http://elec3004.com

Sampling Theory: Aliasing & Anti-Aliasing

ELEC 3004: **Systems**: Signals & Controls

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Lecture 6

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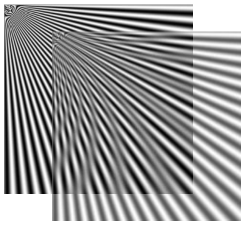
Lecture Schedule:

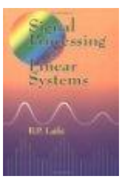
Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Systems: Linear Differential Systems
3	14-Mar	Sampling Theory & Data Acquisition
	16-Mar	Aliasing & Antialiasing
4	21-Mar	Discrete System Analysis
	23-Mar	Convolution Review
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
5	4-Apr	Digital Filters (IIR)
	6-Apr	Digital Windows
6	11-Apr	Digital Filter (FIR)
	13-Apr	FFT
	18-Apr	Holiday
	20-Apr	
	25-Apr	
7	27-Apr	Active Filters & Estimation
8	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	11-May	Digitial Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response
	25-May	Applications in Industry
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Overview (i.e. today we are going to learn …)

- Aliasing
- Spectral Folding
- Anti-Aliasing
  - Low-pass filtering of signals so as to keep things band limited



Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 5:
  Sampling
  - § 5.1 **The Sampling Theorem**
  - § 5.2 Numerical Computation of Fourier Transform: The Discrete Fourier Transform (DFT)

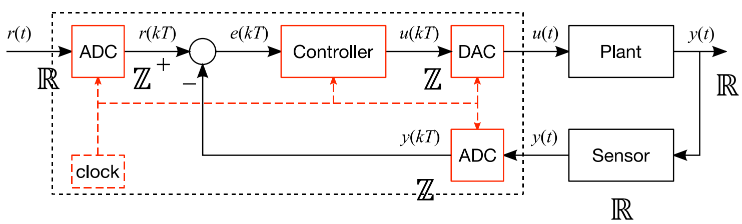
Also:

- § 4.6 Signal Energy

Sampling
&
RECONSTRUCTION

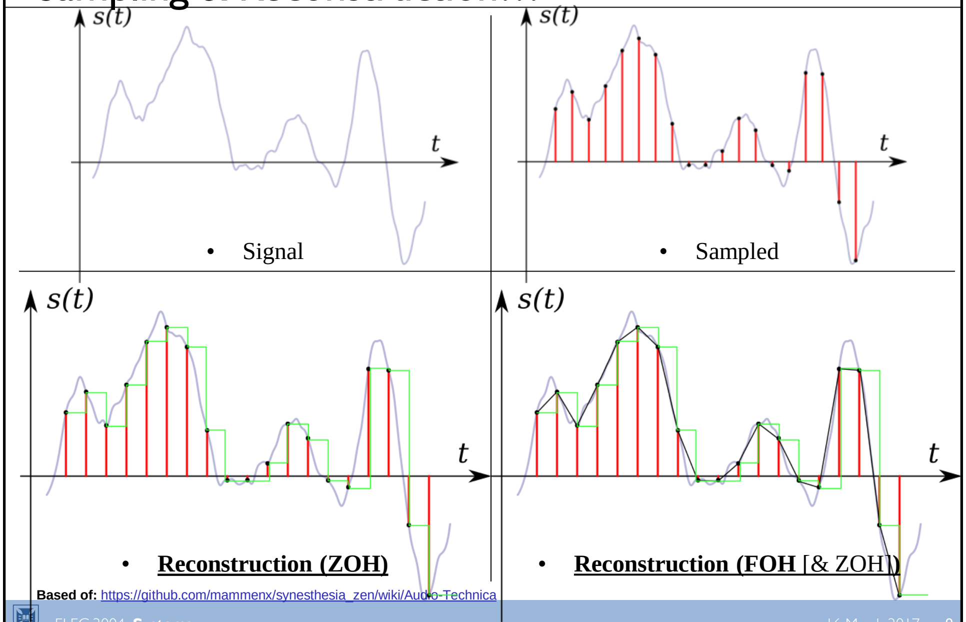
Digital Systems

- Implies something "something discrete" or …
  that a mapping exists to an "integer set" $s \in \mathbb{Z}$
- Often the "state-space" and "time" are discretised.
  *(But they **both** need no be)*



- Why?
  - Beat the noise (e.g., more signal "sharing")
  - Leverage time-keeping (oscillator) precision

