{z}.$$

The argument of the integral has no pole inside the contour if $k - l \geq 1$, and it has zero residue at the pole at $z = 0$ if $k - l < 0$. Only if $k = l$ does the integral have a residue, and that is 1. By (2.108), the integral is zero if $k \neq l$ and is $2\pi j$ if $k = l$. Thus $\mathcal{I} = f(k)$, which demonstrates (2.107).

To illustrate the use of (2.108) to compute the inverse of a z-transform, we will use the function $z/(z-1)$ and consider first the case of convergence for $|z| > 1$ and second the case of convergence for $|z| < 1$. For the first case,

$$f_1(k) = \frac{1}{2\pi j} \oint_{|z|=R>1} \frac{z}{z-1} z^k \frac{dz}{z}, \tag{2.110}$$

where the contour is a circle of radius greater than 1. Suppose $k < 0$. In this case, the argument of the integral has two poles inside the contour: one at $z = 1$ with residue

$$\lim_{z \to 1} (z-1) \frac{z^k}{z-1} = 1,$$

and one pole at $z = 0$ with residue found as in (2.109)(if $k < 0$, then z^{-k} removes the pole):

$$\begin{aligned} z^{-k} \frac{z^k}{z-1} &= -\frac{1}{1-z} \\ &= -(1 + z + z^2 + \cdots + z^{-k-1} + \cdots). \end{aligned}$$

The residue is thus -1 for all k, and the sum of the residues is zero, and

$$f_1(k) = 0, \qquad k < 0. \tag{2.111}$$

For $k \geq 0$, the argument of the integral in (2.110) has only the pole at $z = 1$ with residue 1. Thus

$$f_1(k) = 1, \qquad k \geq 0. \tag{2.112}$$

Equations (2.108) and (2.109) correspond to the unit-step function, as they should. We would write the inverse transform symbolically $\mathcal{Z}^{-1}\{.\}$ as, in this case,

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = 1(k) \tag{2.113}$$

when $z/(z-1)$ converges for $|z| > 1$.

If, on the other hand, convergence is inside the unit circle, then for $k \geq 0$, there are no poles of the integrand contained in the contour, and

$$f_2(k) = 0, \qquad k \geq 0.$$

At $k < 0$, there is a pole at the origin of z, and as before, the residue is equal to -1 there, so

$$f_2(k) = -1, \qquad k < 0.$$

In symbols, corresponding to (2.113), we have

$$\mathcal{Z}^{-1}\left\{\frac{z}{z-1}\right\} = 1(k) - 1$$

when $z/(z-1)$ converges for $|z| < 1$.

Although, as we have just seen, the inverse integral can be used to compute an expression for a sequence to which a transform corresponds, a more effective use of the integral is in more general manipulations. We consider one such case that will be of some interest later. First, we consider an expression for the transform of a product of two sequences. Suppose we have

$$f_3(k) = f_1(k)f_2(k),$$

and f_1 and f_2 are such that the transform of the product exists. An expression for $F_3(z)$ in terms of $F_1(z)$ and $F_2(z)$ can be developed as follows. By definition

$$F_3(z) = \sum_{k=-\infty}^{\infty} f_1(k)f_2(k)z^{-k}.$$

From the inversion integral, (2.107), we can replace $f_2(k)$ by an integral:

$$F_3(z) = \sum_{k=-\infty}^{\infty} f_1(k) z^{-k} \frac{1}{2\pi j} \oint_{C_2} F_2(\zeta)\zeta^k \frac{d\zeta}{\zeta}.$$

We assume that we can find a region where we can exchange the summation with the integration. The contour will be called C_3 in this case:

$$F_3(z) = \frac{1}{2\pi j} \oint_{C_3} F_2(\zeta) \sum_{k=-\infty}^{\infty} f_1(k) \left(\frac{z}{\zeta}\right)^{-k} \frac{d\zeta}{\zeta}.$$

The sum can now be recognized as $F_1(z/\zeta)$ and, when we substitute this,

$$F_3(z) = \frac{1}{2\pi j} \oint_{C_3} F_2(\zeta) F_1\left(\frac{z}{\zeta}\right) \frac{d\zeta}{\zeta}, \tag{2.114}$$

the contour C_3 must be in the overlap of the convergence regions of $F_2(\zeta)$ and $F_1(z/\zeta)$. Then $F_3(z)$ will converge for the range of values of z for which C_3 can be found.

