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Systems Theory: Linear Differential Systems

ELEC 3004: **Systems**: Signals & Controls

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Lecture 4

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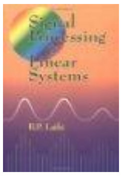
March 9, 2017

Yesterday: UN International Women's Day 2017

Lecture Schedule:

Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Systems: Linear Differential Systems
3	14-Mar	Sampling Theory & Data Acquisition
	16-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	23-Mar	Convolution Review
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
5	4-Apr	Digital Filters (IIR)
	6-Apr	Digital Windows
6	11-Apr	Digital Filter (FIR)
	13-Apr	FFT
	18-Apr	Holiday
	20-Apr	
	25-Apr	
7	27-Apr	Active Filters & Estimation
8	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	11-May	Digitial Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response
	25-May	Applications in Industry
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 2:
  Time-Domain Analysis of Continuous-Time Systems
  - § 2.1 Introduction
  - § 2.3 The Unit Impulse Response
  - § 2.6 System Stability
  - § 2.7 Intuitive Insights into System Behaviour
  - § 2.9 Summary

Linear Differential Systems

Linearity: Linear Equations

- Consider system of linear equations:

$$\begin{array}{rcccccc} y_1 & = & a_{11}x_1 & + & a_{12}x_2 & +\cdots+ & a_{1n}x_n \\ y_2 & = & a_{21}x_1 & + & a_{22}x_2 & +\cdots+ & a_{2n}x_n \\ & \vdots & & & & & \\ y_m & = & a_{m1}x_1 & + & a_{m2}x_2 & +\cdots+ & a_{mn}x_n \end{array}$$

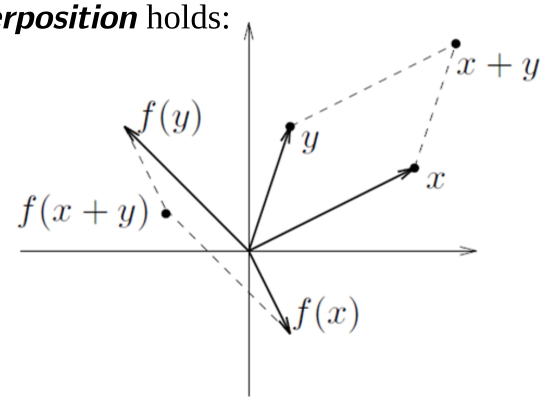
- This can be written in a matrix form as $y = Ax$, where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Source: Boyd, EE263, Slide 2-2

Linearity: Linear Functions

- A function f $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if:
  - $f(x+y) = f(x) + f(y),\ \forall x, y \in \mathbf{R}^n$
  - $f(\alpha x) = \alpha f(x),\ \forall x \in \mathbf{R}^n\ \forall \alpha \in \mathbf{R}$
- That is, ***Superposition*** holds:



Linearity: Linear functions and Matrix Multiplication

Consider a $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$
given by $f(x) = Ax$, where $A \in \mathbb{R}^{m \times n}$

As matrix multiplication function if $\boldsymbol{f}$ is **linear**, we may now say:

- **converse is true**: ***any*** linear function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be written as $f(x) = Ax$, for dome $A \in \mathbb{R}^{m \times n}$
- Representation via matrix multiplication is **unique**:
  for any linear function $\hat{\boldsymbol{f}}$ there is only one matrix $\hat{\boldsymbol{A}}$ for which $\hat{f}(x) = \hat{A}x$ for all x
- $y = Ax$ is a concrete representation of a generic linear function

Linearity: Interpretations

➔ of $y = Ax$:

- y is measurement or observation; x is unknown to be determined
- x is an "input" or "stated action"; y is "output" or "result"
  - In controls this "x" is sometimes "separated" into x and u such that $\boldsymbol{x}$ is the state and the $\boldsymbol{u}$ is the action done by the controller
- A function/transformation that maps $\boldsymbol{x} \in \mathbb{R}^n$ into $\boldsymbol{y} \in \mathbb{R}^m$

➔ of A (or a_{ij}):