Sampling & Reconstruction…



Based of: https://github.com/mammenx/synesthesia_zen/wiki/Audio-Technica

Sampling Theorem

- The Nyquist criterion states:

To prevent aliasing, a ***bandlimited*** *signal of bandwidth* w_B *rad/s must be sampled at a rate greater than* $2w_B$ *rad/s*

$$w_s > 2w_B$$

Note: this is a > sign not a ≥

Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

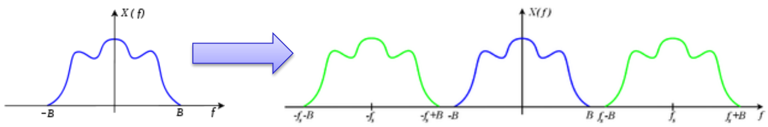
Spectrum Replication

- Sampling with a pulse train ($\delta(t)$)…

$$x(t) = x(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$$

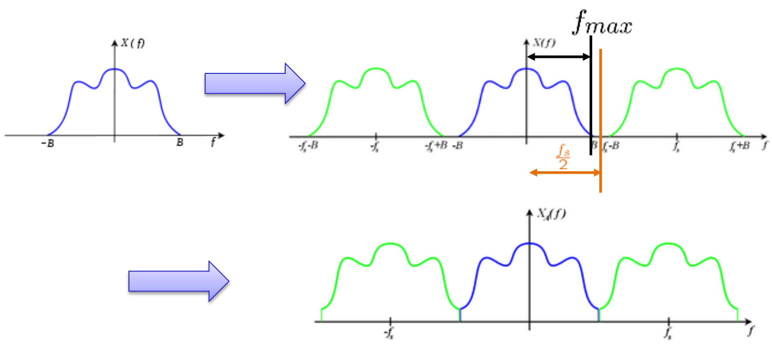
- Gives replication in $X(f)$

$$X(f) = \frac{1}{T_s} \cdot \sum_{k=-\infty}^{+\infty} X\left(f - \frac{k}{T_s}\right)$$



$f_s > 2B$

Spectrum Replication & Nyquist



$f_s > 2B$

$f_s = 2B$

- This suggests a limit:
  - Analog signal spectrum $X(f)$ runs up to f_{max}Hz
  - Spectrum replicas are separated by $f_s = \frac{1}{T_s}$ Hz

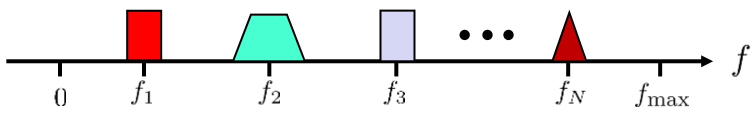
Violating Nyquist? Compressed Sensing

- Not so fast…
  "Exploits" the observation that most signals are sparse

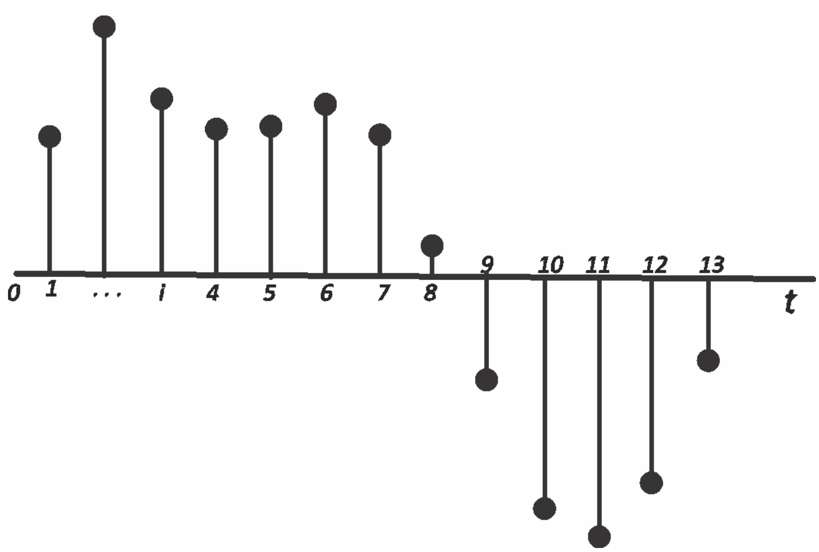
- Why?
  – Note that the Maximum Achievable Rate comes from the Karhunen-Loeve Decomposition or DFT Decomposition
  ➔ This assumes a "dense" signal…

$$C = \frac{1}{2}\int \log\left(\nu \frac{|H(f)|^2}{\mathcal{S}_\eta(f)}\right) df$$