If we let $f_1 = f_2$ and $z = 1$ in (2.114), we have the discrete version of Parseval's theorem, where convergence is on the unit circle:

$$F_3(1) = \sum_{k=-\infty}^{\infty} f_1^2 = \frac{1}{2\pi j} \oint_C F_1(\zeta) F_1\left(\frac{1}{\zeta}\right) \frac{d\zeta}{\zeta}. \tag{2.115}$$

This particular theorem shows how we can compute the sum of squares of a time sequence by evaluating a complex integral in the z-domain. The result is useful in the design of systems by least squares.

2.7.2 Another Derivation of the Transfer Function

Let $\mathcal{D}$ be a discrete system which maps an input sequence, $\{e(k)\}$, into an output sequence $\{u(k)\}$.[26] Then, expressing this as an operator on $e(k)$, we have

$$u(k) = \mathcal{D}\{e(k)\}.$$

If $\mathcal{D}$ is linear, then

$$\mathcal{D}\{\alpha e_1(k) + \beta e_2(k)\} = \alpha\mathcal{D}\{e_1(k)\} + \beta\mathcal{D}\{e_2(k)\}. \tag{2.116}$$

[26]This derivation was suggested by L. A. Zadeh in 1952 at Columbia University.

If the system is constant, a shift in $e(k)$ to $e(k+j)$ must result in no other effects but a shift in the response, u. We write

$$\mathcal{D}\{e(k+j)\} = u(k+j) \qquad \text{for all } j \tag{2.117}$$

if

$$\mathcal{D}\{e(k)\} = u(k).$$

Theorem. If $\mathcal{D}$ is linear and constant and is given an input z^k for a value of z for which the output is finite at time k, then the output will be of the form $H(z)z^k$.

Proof. In general, if $e(k) = z^k$, then an arbitrary finite response can be written

$$u(k) = H(z,k)z^k.$$

Consider $e_2(k) = z^{k+j} = z^j z^k$ for some fixed j. From (2.116), if we let $\alpha = z^j$, it must follow that

$$\begin{aligned} u_2 &= z^j u(k) \\ &= z^j H(z,k)z^k \\ &= H(z,k)z^{k+j}. \end{aligned} \tag{2.118}$$

From (2.117), we must have

$$\begin{aligned} u_2(k) &= u(k+j) \\ &= H(z,j+k)z^{k+j} \qquad \text{for all } j. \end{aligned} \tag{2.119}$$

From a comparison of (2.118) and (2.119), it follows that

$$H(z,k) = H(z,k+j) \qquad \text{for all } j;$$

that is, H does not depend on the second argument and can be written $H(z)$. Thus for the elemental signal $e(k) = z^k$, we have a solution $u(k)$ of the same (exponential) shape but modulated by a ratio $H(z)$, $u(k) = H(z)z^k$.

Can we represent a general signal as a *linear sum* (integral) of such elements? We can, by the inverse integral derived above, as follows:

$$e(k) = \frac{1}{2\pi j} \oint E(z) z^k \frac{dz}{z}, \tag{2.120}$$

where

$$E(z) = \sum_{-\infty}^{\infty} e(k) z^{-k}, \qquad r < |z| < R, \tag{2.121}$$

for signals with $r < R$ for which (2.121) converges. We call $E(z)$ the z-transform of $e(k)$, and the (closed) path of integration is in the annular region of convergence of (2.121). If $e(k) = 0$, $k < 0$, then $R \to \infty$, and this region is the whole z-plane *outside* a circle of finite radius.

