THE UNIVERSITY OF QUEENSLAND
AUSTRALIA

This exam paper must not be removed from the venue

Venue ____________________

Seat Number ________

Student Number ______________________

Name ______________________

School of Information Technology and Electrical Engineering

EXAMINATION

Semester One Final Examinations, 2019

ELEC3004 Signals, Systems and Control

This paper is for St Lucia Campus students.

Examination Duration: 180 minutes

Reading Time: 10 minutes

Exam Conditions:

This is a Central Examination

This is a Closed Book Examination - specified materials permitted

During reading time - write only on the rough paper provided

This examination paper will be released to the Library

Materials Permitted In The Exam Venue:

(No electronic aids are permitted e.g. laptops, phones)

Calculators - Any calculator permitted - unrestricted

One A4 sheet of handwritten or typed notes double sided is permitted

Materials To Be Supplied To Students:

N/A (Everything is here, including our best wishes :-))

Instructions To Students:

Additional exam materials (eg. answer booklets, rough paper) will be provided upon request.

Please be sure to justify all answers. (Solutions without justification are insufficient). Please label final solutions clearly.

For Examiner Use Only

Q	Mark
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Total ________

⇨ PLEASE RECORD ALL ANSWERS ⇦
⇨ IN THE EXAM PAPER ⇦

The white space provided for a question maybe more than needed.

Please **do not** answer a question in the space reserved for another question (e.g. **do not** answer part of Question 1 in space for Question 2).

If you have **exhausted all the pages in this exam paper**, *then* please request an optional answer booklet.

Please mark final answers clearly. *Thank you!*

Some General Exam Taking Tips:

- **Please pace wisely**: spend time proportionately to the question and its mark
- **The questions do not need to be answered sequentially**. For example, one may answer Question 2 before Question 1.
  (However, remember not to answer Question 2 in Question 1's space).

This exam has THREE (3) Sections for a total of 180 Points
(which very roughly, on the whole, corresponds to ~1 Point/Minute)

Section 1: Digital Linear Dynamical **Systems**..60 Points (33 %)
Section 2: Digital Processing/Filtering of **Signals**................................. 60 Points (33 %)
Section 3: Digital & State-Space **Control**..60 Points (33 %)

⇨ Please answer **ALL** questions + **ALL Answers MUST Be Justified** ⇦

⇨ **PLEASE RECORD ALL ANSWERS IN THE EXAM** ⇦

Section 1: Digital Linear Dynamical Systems

Please **Justify and Explain All Answers** (**4 Questions** | **60 Points**)

1. **A Fine Start to LTI Signals and Systems** (12 Points)
   Consider the system with a scalar gain $\boldsymbol{k}$ given by:

$$y_i[n] = \boldsymbol{k_i} x[n] + 3004$$

   A. Is this system linear? Please briefly explain.

   B. Consider a new system consisting of the difference between the responses to two versions of the aforementioned system

$$z[n] = y_1[n] - y_2[n]$$

   Is this second system, $z[n]$, linear? Please briefly explain.

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2. ***A* Naughty Identity** (12 Points)

For a linear function $f(x) = \boldsymbol{A}x$, may it be concluded that $\boldsymbol{A}^{\mathbf{0}} = \boldsymbol{I}$?

Please explain.

Is the statement true or false or "maybe" (i.e. true under certain conditions)?
If possible under certain conditions (i.e. "maybe"), *then* please enumerate what constraints might have to be placed on $\boldsymbol{A}$ to make this true.

3. **Linearity's True Nature** (16 Points)
Please state if the following statements are **TRUE (T)** or **FALSE (F).**
Kindly circle the answer Ⓣ or Ⓕ **and also** provide a **brief justification**.

(a) **A Linear Function ⟹ Matrix** **[T | F]**
A linear function can **always be** represented as a matrix-vector multiplication.
That is, there is a $m \times n$ matrix $\boldsymbol{A}$ such that $f(x) = \boldsymbol{A}x$ for all vectors x ($x \in \mathbb{R}^n$)

(b) **A Matrix ⟹ Linear Function** **[T | F]**
All functions defined by a matrix-vector multiplication are linear.
(i.e., the converse of (a))

(c) **Coefficients** **[T | F]**
For a matrix function $y = \boldsymbol{A}x$, the coefficient A_{ij} describes how y_i depends on x_j.