- Note:
  – Analog Compressed Sensing – Xampling [MishaliEldar'10]
  – Multi-band receivers at sub-Nyquist sampling rates
  – Can be used in low-complexity cognitive radios



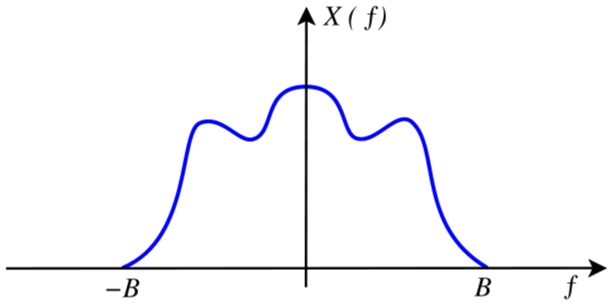
Reconstruction



Reconstruction

- Whittaker–Shannon interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$



Why *sinc*?

Time Domain Analysis of Reconstruction

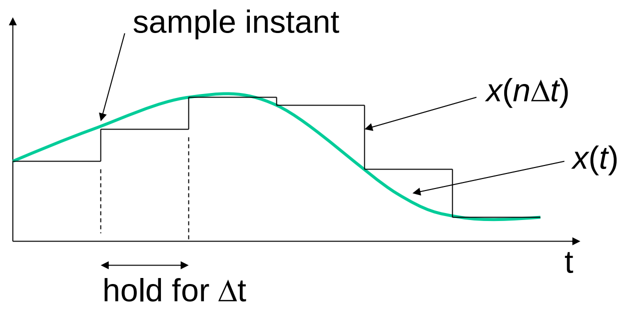
- Frequency domain: multiply by ideal LPF
  - ideal LPF: 'rect' function (gain Δt, cut off w_c)
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with 'sinc' function
  - as $F^{-1}\{\Delta t \ \mathrm{rect}(w/w_c)\} = \Delta t \, w_c \, \mathrm{sinc}(w_c t/\pi)$
  - i.e., weighted sinc on every sample
- Normally, $w_c = w_s/2$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \, \mathrm{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$

Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every Δt seconds
  2. holds that value constant until next sample

- Produces 'staircase' waveform, $x(n\Delta t)$



Practical Reconstruction

Two stage process:

- Digital to analogue converter (D/A)
  - zero order hold filter
  - produces 'staircase' analogue output

- Reconstruction filter
  - non-ideal filter: $\omega_c = \frac{\omega_s}{2}$
  - further reduces replica spectrums
  - usually 4th – 6th order e.g., Butterworth
    - for acceptable phase response

Sampling & Aliasing

Alliasing

- Aliasing - through sampling, two entirely different analog sinusoids take on the same "discrete time" identity

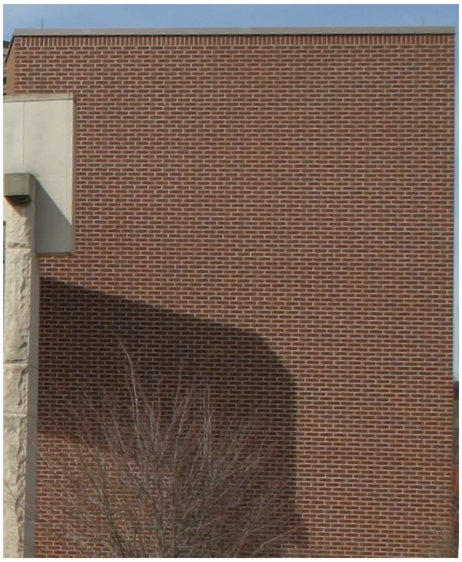
For $f[k] = \cos(\Omega k,)$ $\Omega = \omega T$:

The period has to be less than F_h (highest frequency): $T \leq \frac{1}{2\mathcal{F}_h}$