The consequences of linearity are that the response to a sum of signals is the sum of the responses as given in (2.116). Although (2.120) is the limit of a sum, the result still holds, and we can write

$$u(k) = \frac{1}{2\pi j} \oint E(z)[\text{response to } z^k] \frac{dz}{z},$$

but, by the theorem, the response to z^k is $H(z)z^k$. Therefore we can write

$$\begin{aligned} u(k) &= \frac{1}{2\pi j} \oint E(z)[H(z)z^k] \frac{dz}{z}. \\ &= \frac{1}{2\pi j} \oint H(z)E(z) z^k \frac{dz}{z}. \end{aligned} \tag{2.122}$$

We can define $U(z) = H(z)E(z)$ by comparison with (2.120) and note that

$$U(z) = \sum_{k=-\infty}^{\infty} u(k) z^{-k} = H(z)E(z). \tag{2.123}$$

Thus $H(z)$ is the *transfer function*, which is the ratio of the transforms of $e(k)$ and $u(k)$ as well as the amplitude response to inputs of the form z^k.

This derivation begins with linearity and stationarity and derives the z-transform as the natural tool of analysis from the fact that input signals in the form z^k produce an output that has the same shape.[27] It is somewhat

[27]Because z^k is unchanged in shape by passage through the linear constant system, we say that z^k is an eigenfunction of such systems.

more satisfying to derive the necessary transform than to start with the transform and see what systems it is good for. Better to start with the problem and find a tool than start with a tool and look for a problem. Unfortunately, the direct approach requires extensive use of the inversion integral and more sophisticated analysis to develop the main result, which is (2.123). Chacun à son goût.

2.8 SUMMARY

In this chapter we have shown how systems described by linear difference equations with constant coefficients can be described by transfer functions if the signals are represented by z-transforms. The transfer function was shown to be the z-transform of the unit-pulse response of the system; and, furthermore, the system output was shown to be the convolution of the input with the unit-pulse response. Using this result, we showed a condition for the Bounded-Input-Bounded-Output stability of a linear, constant system. The test developed by Jury for a polynomial to have all roots inside the unit circle was introduced to provide a stability test. We introduced the observer and the control canonical forms for transfer functions and gave rules for block-diagram reduction of transfer functions. We also introduced the state descriptions of these canonical forms and showed how to derive the matrices of the dynamic system in state form. We then derived the discrete transform for a sampled-data system both by transform and by state-space methods. The latter are especially well suited for computer implementation.

We studied the dynamic response of discrete systems, including especially the step response of a second-order system. The effects of the location of a zero and of a third pole were plotted, largely for future reference in design.

Several of the properties of the z-transform were demonstrated, and the calculation of the inverse of a z-transform was presented by long division, by partial-fraction expansion, and by evaluation of the inverse transform integral.

PROBLEMS AND EXERCISES

2.1 Check the following for stability:

a) $u(k) = 0.5u(k-1) - 0.3u(k-2)$

b) $u(k) = 1.6u(k-1) - u(k-2)$

c) $u(k) = 0.8u(k-1) + 0.4u(k-2)$

2.2 a) Derive the difference equation corresponding to the approximation of integration found by fitting a parabola to the points e_{k-2}, e_{k-1}, e_k and taking the area under this parabola between $t = kT - T$ and $t = kT$ as the approximation to the integral of $e(t)$ over this range.

b) Find the transfer function of the resulting discrete system and plot the poles and zeros in the z-plane.

2.3 Verify that the transfer function of the system of Fig. 2.8(c) is given by the same $H(z)$ as the system of Fig. 2.9(c).

2.4 a) Compute and plot the unit-pulse response of the system derived in Problem 2.2.

b) Is this system BIBO stable?