(d) **Linear Systems are Smooth** **[T | F]**
Linear Systems also satisfy the condition of smoothness, where small changes in the system's inputs result in a small change in its outputs.

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(e) A Singularity is Inconsistent **[T | F]**

For a matrix function $y = \boldsymbol{A}x$, if A is singular it means that the system is redundant or inconsistent.

(f) Nothing Ventured, Nothing Gained **[T | F]**

For a consistent Linear System, zero output means there is zero input.

(g) QR: Quickly Realize Eigenvalues? **[T | F]**

The QR decomposition of a $n \times n$ system matrix $\boldsymbol{A}$ gives a mechanism to calculate the eigenvalues of $\boldsymbol{A}$ quickly.

(h) Matrix Multiplication FOILed? **[T | F]**

Consider two $n \times n$ system matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, may we simplify them as follows:

$$(\boldsymbol{A} + \boldsymbol{B})^2 = \boldsymbol{A}^2 + 2\boldsymbol{A}\boldsymbol{B} + \boldsymbol{B}^2$$

4. **A Sound Sampling Strategy?** (20 Points)

Two recording mangers at LectopiaLand Records, Bobbie Ray (henceforth, "BR") and Charlie Davidson (henceforth "CD"), and are wondering what sampling rate to use for their fine **stereo audio recordings** that have a bandwidth of 20 Hz to 18 kHz.

- **BR** advocates for 192 kHz, 24-bit sampling (Blue-Ray audio standard). BR claims that the higher frequency sounds in the recording would be "corrupted" if they only have a couple of samples points per wave.

- **CD** suggests that 44.1 kHz, 16-bits sampling (CD audio standard) is sufficient. CD claims that the sampling of a band-limited signal is a lossless and rather that any "loss" or "corruption" is from the anti-aliasing, low-pass filters on the input.

We assume the audiences' hearing range has a bandwidth of 20 Hz to 18kHz and that the source recording has 12-bits of dynamic range. Given your understanding of signal-processing and sampling, let us explore, which sampling policy should be used.
Please remember to justify all answers.

A. CD asserts that "(bandlimited) sampling is a lossless process."
(1) Is this correct? **(2)** Can the audio signal be perfectly reconstructed using a CD audio standard?

B. BR adds that the anti-aliasing filters have a transition band. **(1)** How large a transition band does each sampling strategy (Blue-ray audio and CD audio) allow in this case? **(2)** What sampling strategy does this suggest?

C. BR also claims that "higher sampling rates have better time resolution."
CD counters that "sampling bandlimited signals has nothing to do with time resolution (i.e., it has infinite time resolution)".
Who is correct? Please explain.

D. Finally, BR contends that higher sampling rate will result in bigger files that take more time, and thus are more arduous to copy. For a recording that is 3004 seconds long (50:04 min:sec),
(1) How much bigger will the resulting file be?
(2) How much longer will it take to transmit on an average Australian Internet Connection (11.1 Mbps*)?

[**Please assume**: no compression, negligible codec effects, negligible network overhead, negligible packet loss, etc.]

* Source: Akamai's State of the Internet Report, 2017, URL: `https://www.akamai.com/us/en/multimedia/documents/state-of-the-internet/q1-2017-state-of-the-internet-connectivity-report.pdf`

…Question 4. A Sound Sampling Strategy **[Extra Leaf 1/2]** (Answer Space Continued)

…**Question 4**. A Sound Sampling Strategy **[Extra Leaf 2/2]** (Answer Space Continued)

<u>Section 2: Digital Processing & Filtering of Signals</u>

Please **<u>Justify and Explain All Answers</u>** (**6 Questions** | **60 Points**)

5. **Limited-edition Signals** (5 Points)
   Is it possible to have a non-trivial signal that is **both** time-limited and band-limited? Please briefly explain (i.e. provide a brief proof).

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6. **Analog Filtering on Tap** (10 Points)

State the main characteristics of the following analogue prototype filters:

- Butterworth
- Chebyshev (Type I)
- Chebyshev (Type II)
- Elliptic (Cauer)

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7. **Nyquist's Call** (10 Points)
On one side of a telephone call (master) there is a buzzer playing a note at **11,004 Hz**. On the other side (remote) is a crystal glass with a resonant frequency of **3004 Hz**. The telephone is a **band-limited** digital system that samples at **8000 Hz**.