Thus: $0 \leq \mathcal{F} \leq \frac{\mathcal{F}_s}{2}$

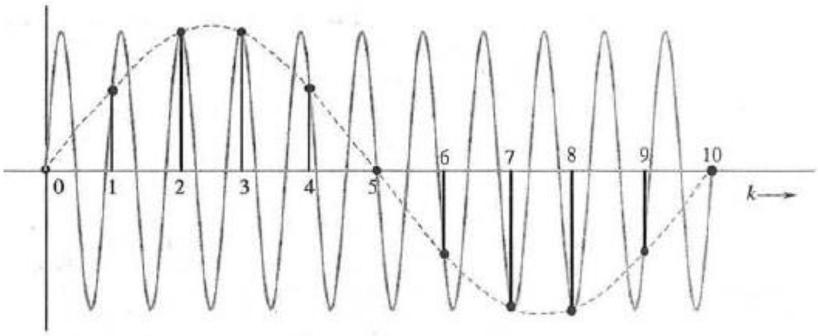
ω_f: aliased frequency: $\omega T = \omega_f T + 2\pi m$

Ex: Moire Effects

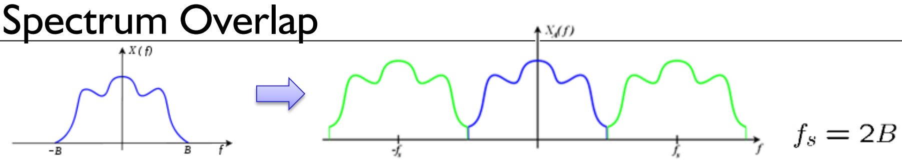


Source: Wikimedia https://en.wikipedia.org/wiki/Aliasing#/media/File:Moire_pattern_of_bricks.jpg (and aliased)

Aliasing: Another view of this

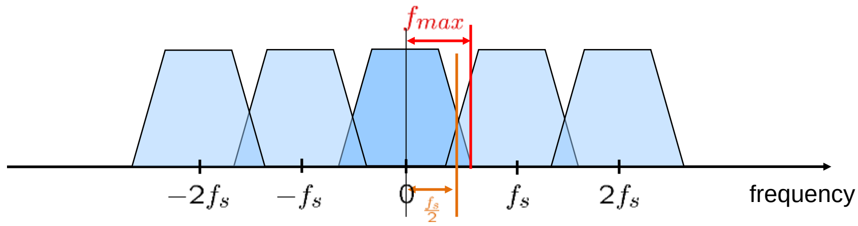


Spectrum Overlap

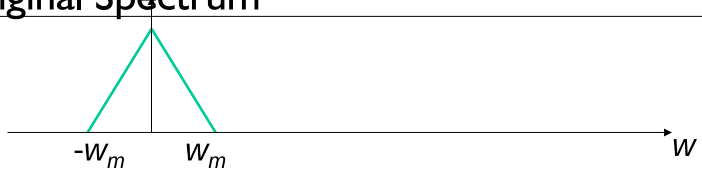


➔ if $f_s < 2B$

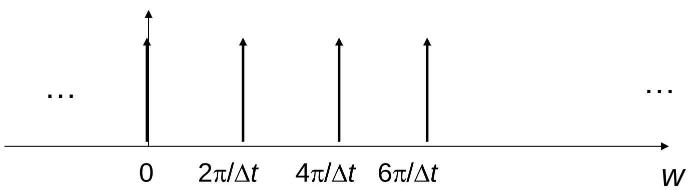
➔ *then* "**Folding**" or "**aliasing**":



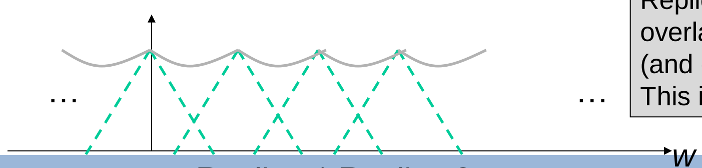
Original Spectrum



Fourier transform of impulse train (sampling signal)