2.5 Consider the difference equation

$$u(k+2) = 0.25u(k).$$

a) Assume a solution $u(k) = A_i z^k$ and find the characteristic equation in z.

b) Find the characteristic roots z_1 and z_2 and decide if the equation solutions are stable or unstable.

c) Assume a general solution of the form

$$u(k) = A_1 {z_1}^k + A_2 {z_2}^k$$

and find A_1 and A_2 to match the initial conditions $u(0) = 0, u(1) = 1$.

d) Repeat parts (a), (b), and (c) for the equation

$$u(k+2) = -0.25u(k)$$

e) Repeat parts (a), (b), and (c) for the equation

$$u(k+2) = u(k+1) - 0.5u(k).$$

2.6 Show that the characteristic equation

$$z^2 - 2r\cos(\theta)z + r^2$$

has the roots

$$z_{1,2} = re^{\pm j\theta}.$$

2.7 a) Use the method of block-diagram reduction, applying Figs 2.5, 2.6, and 2.7 to compute the transfer function of Fig 2.8(c).

b) Repeat part (a) for the diagram of Fig. 2.9(c).

2.8 Apply Jury's test to determine if the following characteristic equations have any roots outside the unit circle.

a) $z^2 + 0.25$
b) $z^3 - 1.1z^2 + 0.01z + 0.405$
c) $z^3 - 3.6z^2 + 4z - 1.6$

2.9 Compute by hand and table look-up the discrete transfer function if the $G(s)$ in Fig. 2.13 is

a) $\dfrac{K}{s}$ **b)** $\dfrac{3}{s(s+3)}$

c) $\dfrac{3}{(s+1)(s+3)}$ **d)** $\dfrac{(s+1)}{s^2}$

e) $\dfrac{e^{sT/2}}{s^2}$ **f)** $\dfrac{(1-s)}{s^2}$

g) $\dfrac{3e^{-1.5Ts}}{(s+1)(s+3)}$

h) Repeat the calculation of these discrete transfer functions using a CAD tool. Compute for the sampling period $T = 0.05$ and $T = 0.5$ and plot the location of the poles and zeros in the z-plane.

2.10 Use a CAD tool to compute the discrete transfer function if the $G(s)$ in Fig. 2.13 is

a) the two-mass system with the noncolocated actuator and sensor of (1.4) with sampling periods $T = 0.02$ and $T = 0.1$. Plot the zeros and poles of the results in the z-plane. Let $w_p = 5$, $\zeta_p = 0.01$.
b) the two-mass system with the colocated actuator and sensor given by (1.5). Use $T = 0.02$ and $T = 0.1$. Plot the zeros and poles of the results in the z-plane. Let $w_p = 5$, $w_z = 3$, $\zeta_p = \zeta_z = 0$.
c) the two-input–two-output paper machine described on page 788. Let $T = 0.1$ and $T = 0.5$.

2.11 Consider the system described by the transfer function

$$\frac{Y}{U} = G(s) = \frac{3}{(s+1)(s+3)}.$$

a) Draw the block diagram corresponding to this system in control canonical form, define the state vector, and give the corresponding description matrices $\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}$.
b) Write $G(s)$ in partial fractions and draw the corresponding parallel block diagram with each component part in control canonical form. Define the state ξ and give the corresponding state description matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$.
c) By finding the transfer functions X_1/U and X_2/U of part (a) in partial fraction form, express x_1 and x_2 in terms of ξ_1 and ξ_2. Write these relations as the two-by-two transformation $\mathbf{T}$ such that $\mathbf{x} = \mathbf{T}\xi$.

d) Verify that the matrices you have found are related by the formulas

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T},$$
$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{G},$$
$$\mathbf{C} = \mathbf{H}\mathbf{T},$$
$$\mathbf{D} = \mathbf{J}.$$

2.12 The first-order system $(z-\alpha)/(1-\alpha)z$ has a zero at $z=\alpha$.

a) Plot the step response for this system for $\alpha = 0.8, 0.9, 1.1, 1.2, 2$.

b) Plot the overshoot of this system on the same coordinates as those appearing in Fig. 2.30 for $-1 < \alpha < 1$.

c) In what way is the step response of this system unusual for $\alpha > 1$?