A. If the buzzer's note (treat it as a pure tone or sine wave) is played, will the glass shatter on the other side? [**Please remember to briefly justify**]

B. Now if we add a whistle **at 4,996 Hz**, will this make a difference? [**Please remember to briefly justify**]

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8. **Josiah Gibbs's Great Invention on Bounded Basis and Systems** (10 Points)

   In 1899 Gibbs was doing peer-review (without Platypus) on Michelson's harmonic analyser (i.e. a Fourier Series approximator).

   A. What is Gibbs phenomenon?
      Please briefly explain.

   B. Please illustrate this in relation to the digital processing of square wave.

   C. For a digital-system, would passing the signal through the another anti-aliasing / smoothing filter help remove these phenomena?

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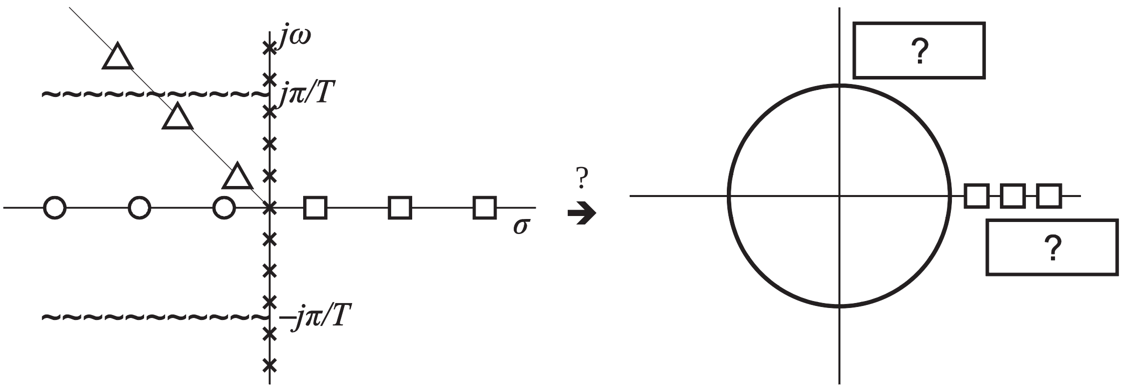
9. **An E-Z Correspondence** (10 points)
In mapping from the ***s-plane*** to the ***z-plane***, recall that the duration of a time signal is related to the radius (of the pole location) and the sample rate is related to the angle by $z = e^{sT}$. From this we can sketch major features of the ***s-plane*** to the ***z-plane*** such that they have the same features.

For the following poles marked on the ***s-plane***:
a) "○"
b) "△"
c) "×",
d) "~", and
e) The axis labels (σ and $j\omega$)

please draw their corresponding locations (and/or terms) on the ***z-plane***.
Please also **briefly justify** your mapping/answer.



s-plane

Corresponding points on the ***z-plane***
(e.g. the ☐ on the σ-$j\omega$ maps to points outside the unit circle in the **z-plane** as marked)

10. **Making a Difference via Filtering** (15 Points)

A Causal LTI System is described by the difference equation

$$y[n] = y[n-a] + y[n-b] + x[n-c]$$

A. Find the system function $H(z) = \frac{Y(z)}{X(z)}$

B. Is this filter IIR or FIR or both?

C. For $a = 1$, $b = 2$, and $c = 1$, Plot the poles and zeros of $H(z)$ on the γ-Plane

D. What is the Region of Convergence for $H(z)$

E. Is the system stable or unstable?
If the system is stable, what type of filter is it? (high-pass, low-pass, etc.)
If the system is unstable, what values of a, b, and c might make it stable again.

…**Question 10**. Making a Difference via Filtering **[Extra Leaf 1/2]** (Answer Space Continued)

…**Question 10**. Making a Difference via Filtering **[Extra Leaf 2/2]** (Answer Space Continued)

Section 3: Digital & State-Space Control

Please **Justify and Explain All Answers** (**5 Questions** | **60 Points**)

11. **I ♥ Digital Systems** (5 Points)

For discrete time second order systems, a curve of constant damping (ζ) is a logarithmic spiral (i.e. heart-shape) in the γ plane (z-domain). **Please prove why this is**.

[**Hint**: The function $z = e^{sT}$ may be thought of as a mapping. If one takes that view, then consider what a constant ζ would be the Laplace (s) domain.]

12. **Closing the Digital Divide** (10 Points)

A system's signal(s) maybe may said to consist of two forms:
Continuous ($f(t)$) and Discrete ($f[k]$).