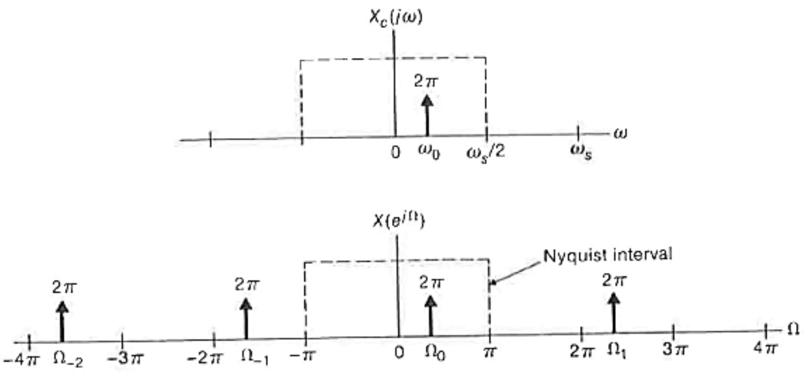
Amplitude spectrum of sampled signal



Replica spectrums overlap with original (and each other)
This is **Aliasing**

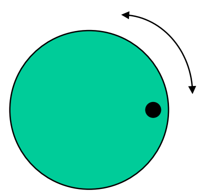
Original Replica 1 Replica 2 …

Original Spectrum

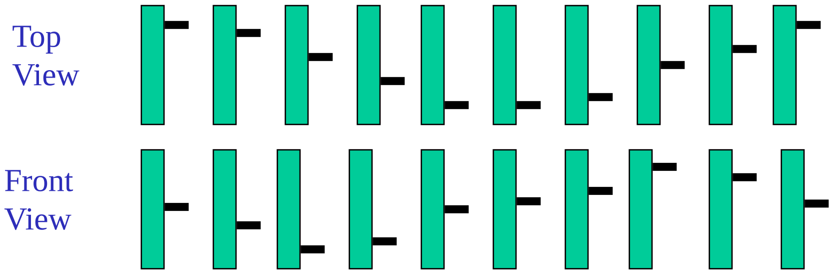


Another way to see Aliasing Too!

Rotating wheel and peg



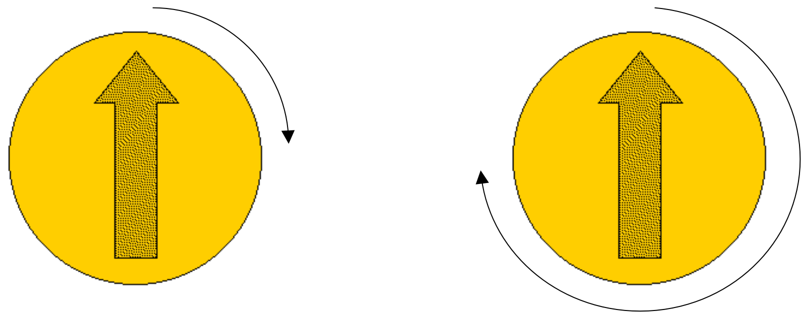
Need both top and front view to determine rotation



Temporal Aliasing

90^o clockwise rotation/frame
clockwise rotation perceived

270^o clockwise rotation/frame
(90^o) anticlockwise rotation
perceived i.e., aliasing