- a_{ij} is a gain factor from j^{th} input (x_j) to i^{th} output (y_i)
- i^{th} row of A concerns i^{th} **output** *("row-out to sea")*
- j^{th} column of A concerns j^{th} **input** *("col-in to land")*
- $a_{34} = 0$ means 3rd output (y_3) doesn't depend on 4th input (x_4)
- $|a_{34}| \gg |a_{3j}|$ for $\boldsymbol{j} \neq \boldsymbol{4}$ means $\boldsymbol{y_3}$ ***depends*** mainly on x_4
- $|a_{34}| \gg |a_{i4}|$ for $i \neq 3$ means x_4 ***affects*** mainly y_3
- *If A is* **diagonal**, *then* i^{th} output depends only on i^{th} input
- If A is lower triangular [i.e., $a_{ij} = 0$ for $i < j$], then the y_i only depends on $x_1, \ldots, x_i$

➔ Nothing tells you something:

- The sparsity pattern of A [i.e, zero/nonzero entries], shows which x_j affect which y_i
- `Matlab`: **`spy(A)`** [or just try **`spy`**]

Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

Laplace:

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s) Y(s) = B(s) X(s)$$

- Total response = Zero-input response + Zero-state response

Initial conditions (Zero-input response) **External Input** (Zero-state response)

Linear Systems and ODE's

- Linear system described by differential equation

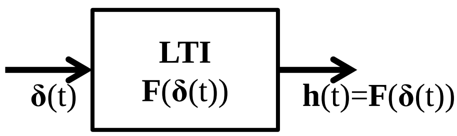
$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

  - Which using Laplace Transforms can be written as

$$a_0 Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

where *A(s)* and *B(s)* are polynomials in *s*

Unit Impulse Response



- **δ**(t): Impulsive excitation
- h(t): characteristic mode terms

Ex:

EXAMPLE 2.4

Determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)\, y(t) = Dx(t) \qquad (2.25)$$

This is a second-order system ($N = 2$) having the characteristic polynomial
$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2)$

The characteristic roots of this system are $\lambda = -1$ and $\lambda = -2$. Therefore

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t} \qquad (2.26a)$$

Differentiation of this equation yields

$$\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \qquad (2.26b)$$

The initial conditions are [see Eq. (2.24b) for $N = 2$]
$\dot{y}_n(0) = 1$ and $y_n(0) = 0$

Setting $t = 0$ in Eqs. (2.26a) and (2.26b), and substituting the initial conditions just given, we obtain
$0 = c_1 + c_2$

$1 = -c_1 - 2c_2$

Solution of these two simultaneous equations yields
$c_1 = 1$ and $c_2 = -1$

Therefore
$y_n(t) = e^{-t} - e^{-2t}$

Moreover, according to Eq. (2.25), $P(D) = D$, so that
$P(D)y_n(t) = Dy_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}$

Also in this case, $b_0 = 0$ [the second-order term is absent in $P(D)$]. Therefore
$h(t) = [P(D)y_n(t)]u(t) = (-e^{-t} + 2e^{-2t})u(t)$

First Order Systems

First order systems

$$ay' + by = 0 \quad (\text{with } a \neq 0)$$

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

$$Ty' + y = 0 \quad \text{or} \quad y' + ry = 0$$

where

- $T = a/b$ is a *time* (units: seconds)
- $r = b/a = 1/T$ is a *rate* (units: $1/\text{sec}$)

First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT + 1} = \frac{y(0)}{s + 1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, y *decays* exponentially

- T gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half $(0.693 = \log 2)$
- $4.6T$ gives time to decay by 0.01 $(4.6 = \log 100)$

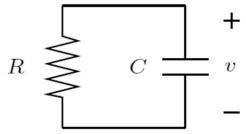
if $T < 0$, y *grows* exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100

First Order Systems

Examples

simple RC circuit:



circuit equation: $RCv' + v = 0$

solution: $v(t) = v(0)e^{-t/(RC)}$

population dynamics:

- $y(t)$ is population of some bacteria at time t
- growth (or decay if negative) rate is $y' = by - dy$ where b is birth rate, d is death rate
- $y(t) = y(0)e^{(b-d)t}$ (grows if $b > d$; decays if $b < d$)

Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a(\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}(y'')}) + b(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

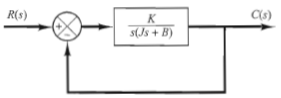
Second Order Systems

so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

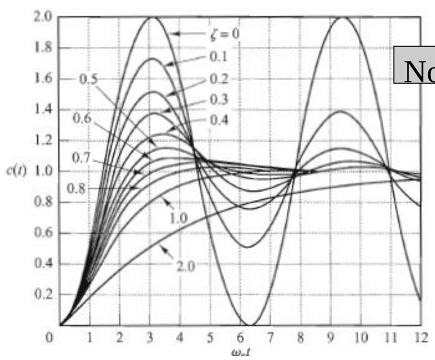
Second Order Response

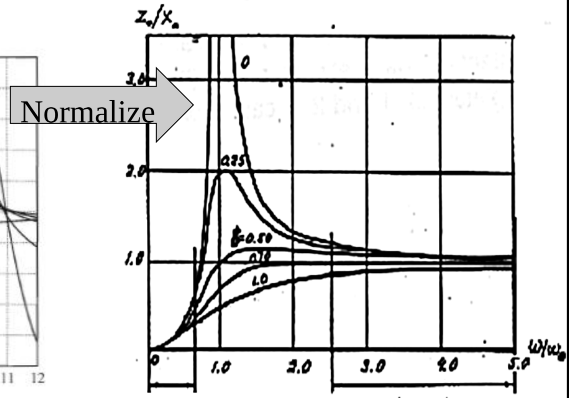


$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

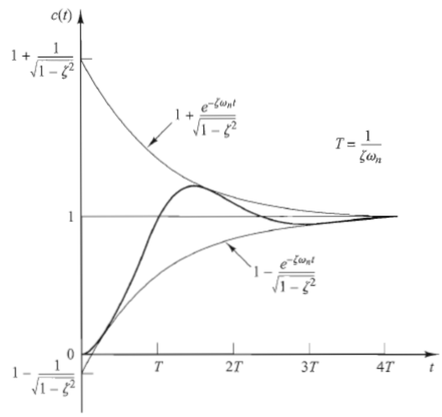
$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

Unit-Step Response

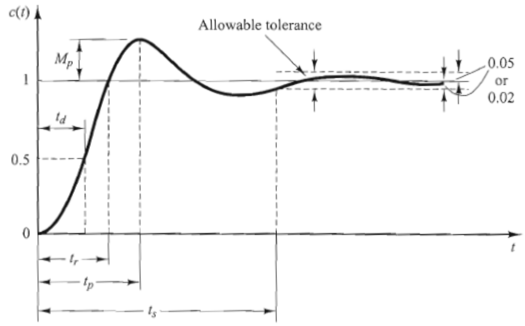




Second Order Response
Envelope Curves



Second Order Response
Unit Step Response Terms



- Delay time, t_d: The time required for the response to reach half the final value
- Rise time, t_r: The time required for the response to rise from 10% to 90%
- Peak time, t_p:The time required for the response to reach the first peak of the overshoot
- Maximum (percent) overshoot, Mp:

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

- Settling time, t_s: The time to be within 2-5% of the final value

Second Order Response
Seeing this on the S-plane

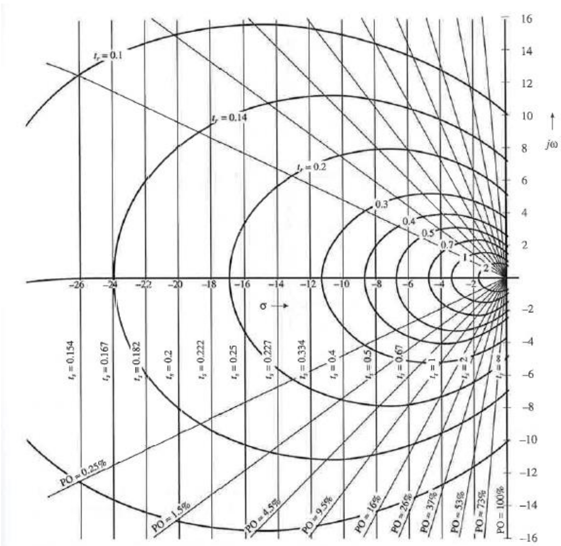
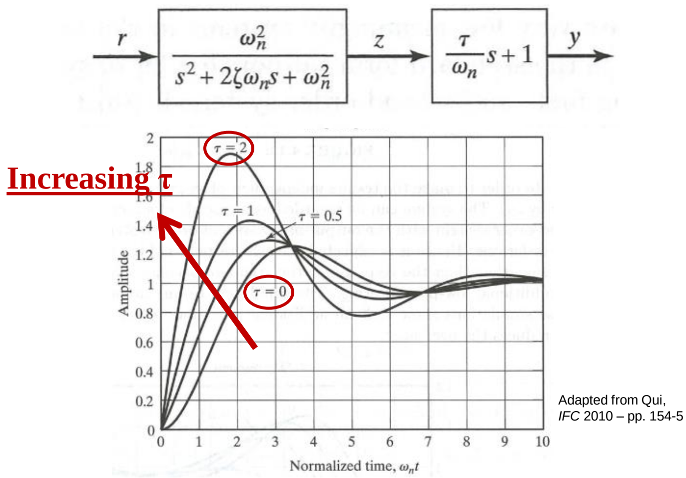


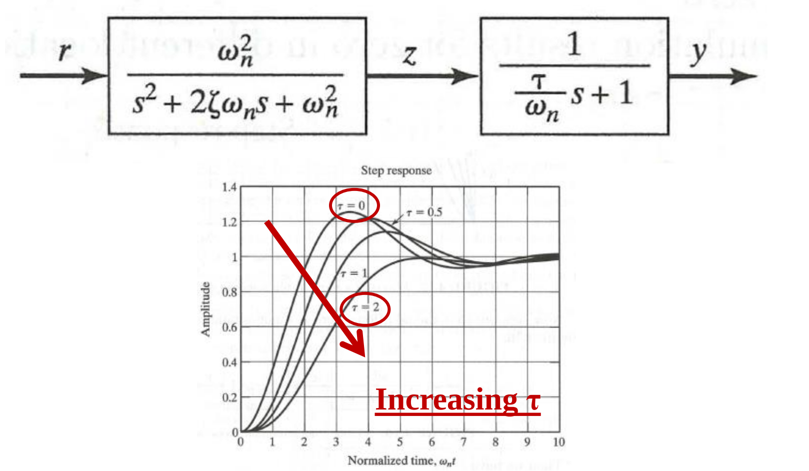
Fig. 6.40 Contours of second-order system pole location for constant PO, constant t_s, and constant t_r in s plane.

Second Order Response
The Case of **Adding a Zero**



- The addition of a zero (a **s** term) gives a system with a shorter rise time, a shorter peak time, and a larger overshoot

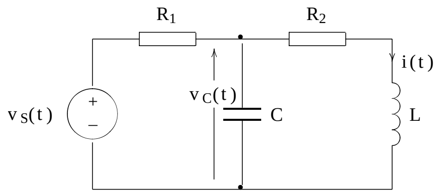
Second Order Response
The Case of **Adding a Zero**



- The addition of a pole (a **1/s** term) **slows down** the system response and reduces the overshoot.

Adapted from Qui,
IFC 2010 – pp. 154-5

Example of 2nd Order: RLC Circuits



- KCL:

$$\frac{V_s(t) - V_c(t)}{R_1} = C\frac{d}{dt}V_c(t) + i(t)$$

- KVL:

$$V_c(t) = L\frac{d}{dt}i(t) + R_2 i(t)$$

- Combining:

$$V_s(t) = R_1 LC\frac{d^2}{dt^2}i(t) + (L + R_1R_2C)\frac{d}{dt}i(t) + (R_1 + R_2)\,i(t)$$