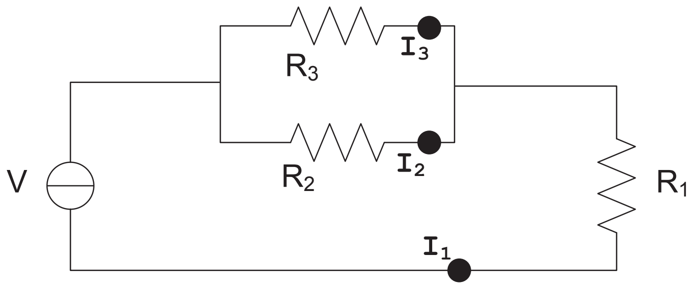
Thus, a SISO system may be seen to consist of four types of elements (as below). For **each** of the remaining types (labelled I, II, III) of possible elements, please:

1. Write an appropriate system equation for a second order system using typical notation. One may use the Laplace (s) or Z-Transform (z) as needed. If needed you may introduce variables, but please define them.
2. Determine the relative phase lag compared to a continuous-continuous system (may be left in terms of the sampling period, T_s)
3. Give a specific and unique example for each of the cases.

Output→ Input ↴	Continuous	Discrete
Continuous	$A(s) = s^2 + \mathrm{B}s + \mathrm{C}$ Relative Phase Lag = 0 (by definition) Example: RLC Circuit	I
Discrete	II	III

…Question 12. Closing the Digital Divide **[Extra Leaf 1/1]** (Answer Space Continued)

13. **A Stately Circuit** (15 Points)
Consider the simple resistive circuit given below with voltage source V and currents I_1, I_2 and I_3.



A. If we consider V, R_1, R_2, and R_3 as constants. Please determine linear system model as a function of the currents for a voltage output

B. Based on the solution to (a), if this system were to be expressed in Linear Systems form, $y = \boldsymbol{A}x$, then what would the Matrix $\boldsymbol{A}$ and the vector x be?

C. Use this to solve for the currents I_1, I_2 and I_3.

[**Hint**: for partial credit, one may leave this in terms of y, $\boldsymbol{A}$ and x]

…**Question 13**. A Stately Circuit [**Extra Leaf 1/1**] (Answer Space Continued)

14. **Using State-Space to Shine Light on Business** (15 Points)

SNaF University wants to be "a leader in sustainability and the renewable energy field". Much to the chagrin of management, the fraction of the solar energy at SNaF is **1%**. While its management struggles with "leadership" (including its very definition), it does not struggle with hapless marketing. In response to another university's advertisement, the management has proclaimed that **20%** of SNaF's energy should come from solar.

Linear systems theory might be able to help here. We may consider the fraction of energy at SNaF at any given month to be a state-vector, $\boldsymbol{x_k}$, where $x_k = \begin{bmatrix} solar \\ non - solar \end{bmatrix}$; thus, for example, $x_0 = \begin{bmatrix} 0.01 \\ 0.99 \end{bmatrix}$. With the help of massive spending (as marketing budgets are large and discretionary), new panels are constantly being added. However, some panels are also lost due to lack of maintenance. With these two factors, the fraction of energy from solar each month is multiplied by the following (Markov†) matrix $\boldsymbol{A}$:

$$\boldsymbol{A} = \begin{bmatrix} 0.8 & 0.05 \\ 0.2 & 0.95 \end{bmatrix}$$

This is also a single step of a Markov chain, such that the vectors x_0, x_1, x_2, etc. are multiplied by $\boldsymbol{A}$ for each step. Thus giving, $x_1 = \boldsymbol{A}x_0$, $x_2 = \boldsymbol{A^2}x_0$,…, $x_k = \boldsymbol{A^k}x_0$.

Given the problem description, please describe the state vector x_k.

A. Will all the vectors of x_k (x_0, …, x_k) be nonnegative?
   If so, why? If not, why not?

B. What is the value of x_2? And , x_{12}? (i.e., after one year)

C. What is the final, steady-state value (x_∞)?

D. Thus, is the management's 20% goal achievable? When will it be achieved?

E. If SNaF-U‡ wanted to get to its goal faster, should it increase the rate it purchases solar cells or reduce the loss rate (i.e. improve the facilities group)?