Require LPF to ‘blur’ motion

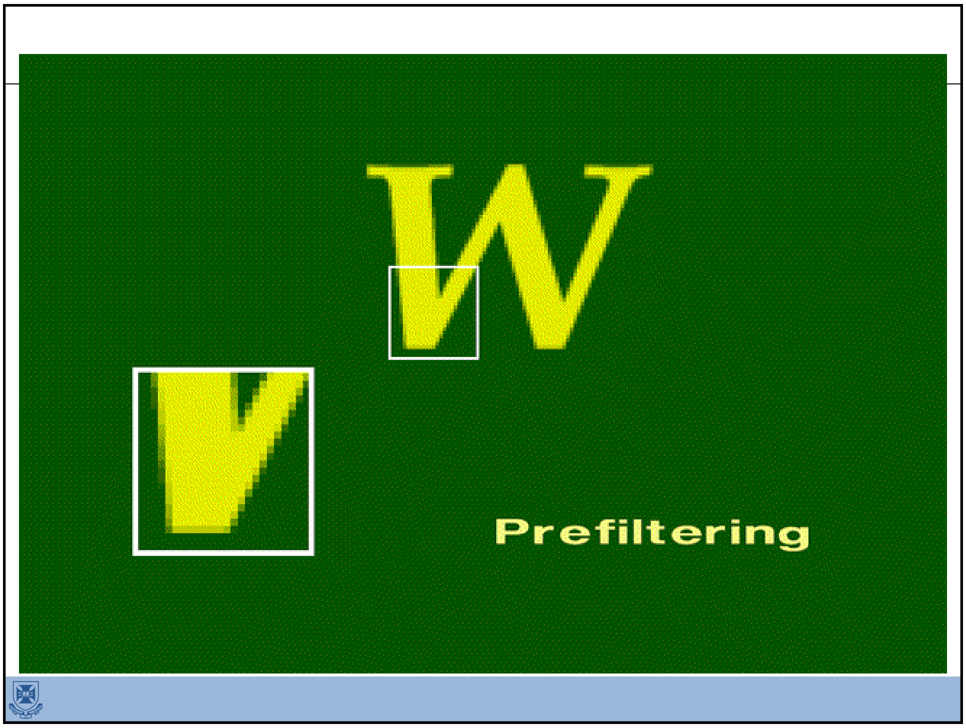
BREAK

Matlab Example

```
%% Sample PSD

%% Set Values
f=1;
phi=0;
fs=1e2;
t0=0;
tf=1;

%% Generate Signal
t=linspace(t0,tf,(fs*(tf-t0)));
x1=cos(2*pi*f*t + phi);
figure(10); plot(t, x1);

%% PSD
[p_x1, f_x1] = pwelch(x1,[],[],[],fs);
figure(20); plot(f_x1, pow2db(p_x1));
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');

%% PSD (Centered)
[p_x1, f_x1] = pwelch(x1,[],[],[],fs, 'centered','power');
figure(30); plot(f_x1, pow2db(p_x1));
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');
```



No antialiasing

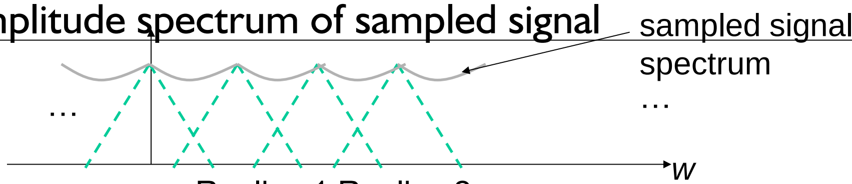

Prefiltering

Sampling & Antialiasing

Practical Anti-aliasing Filter

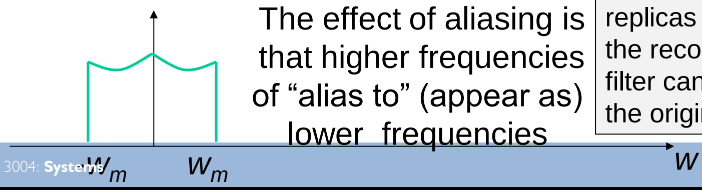
- Non-ideal filter
  - ➢ $w_c = \frac{w_s}{2}$
- Filter usually 4th – 6th order (e.g., Butterworth)
  - so frequencies > w_c may still be present
  - not higher order as phase response gets worse
- Luckily, most real signals
  - are lowpass in nature
    - signal power reduces with increasing frequency
  - e.g., speech naturally bandlimited (say < 8KHz)
  - Natural signals have a $\sim\frac{1}{f}$ spectrum
  - so, in practice aliasing is not (usually) a problem

Amplitude spectrum of sampled signal



Reconstruction filter (ideal lowpass filter)

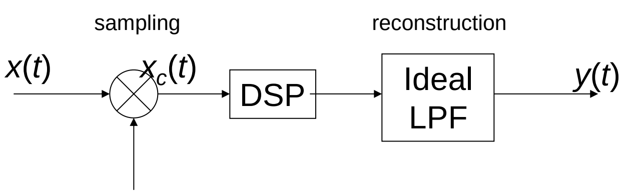
Spectrum of reconstructed signal



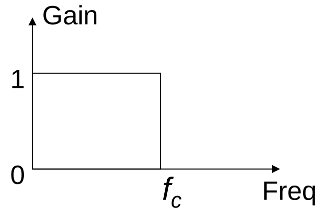
The effect of aliasing is that higher frequencies of "alias to" (appear as) lower frequencies

Due to overlapping replicas (aliasing) the reconstruction filter cannot recover the original spectrum

Mathematics of Sampling and Reconstruction



Sampling frequency $f_s = 1/\Delta t$



Cut-off frequency = f_c

Frequency Domain Analysis of Sampling

- Consider the case where the DSP performs no filtering operations
  - i.e., only passes $x_c(t)$ to the reconstruction filter
- To understand we need to look at the frequency domain
- Sampling: we know
  - multiplication in time ≡ convolution in frequency
  - $F\{x(t)\} = X(w)$
  - $F\{\delta T(t)\} = \sum\delta(w - 2\pi n/\Delta t)$,
  - i.e., an impulse train in the frequency domain