BREAK

Multi-Domain-sional Nature of Multidimensional Signals & Systems

Equivalence Across Domains

Table 2.1 Summary of Through- and Across-Variables for Physical Systems

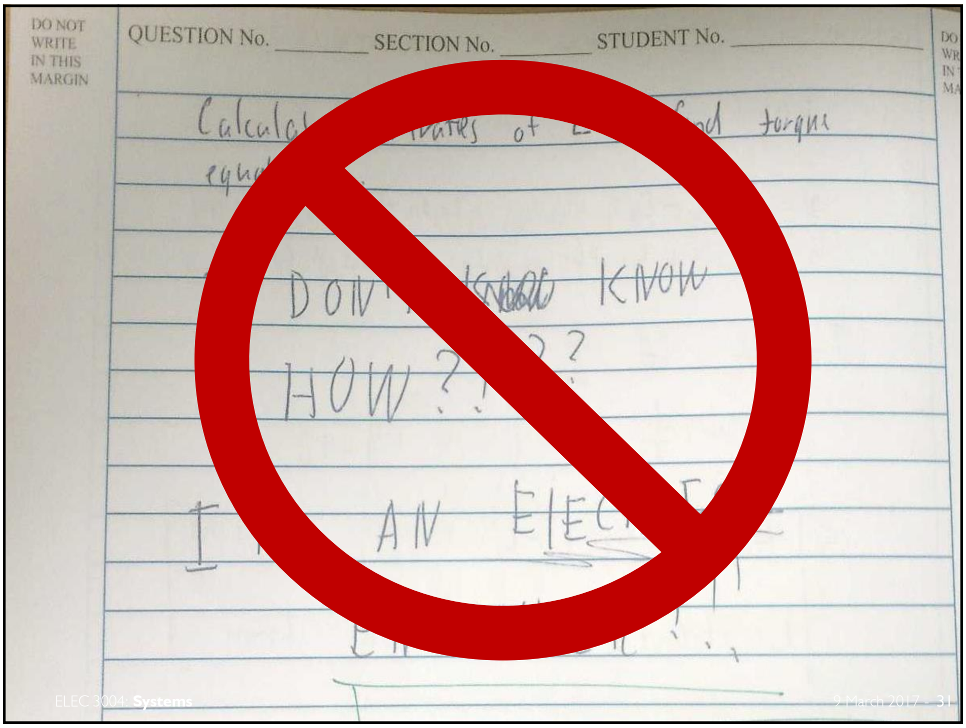
System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Source: Dorf & Bishop, *Modern Control Systems*, 12th Ed., p. 73

Table 2.2 Summary of Governing Differential Equations for Ideal Elements

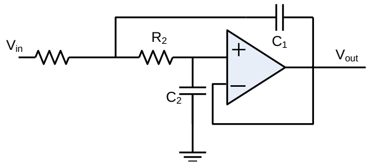
Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	
Capacitive storage	Electrical capacitance	$i = C\frac{dv_{21}}{dt}$	$E = \frac{1}{2}Cv_{21}^2$	
	Translational mass	$F = M\frac{dv_2}{dt}$	$E = \frac{1}{2}Mv_2^2$	
	Rotational mass	$T = J\frac{d\omega_2}{dt}$	$E = \frac{1}{2}J\omega_2^2$	
	Fluid capacitance	$Q = C_f\frac{dP_{21}}{dt}$	$E = \frac{1}{2}C_fP_{21}^2$	
	Thermal capacitance	$q = C_t\frac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	
Energy dissipators	Electrical resistance	$i = \frac{1}{R}v_{21}$	$\mathcal{P} = \frac{1}{R}v_{21}^2$	
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}^2$	
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}^2$	
	Fluid resistance	$Q = \frac{1}{R_f}P_{21}$	$\mathcal{P} = \frac{1}{R_f}P_{21}^2$	
	Thermal resistance	$q = \frac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t}\mathcal{T}_{21}$	

Source: Dorf & Bishop, *Modern Control Systems*, 12th Ed., p. 74



Example: 2nd Order Active RC Filter (Sallen–Key)

- 2^{nd} Order System Sallen–Key Low-Pass Topology:



Build this for Real in **ELEC 4403**

- KCL: $\frac{v_{\text{in}}-v_x}{R_1} = C_1 s\,(v_x - v_{\text{out}}) + \frac{v_x - v_{\text{out}}}{R_2}$

- Combined with Op-Amp Law:

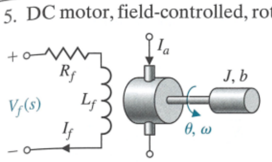
$$\frac{v_{\text{in}}-v_{\text{out}}(C_2 s R_2+1)}{R_1} = C_1 s v_{\text{out}}\,(C_2 s R_2+1) - v_{\text{out}} + \frac{v_{\text{out}}(C_2 s R_2+1)-v_{\text{out}}}{R_2}$$

- Solving for Gives a 2^{nd} order System:

$$\frac{v_{out}}{v_{in}} = \frac{1}{C_1C_2R_1R_2s^2+C_2(R_1+R_2)s+1}$$

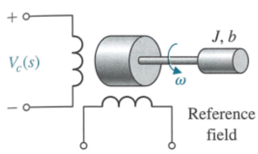
Motors

5. DC motor, field-controlled, rotational actuator



$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js+b)(L_f s + R_f)}$$

7. AC motor, two-phase control field, rotational actuator



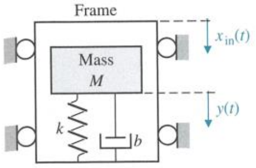
$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s+1)}$$

$$\tau = J/(b-m)$$

m = slope of linearized torque-speed curve (normally negative)

Mechanical Systems

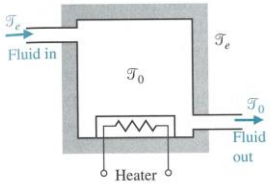
15. Accelerometer, acceleration sensor



$$x_o(t) = y(t) - x_{in}(t),$$
$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$
For low-frequency oscillations, where $\omega < \omega_n$,
$$\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

Thermal Systems

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$ = temperature difference due to thermal process

C_t = thermal capacitance

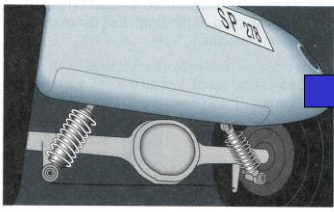
Q = fluid flow rate = constant

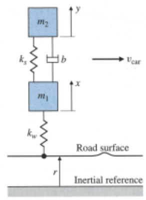
S = specific heat of water

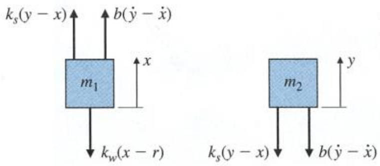
R_t = thermal resistance of insulation

$q(s)$ = transform of rate of heat flow of heating element

Example: Quarter-Car Model







Example: Quarter-Car Model (2)

$$\ddot{x} + \frac{b}{m_1}(\dot{x}-\dot{y}) + \frac{k_s}{m_1}(x-y) + \frac{k_w}{m_1}x = \frac{k_w}{m_1}r,$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y}-\dot{x}) + \frac{k_s}{m_2}(y-x) = 0.$$

$$s^2X(s) + s\frac{b}{m_1}(X(s)-Y(s)) + \frac{k_s}{m_1}(X(s)-Y(s)) + \frac{k_w}{m_1}X(s) = \frac{k_w}{m_1}R(s),$$

$$s^2Y(s) + s\frac{b}{m_2}(Y(s)-X(s)) + \frac{k_s}{m_2}(Y(s)-X(s)) = 0,$$

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}.$$

Economics: Cost of Production

Materials, parts, labour, etc. (***inputs***) are combined to make a number of products (***outputs***):

- x_j: price per unit of production input j
- a_{ij}: input j required to manufacture one unit of product i
- y_i: production cost per unit of product i
- For $y = Ax$:
  - i^{th} row of A is bill of materials for unit of product i
- Production inputs needed:
  - q_i is quantity of product i to be produced
  - r_j is total quantity of production input j needed