† A Markov matrix is one with entries $A_{ij} > 0$ and where each column adds to 1

‡ *Scientia Nummus ac Fortuna* (Know Money and Luck) - University [or Some Nearby Academic Facility - University]

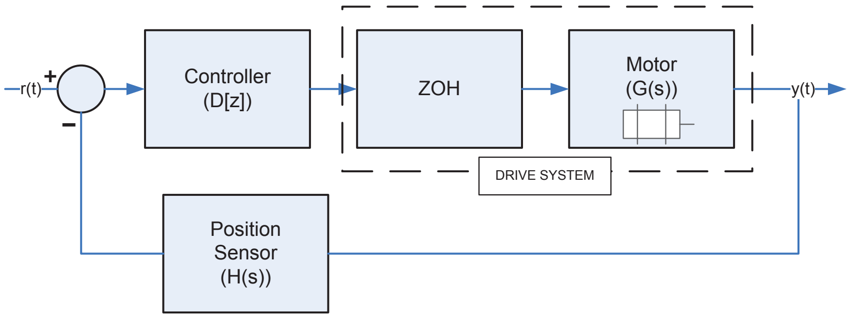
…Question 14. Using State-Space to Shine Light on Business **[1/2]** (Answer Space Continued)

…Question 14. Using State-Space to Shine Light on Business **[2/2]** (Answer Space Continued)

15. **End on a High Note (♩♪♫)** (15 Points)

LectopiaLand Records is designing a new phonograph (record player), *The Aria*. To improve on previous units, the manager suggests that the turntable be built with digital motor control so that the record rotates at the desired speed; and thus, high notes are correctly played "on a high note".

Chief Engineer, Eliza, proposes the following block diagram:



Eliza has selected a **1 kHz sample rate** for the digital controller and has characterised the turntable and sensor as: $G(s) = \frac{5}{s(s+20)}$ and $H(s) = 1$.

A. Eliza suggests that the fast pole (at $s = -20$) has an insignificant effect and that the model could be reduced to a first order model $G^* \approx \frac{k_1}{s}$
Is this correct? **Please explain.**
If so, what is the value of k_1 for this reduced system model G^*?

[**Hint**: Consider the magnitude for the fast pole; i.e. the value for e^{-sT}]

B. What is the Z-Transform of the Drive System (see also above figure)?
(i.e., as shown above, please consider the ZOH and the model from part (a)).

[**Hint**: The ZOH may be modelled as $G_{ZOH} = \frac{1-e^{-sT}}{s}$]

C. What is $D[z]$ if the controller is Proportional with constant gain of **D**?

D. Eliza first nominates $D = 4{,}000$, in this case what is the closed loop transfer function $T[z]$ for this system?

E. For the case above, plot the location of the open-loop and close loop poles on the γ plane. Discuss if the system is stable.

F. What happens if the sensor now has a large delay of $T_d = 123\ ms$?
What is the closed loop transfer $T_{delay}[z]$?

[**Hint**: If time is limited, solve for $T_{delay}[z]$ in terms of T_d]

G. What is your opinion of how Eliza's Aria will sound?

…Question 15. End on a High Note **[Extra Leaf 1/3]** (Answer Space Continued)

…Question 15. End on a High Note **[Extra Leaf 2/3]** (Answer Space Continued)

…Question 15. End on a High Note **[Extra Leaf 3/3]** (Answer Space Continued)

☺ **END OF EXAMINATION — *Thank you !!!*** ☺

Vanquish the spectre. Away with the tricks,
"A Signal is a Vector. A System is a Matrix!"

Let Linear Systems bring light.
Oh, Indeed!
Forevermore, shall your future be bright.
Godspeed! ☺

Additional Answer Space **[QUESTION: _______]**

Additional Answer Space **[QUESTION: _______]**

Additional Answer Space **[QUESTION: _______]**

Table 1: Commonly used Formulae

The Laplace Transform

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

The $\mathscr{Z}$ Transform

$$F(z) = \sum_{n=0}^{\infty} f[n]z^{-n}$$

IIR Filter Pre-warp

$$\omega_a = \frac{2}{\Delta t}\tan\left(\frac{\omega_d \Delta t}{2}\right)$$

Bi-linear Transform

$$s = \frac{2(1-z^{-1})}{\Delta t(1+z^{-1})}$$

FIR Filter Coefficients

$$c_n = \frac{\Delta t}{\pi}\int_0^{\pi/\Delta t} H_d(\omega)\cos(n\omega\Delta t)\,d\omega$$

Table 2: Comparison of Fourier representations.