Frequency Space

Signal	*Time domain*	*Transform*
Impulse	$\delta[n]$	1
	$\delta[n - n_0]$	$e^{-j\Omega n_0}$
Unit step	$u[n]$	$\frac{1}{1 - e^{-j\Omega}} + \pi\,\delta(\Omega), \quad \lvert\Omega\rvert \le \pi$
	$-u[-n-1]$	$\frac{1}{1 - e^{-j\Omega}} - \pi\,\delta(\Omega), \quad \lvert\Omega\rvert \le \pi$
Exponential	$a^n u[n]$	$\frac{1}{1 - ae^{-j\Omega}}, \quad \lvert a\rvert < 1$
	$-a^n u[-n-1]$	$\frac{1}{1 - ae^{-j\Omega}}, \quad \lvert a\rvert > 1$
Weighted exponential	$(n+1)a^n u[n]$	$\frac{1}{(1 - ae^{-j\Omega})^2}, \quad \lvert a\rvert < 1$
DC signal	1, for all n	$2\pi\,\delta(\Omega), \quad \lvert\Omega\rvert \le \pi$
Complex sinusoid	$e^{j\Omega_0 n}$	$2\pi\,\delta(\Omega - \Omega_0), \quad \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
Sine wave	$\sin\Omega_0 n$	$-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)], \quad \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$
Cosine wave	$\cos\Omega_0 n$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)], \quad \lvert\Omega\rvert, \lvert\Omega_0\rvert \le \pi$

Frequency Domain Analysis of Sampling

- In the frequency domain we have

$$X_c(w) = \frac{1}{2\pi}\left(X(w) * \frac{2\pi}{\Delta t}\sum_n \delta\left(w - \frac{2\pi n}{\Delta t}\right)\right)$$
$$= \frac{1}{\Delta t}\sum_n X\left(w - \frac{2\pi n}{\Delta t}\right)$$

Remember convolution with an impulse? Same idea for an impulse train

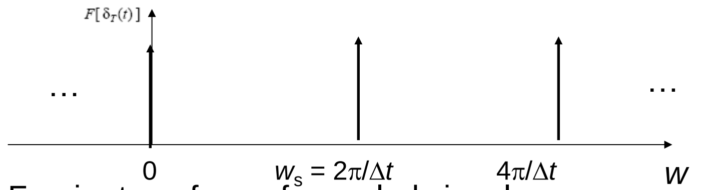
- Let's look at an example
  - where X(w) is triangular function
  - with maximum frequency w_m rad/s
  - being sampled by an impulse train, of frequency w_s rad/s

Sampling Frequency

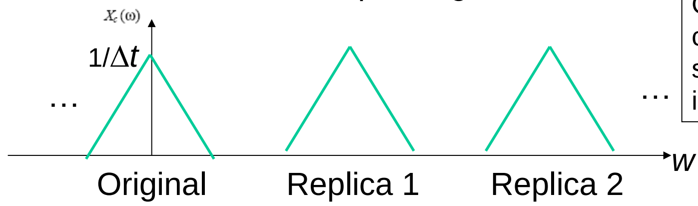
- In this example it was possible to recover the original signal from the discrete-time samples
- But is this always the case?
- Consider an example where the sampling frequency w_s is reduced
  - i.e., Δt is increased

Fourier transform of original signal X(ω) (signal spectrum)

Fourier transform of impulse train $\delta_T(\omega/2\pi)$ (sampling signal)

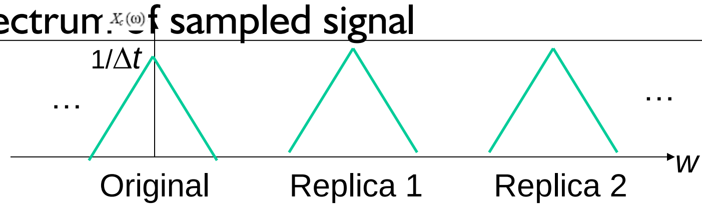


Fourier transform of sampled signal

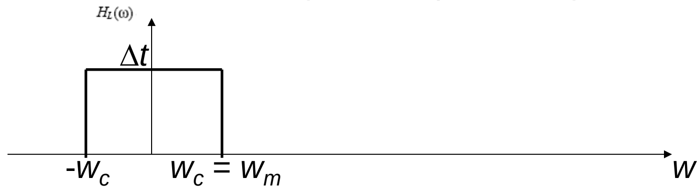


Original spectrum convolved with spectrum of impulse train

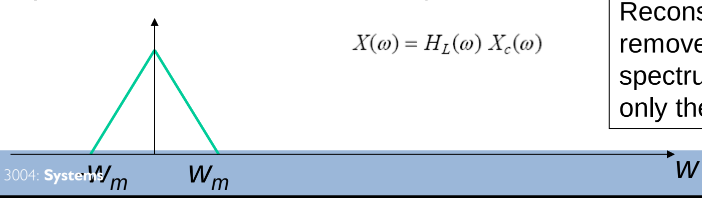
Spectrum of sampled signal



Reconstruction filter (ideal lowpass filter)