2.13 The one-sided z-transform is defined as

$$F(z) = \sum_{0}^{\infty} f(k) z^{-k}.$$

a) Show that the one-sided transform of $f(k+1)$ is

$$\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0).$$

b) Use the one-sided transform to solve for the transforms of the Fibonacci numbers by writing (2.4) as $u_{k+2} = u_{k+1} + u_k$. Let $u_0 = u_1 = 1$. [You will need to compute the transform of $f(k+2)$.]

c) Compute the location of the poles of the transform of the Fibonacci numbers.

d) Compute the inverse transform of the numbers.

e) Show that if u_k is the kth Fibonacci number, then the ratio u_{k+1}/u_k will go to $(1+\sqrt{5})/2$, the golden ratio of the Greeks.

f) Show that if we add a forcing term, $e(k)$, to (2.4) we can generate the Fibonacci numbers by a system that can be analyzed by the two-sided transform; i.e., let $u_k = u_{k-1} + u_{k-2} + e_k$ and let $e_k = \delta_0(k)[\delta_0(k) = 1$ at $k = 0$ and zero elsewhere]. Take the two-sided transform and show that the same $U(z)$ results as in part (b).

2.14 Substitute $u = Az^k$ and $e = Bz^k$ into (2.2) and (2.7) and show that the transfer functions, (2.15) and (2.14), can be found in this way.

2.15 Consider the transfer function

$$H(z) = \frac{(z+1)(z^2 - 1.3z + 0.81)}{(z^2 - 1.2z + 0.5)(z^2 - 1.4z + 0.81)}.$$

Draw a *cascade* realization, using observer canonical forms for second-order blocks and in such a way that the coefficients as shown in $H(z)$ above are the parameters of the block diagram.

2.16 a) Write the $H(z)$ of Exercise 2.15 in partial fractions in two terms of second order each, and draw a *parallel* realization, using the observer canonical form for each block and showing the coefficients of the partial-fraction expansion as the parameters of the realization.

b) Suppose the two factors in the denominator of $H(z)$ were identical (say we change the 1.4 to 1.2 and the 0.81 to 0.5). What would the parallel realization be in this case?

2.17 Show that the observer canonical form of the system equations shown in Fig. 2.9 can be written in the state-space form as given by (2.25).

2.18 Draw out each block of Fig. 2.10 in (a) control and (b) observer canonical form. Write out the state-description matrices in each case.

2.19 For a second-order system with damping ratio 0.5 and poles at an angle in the z-plane of $\theta = 30°$, what percent overshoot to a step would you expect if the system had a zero at $z_2 = 0.6$?

2.20 Consider a signal with the transform (which converges for $|z| > 2$)

$$U(z) = \frac{z}{(z-1)(z-2)}.$$

a) What value is given by the formula (final-value theorem) of (2.100) applied to this $U(z)$?

b) Find the final value of $u(k)$ by taking the inverse transform of $U(z)$, using partial-fraction expansion and the tables.

c) Explain why the two results of (a) and (b) differ.

2.21 a) Find the z-transform and be sure to give the region of convergence for the signal

$$u(k) = r^{+|k|}, \qquad r < 1.$$

[*Hint:* Write u as the sum of two functions, one for $k \geq 0$ and one for $k < 0$, find the individual transforms, and determine values of z for which *both* terms converge.]

b) If a rational function $U(z)$ is known to converge on the unit circle $|z| = 1$, show how partial-fraction expansion can be used to compute the inverse transform. Apply your result to the transform you found in part (a).

2.22 Compute the inverse transform, $f(k)$, for each of the following transforms:

a) $F(z) = \dfrac{1}{1 + z^{-2}}, \qquad |z| > 1;$

b) $F(z) = \dfrac{z(z-1)}{z^2 - 1.25z + 0.25}, \qquad |z| > 1;$

c) $F(z) = \dfrac{z}{z^2 - 2z + 1}, \qquad |z| > 1;$

d) $F(z) = \dfrac{z}{(z - \frac{1}{2})(z - 2)}, \qquad 1/2 < |z| < 2.$

2.23 Use the z-transform to solve the difference equation

$$y(k) - 3y(k-1) + 2y(k-2) = 2u(k-1) - 2u(k-2),$$

$$u(k) = \begin{cases} k, & k \geq 0, \\ 0, & k < 0, \end{cases}$$

$$y(k) = 0, \qquad k < 0.$$