Time Domain	**Periodic**	**Non-periodic**	
Discrete	**Discrete Fourier Transform** $\tilde{X}[k] = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-j2\pi kn/N}$ $\tilde{x}[n] = \sum_{k=0}^{N-1}\tilde{X}[k]e^{j2\pi kn/N}$	**Discrete-Time Fourier Transform** $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ $x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\omega}\right)e^{j\omega n}\,d\omega$	**Periodic**
Continuous	**Complex Fourier Series** $X[k] = \frac{1}{T}\int_{-T/2}^{T/2}\tilde{x}(t)e^{-j2\pi kt/T}\,dt$ $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} X[k]e^{j2\pi kt/T}$	**Fourier Transform** $X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$ $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$	**Non-periodic**
	Discrete	**Continuous**	**Freq. Domain**

Table 3: Selected Fourier, Laplace and z-transform pairs.

Signal	$\longleftrightarrow$	Transform	ROC
$\tilde{x}[n] = \sum_{p=-\infty}^{\infty} \delta[n-pN]$	$\xleftrightarrow{DFT}$	$\tilde{X}[k] = \frac{1}{N}$	
$x[n] = \delta[n]$	$\xleftrightarrow{DTFT}$	$X(e^{j\omega}) = 1$	
$\tilde{x}(t) = \sum_{p=-\infty}^{\infty} \delta(t-pT)$	$\xleftrightarrow{FS}$	$X[k] = \frac{1}{T}$	
$\delta_T[t] = \sum_{p=-\infty}^{\infty} \delta(t-pT)$	$\xleftrightarrow{FT}$	$X(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$	
$\cos(\omega_0 t)$	$\xleftrightarrow{FT}$	$X(j\omega) = \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$	
$\sin(\omega_0 t)$	$\xleftrightarrow{FT}$	$X(j\omega) = j\pi\delta(\omega+\omega_0) - j\pi\delta(\omega-\omega_0)$	
$x(t) = \begin{cases} 1 & \text{when } \lvert t\rvert \leqslant T_0, \\ 0 & \text{otherwise.} \end{cases}$	$\xleftrightarrow{FT}$	$X(j\omega) = \frac{2\sin(\omega T_0)}{\omega}$	
$x(t) = \frac{1}{\pi t}\sin(\omega_c t)$	$\xleftrightarrow{FT}$	$X(j\omega) = \begin{cases} 1 & \text{when } \lvert\omega\rvert \leqslant \lvert\omega_c\rvert, \\ 0 & \text{otherwise.} \end{cases}$	
$x(t) = \delta(t)$	$\xleftrightarrow{FT}$	$X(j\omega) = 1$	
$x(t) = \delta(t-t_0)$	$\xleftrightarrow{FT}$	$X(j\omega) = e^{-j\omega t_0}$	
$x(t) = u(t)$	$\xleftrightarrow{FT}$	$X(j\omega) = \pi\delta(w) + \frac{1}{jw}$	
$x[n] = \frac{\omega_c}{\pi} \operatorname{sinc} \omega_c n$	$\xleftrightarrow{DTFT}$	$X(e^{j\omega}) = \begin{cases} 1 & \text{when } \lvert\omega\rvert < \lvert\omega_c\rvert, \\ 0 & \text{otherwise.} \end{cases}$	
$x(t) = \delta(t)$	$\xleftrightarrow{\mathcal{L}}$	$X(s) = 1$	all s
(unit step) $x(t) = u(t)$	$\xleftrightarrow{\mathcal{L}}$	$X(s) = \frac{1}{s}$	
(unit ramp) $x(t) = t$	$\xleftrightarrow{\mathcal{L}}$	$X(s) = \frac{1}{s^2}$	
$x(t) = \sin(s_0 t)$	$\xleftrightarrow{\mathcal{L}}$	$X(s) = \frac{s_0}{(s^2 + {s_0}^2)}$	
$x(t) = \cos(s_0 t)$	$\xleftrightarrow{\mathcal{L}}$	$X(s) = \frac{s}{(s^2 + {s_0}^2)}$	
$x(t) = e^{s_0 t} u(t)$	$\xleftrightarrow{\mathcal{L}}$	$X(s) = \frac{1}{s - s_0}$	$\mathfrak{Re}\{s\} > \mathfrak{Re}\{s_0\}$
$x[n] = \delta[n]$	$\xleftrightarrow{z}$	$X(z) = 1$	all z
$x[n] = \delta[n-m]$	$\xleftrightarrow{z}$	$X(z) = z^{-m}$	
$x[n] = u[n]$	$\xleftrightarrow{z}$	$X(z) = \frac{z}{z-1}$	
$x[n] = z_0^n u[n]$	$\xleftrightarrow{z}$	$X(z) = \frac{1}{1 - z_0 z^{-1}}$	$\lvert z\rvert > \lvert z_0\rvert$
$x[n] = -z_0^n u[-n-1]$	$\xleftrightarrow{z}$	$X(z) = \frac{1}{1 - z_0 z^{-1}}$	$\lvert z\rvert < \lvert z_0\rvert$
$x[n] = a^n u[n]$	$\xleftrightarrow{z}$	$X(z) = \frac{z}{z-a}$	$\lvert z\rvert < \lvert a\rvert$