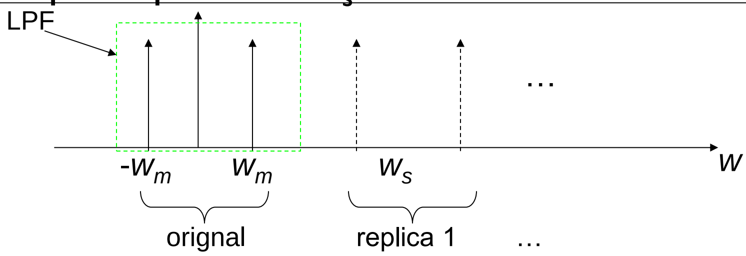
Spectrum of reconstructed signal



$$X(\omega) = H_L(\omega)\, X_c(\omega)$$

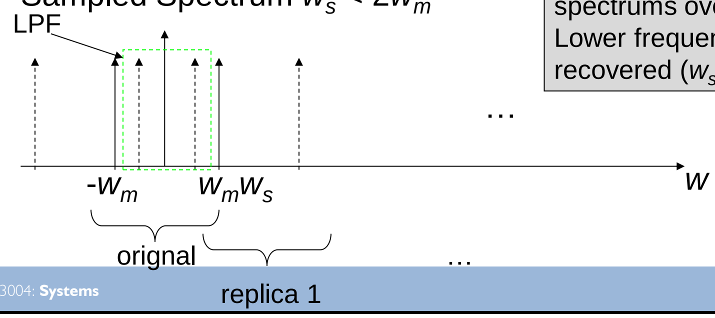
Reconstruction filter removes the replica spectrums & leaves only the original

Sampled Spectrum $w_s > 2wm$



original freq recovered

Sampled Spectrum $w_s < 2w_m$



Original and replica spectrums overlap
Lower frequency recovered ($w_s - w_m$)

Taking Advantage of the Folding

5.1 The Sampling Theorem

We now show that a real signal whose spectrum is bandlimited to B Hz $[F(\omega) = 0$ for $|\omega| > 2\pi B]$ can be reconstructed exactly (without any error) from its samples taken uniformly at a rate $\mathcal{F}_s > 2B$ samples per second. In other words, the minimum sampling frequency is $\mathcal{F}_s = 2B$ Hz.†

To prove the sampling theorem, consider a signal $f(t)$ (Fig. 5.1a) whose spectrum is bandlimited to B Hz (Fig. 5.1b).‡ For convenience, spectra are shown as functions of ω as well as of $\mathcal{F}$ (Hz). Sampling $f(t)$ at a rate of $\mathcal{F}_s$ Hz ($\mathcal{F}_s$ samples per second) can be accomplished by multiplying $f(t)$ by an impulse train $\delta_T(t)$(Fig. 5.1c), consisting of unit impulses repeating periodically every T seconds, where $T = 1/\mathcal{F}_s$. The result is the sampled signal $\overline{f}(t)$ resented in Fig. 5.1d. The sampled signal consists of impulses spaced every T seconds (the sampling interval). The nth impulse, located at $t = nT$, has a strength $f(nT)$, the value of $f(t)$ at $t = nT$.

$$\overline{f}(t) = f(t)\delta_T(t) = \sum_n f(nT)\delta(t - nT) \quad (5.1)$$

†The theorem stated here (and proved subsequently) applies to lowpass signals. A bandpass signal whose spectrum exists over a frequency band $\mathcal{F}_c - \frac{B}{2} < |\mathcal{F}| < \mathcal{F}_c + \frac{B}{2}$ has a bandwidth of B Hz. Such a signal is uniquely determined by $2B$ samples per second. In general, the sampling scheme is a bit more complex in this case. It uses two interlaced sampling trains, each at a rate of B samples per second (known as second-order sampling). See, for example, the references.[1,2]

‡The spectrum $F(\omega)$ in Fig. 5.1b is shown as real, for convenience. However, our arguments are valid for complex $F(\omega)$ as well.

319

Next Time…

- **Digital Systems**

- Review:
  - Chapter 8 of Lathi

- A signal has many signals ☺
  [Unless it's bandlimited. Then there is the one ω]