PS 3 – Q5 Hints

ELEC3004 — S. Singh May 21, 2019

The idea here is that there are spectrums of light frequencies that are combined to make to make a colour's response.

For example, for the cone cells in the eye (from Wikipedia) one gets:

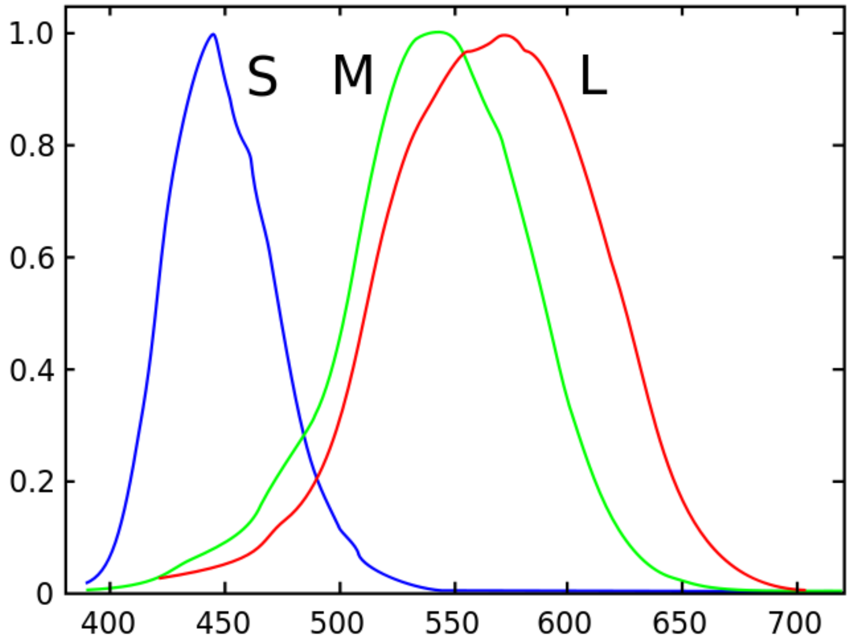

Figure 1: Normalized responsivity spectra of human cone cells, S, M, and L types (light wavelength in nm)

This means the camera (and the eye) are not just responding to one very specific spectral values, but a host of colour values. That is, the red, green, and blue "dyes" that make up the colour filter of a camera's Bayer mask cover a range of values.

The response curves can be thought of as consisting of D discrete values (or "samples"). For example, for the human cone curves above taken with 50-nm bins would give 8 bins or D=8. This is related to the dual question of how to design a series of spectral response curves for a camera.

1. Metamers

Visually identical cases with different spectral power compositions are called metamers Let's encode the spectrum response curves as consisting of D elements. Then we can assemble them into a $3 \times D$ matrix:

$$A = \begin{bmatrix} r_1 & \cdots & r_D \\ g_1 & \cdots & g_D \\ b_1 & \cdots & b_D \end{bmatrix}$$

Where the spectrum of light coming in then encoded as:

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_D \end{bmatrix}$$

Thus, the output response will be:

$$O = \begin{bmatrix} R \\ G \\ B \end{bmatrix} = AP$$

Thus, for the output due to p and p to be visually indistinguishable, we would require their respective outputs, which we could label as O and O, to be the same.

If $O = \hat{O}$, then:

$$Ap = A\hat{p}$$
$$A(p - \hat{p}) = 0$$

Thus, the outputs are the same for the spectrums (i.e. p and $\hat{p}$ are metamers) when (p-$\hat{p}$) in in the null space of A. (Note that the this would also include the "trivial" case when $p = \hat{p}$).

2. Colour Matching

In a colour matching operation, P_{test} is equal to P_{match}. We can describe the colour matching operation in the notation used in the previous problem. Thus:

$$O_{match} = AP_{match}$$
$$O_{match} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = (A[\lambda_u \quad \lambda_v \quad \lambda_w]) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Let's define:

$$B = (A[\lambda_u \quad \lambda_v \quad \lambda_w])$$

Then:

$$O_{match} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix} = (B) \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Or:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = B^{-1} \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$$

In this light, the question may be seen as asking if it is always possible to find a series of values a_1, a_2, and a_3 such the above equation holds. This would require that B would be invertible.

Is B necessarily invertible? Physically, A is full rank if the R, G, and B responses are linearly independent. (Are they?) The matrix P is full rank if the spectra of the primary lights are independent. (Are they?) And, even if both A and P are full rank, B could still be singular. This is because B is not invertible if its component matrices are rank deficient. For example, if $rank(A) < 3$ or $rank(P) < 3$ then B will be singular (and not invertible). FYI, primary lights that generate an invertible B are called *visually independent*.

If B is invertible, then an a_1, a_2, and a_3 could be found via:

$$O_{test} = AP_{test} = (B)\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

The question then asks how this process varies based on the number of bands, which would affect the dimension of A. It then asks if the number of bits of digitization (Q) would affect if the spectrum stays linearly independent.

3. Illumination Effects

There are two different objects (α and β) and two different sets of lights – L_{led} and L_{candle}. The question asks if it is possible that the two objects can *appear* the same under one set of lights and be different under another. A quick thought experiment to consider is if the object is green leaf under green light and then red light. But, one can (and should) treat this more generally.

The essence of appearance is reflectance. Consider the case where the reflectance is given by $O_{output} = ARP$. Thus if the two objects appear the same under a given lighting L, then

$$(AR_\alpha P_L) = (AR_\beta P_L)$$

We can substitute R_α and R_β with $\boldsymbol{R}$ and $\boldsymbol{S}$ respectively and treat P_L as fixed for now (thus P):

$$(A\boldsymbol{R}P) = (A\boldsymbol{S}P)$$

Thus, the two objects (α and β) looking the same, implies:

$$\left(A \underbrace{\begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_D \end{bmatrix}}_{\boldsymbol{R}} P \right) = \left(A \underbrace{\begin{bmatrix} s_1 & 0 & \dots & 0 \\ 0 & s_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & s_D \end{bmatrix}}_{\boldsymbol{S}} P \right)$$

This is equivalent to $(\boldsymbol{R} - \boldsymbol{S})P \in N(A)$, where $N(A)$ is the null space of A. Thus for LED light: $(\boldsymbol{R} - \boldsymbol{S})P_{LED} \in N(A)$.

Now, let us change the illumination to P_{candle}. For the objects ($\boldsymbol{R}$ and $\boldsymbol{S}$) look different under P_{candle}, then that would mean

$$(A\boldsymbol{R}P_{candle}) \neq (A\boldsymbol{S}P_{candle})$$

This would imply $(\boldsymbol{R} - \boldsymbol{S})P_{candle} \notin N(A)$.

Consider the dimension of A. Is it possible that both statements ($(\boldsymbol{R} - \boldsymbol{S})P_{LED} \in N(A)$ and $(\boldsymbol{R} - \boldsymbol{S})P_{candle} \notin N(A)$) could be met? For a certain D, Sure. What size of D though? ☺

4. Design

This question is basically asking to “design” the elements in the matrix A. Of course, we could (and often do) see multiple A matrices for different lighting types (the “sunlight”, “fluorescent”, “candlelight”, etc. colour modes on a pocket/SLR camera). Enjoy! ☺ And, to the claim in the *NYT* article, does this process risk the introduction of a bias?