Table 4: Properties of the Discrete-time Fourier Transform.

Property	Time domain	Frequency domain
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
Differentiation (frequency)	$nx[n]$	$j\dfrac{dX(e^{j\omega})}{d\omega}$
Time-shift	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency-shift	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Convolution	$x_1[n] * x_2[n]$	$X_1(e^{j\omega})X_2(e^{j\omega})$
Modulation	$x_1[n]x_2[n]$	$\frac{1}{2\pi}X_1(e^{j\omega}) \circledast X_2(e^{j\omega})$
Time-reversal	$x[-n]$	$X(e^{-j\omega})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Symmetry (real)	$\mathfrak{Im}\{x[n]\} = 0$	$X(e^{j\omega}) = X^*(e^{-j\omega})$
Symmetry (imag)	$\mathfrak{Re}\{x[n]\} = 0$	$X(e^{j\omega}) = -X^*(e^{-j\omega})$
Parseval	$\displaystyle\sum_{n=-\infty}^{\infty} \|x[n]\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left\|X(e^{j\omega})\right\|^2 d\omega$	

Table 5: Properties of the Fourier series.

Property	Time domain	Frequency domain
Linearity	$a\tilde{x}_1(t) + b\tilde{x}_2(t)$	$aX_1[k] + bX_2[k]$
Differentiation (time)	$\dfrac{d\tilde{x}(t)}{dt}$	$\dfrac{j2\pi k}{T}X[k]$
Time-shift	$\tilde{x}(t - t_0)$	$e^{-j2\pi k t_0/T}X[k]$
Frequency-shift	$e^{j2\pi k_0 t/T}\tilde{x}(t)$	$X[k - k_0]$
Convolution	$\tilde{x}_1(t) \circledast \tilde{x}_2(t)$	$TX_1[k]X_2[k]$
Modulation	$\tilde{x}_1(t)\tilde{x}_2(t)$	$X_1[k] * X_2[k]$
Time-reversal	$\tilde{x}(-t)$	$X[-k]$
Conjugation	$\tilde{x}^*(t)$	$X^*[-k]$
Symmetry (real)	$\mathfrak{Im}\{\tilde{x}(t)\} = 0$	$X[k] = X^*[-k]$
Symmetry (imag)	$\mathfrak{Re}\{\tilde{x}(t)\} = 0$	$X[k] = -X^*[-k]$
Parseval	$\displaystyle\frac{1}{T}\int_{-T/2}^{T/2} \|\tilde{x}(t)\|^2 dt = \sum_{k=-\infty}^{\infty} \|X[k]\|^2$	

Table 6: Properties of the Fourier transform.

Property	Time domain	Frequency domain
Linearity	$a\tilde{x}_1(t) + b\tilde{x}_2(t)$	$aX_1(j\omega) + bX_2(j\omega)$
Duality	$X(jt)$	$2\pi x(-\omega)$
Differentiation	$\dfrac{dx(t)}{dt}$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\frac{1}{j\omega}X(j\omega) + \pi X(j0)\delta(\omega)$
Time-shift	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency-shift	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Convolution	$x_1(t) * x_2(t)$	$X_1(j\omega)X_2(j\omega)$
Modulation	$x_1(t)x_2(t)$	$\frac{1}{2\pi}X_1(j\omega) * X_2(j\omega)$
Time-reversal	$x(-t)$	$X(-j\omega)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Symmetry (real)	$\mathfrak{Im}\{x(t)\} = 0$	$X(j\omega) = X^*(-j\omega)$
Symmetry (imag)	$\mathfrak{Re}\{x(t)\} = 0$	$X(j\omega) = -X^*(-j\omega)$
Scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert}X\left(\dfrac{j\omega}{a}\right)$
Parseval	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \dfrac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2\,d\omega$	

Table 7: Properties of the z-transform.

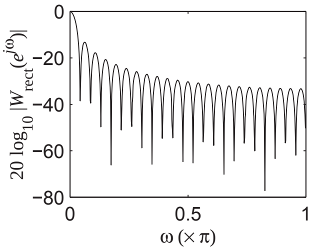
Property	Time domain	z-domain	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\subseteq R_{x_1} \cap R_{x_2}$
Time-shift	$x[n - n_0]$	$z^{-n_0}X(z)$	$R_x^\dagger$
Scaling in z	$z_0^n x[n]$	$X(z/z_0)$	$\lvert z_0\rvert R_x$
Differentiation in z	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	$R_x^\dagger$
Time-reversal	$x[-n]$	$X(1/z)$	$1/R_x$
Conjugation	$x^*[n]$	$X^*(z^*)$	R_x
Symmetry (real)	$\mathfrak{Im}\{x[n]\} = 0$	$X(z) = X^*(z^*)$	
Symmetry (imag)	$\mathfrak{Re}\{X[n]\} = 0$	$X(z) = -X^*(z^*)$	
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	$\subseteq R_{x_1} \cap R_{x_2}$
Initial value	$x[n] = 0, n < 0 \Rightarrow x[0] = \lim_{z\to\infty} X(z)$		

† $z = 0$ or $z = \infty$ may have been added or removed from the ROC.

Table 8: Commonly used window functions.

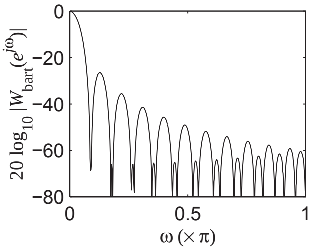
Rectangular:

$$w_{\text{rect}}[n] = \begin{cases} 1 & \text{when } 0 \leqslant n \leqslant M, \\ 0 & \text{otherwise.} \end{cases}$$



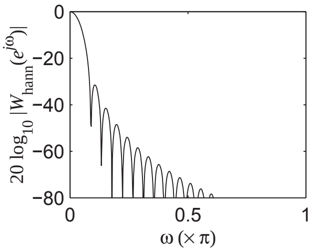
Bartlett (triangular):

$$w_{\text{bart}}[n] = \begin{cases} 2n/M & \text{when } 0 \leqslant n \leqslant M/2, \\ 2 - 2n/M & \text{when } M/2 \leqslant n \leqslant M, \\ 0 & \text{otherwise.} \end{cases}$$



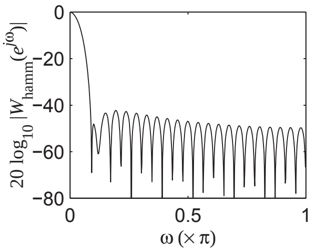
Hanning:

$$w_{\text{hann}}[n] = \begin{cases} \frac{1}{2} - \frac{1}{2}\cos(2\pi n/M) & \text{when } 0 \leqslant n \leqslant M, \\ 0 & \text{otherwise.} \end{cases}$$



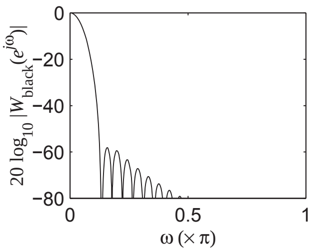
Hamming:

$$w_{\text{hamm}}[n] = \begin{cases} 0.54 - 0.46\cos(2\pi n/M) & \text{when } 0 \leqslant n \leqslant M, \\ 0 & \text{otherwise.} \end{cases}$$



Blackman:

$$w_{\text{black}}[n] = \begin{cases} 0.42 - 0.5\cos(2\pi n/M) \\ \quad + 0.08\cos(4\pi n/M) & \text{when } 0 \leqslant n \leqslant M, \\ 0 & \text{otherwise.} \end{cases}$$



Type of Window	Peak Side-Lobe Amplitude (Relative; dB)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)
Rectangular	-13	$4\pi/(M+1)$	-21
Bartlett	-25	$8\pi/M$	-25
Hanning	-31	$8\pi/M$	-44
Hamming	-41	$8\pi/M$	-53
Blackman	-57	$12\pi/M$	-74