$\therefore \; r = A^T q$

& Total production cost is:

$$r^T x = (A^T q)^T x = q^T A x$$

Source: Boyd, EE263, Slide 2-18

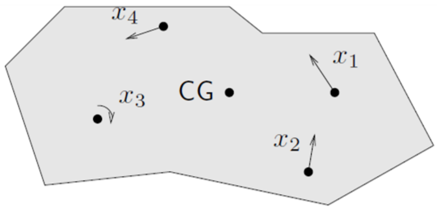
Estimation (or inversion)

$$y = Ax$$

- y_i is i^{th} measurement or sensor reading (which we have)
- x_j is j^{th} parameter to be estimated or determined
- a_{ij} is sensitivity of i^{th} sensor to j^{th} parameter
- sample problems:
  - find x, given y
  - find all x's that result in y (i.e., all x's consistent with measurements)
  - if there is no x such that $y = Ax$, find x s.t. $y \approx Ax$
    (i.e., if the sensor readings are inconsistent, find x which is almost consistent)

Source: Boyd, EE263, Slide 2-26

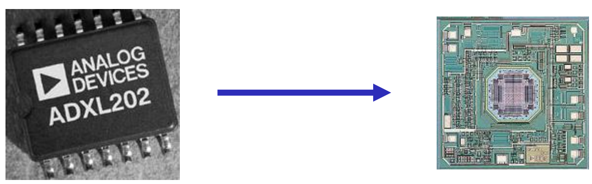
Mechanics: Total force/torque on rigid body



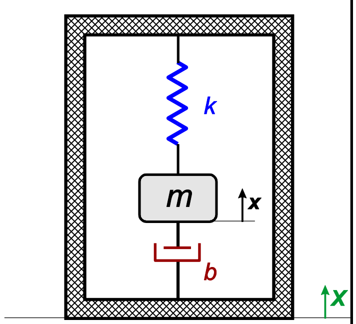
- x_j is external force/torque applied at some point/direction/axis
- $y \in \mathbf{R}^6$ is resulting total force & torque on body
  (y_1, y_2, y_3 are x-, y-, z- components of total force,
  y_4, y_5, y_6 are x-, y-, z- components of total torque)
- we have $y = Ax$
- A depends on geometry
  (of applied forces and torques with respect to center of gravity CG)
- jth column gives resulting force & torque for unit force/torque j

Source: Boyd, EE263, Slide 2-9

Another 2^{nd} Order System: Accelerometer or Mass Spring Damper (MSD)



- General accelerometer:
  – Linear spring (k) (0^{th} order w/r/t o)
  – Viscous damper (b) (1^{st} order)
  – Proof mass (m) (2^{nd} order)



➔ Electrical system analogy:
  – resistor (R) : damper (b)
  – inductance (L) : spring (k)
  – capacitance (C) : mass (m)

Measuring Acceleration: Sense *a* by measuring spring motion *Z*

- Start with Newton's 2nd Law:

$$ma = F$$

- Substitute:

$$m\frac{d^2x}{dt^2} = k\left(X - x\right) + b\frac{d\left(X - x\right)}{dt}$$

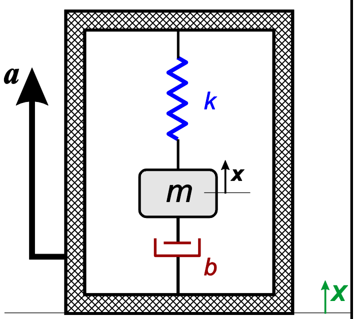
$$\mathbf{Z} \equiv \left(X - x\right) \rightarrow x = X - \mathbf{Z}$$

$$\Rightarrow m\frac{d^2X}{dt^2} = m\frac{d^2\mathbf{Z}}{dt^2} + k\mathbf{Z} + b\frac{d\mathbf{Z}}{dt}$$

The "displacement" measured by the unit (the motion of *m* relative the accelerometer frame)

- Solve ODE:

$$X\left(t\right) = X_0e^{i\omega t} \qquad \mathbf{Z}\left(t\right) = Z_0e^{i\omega t}$$



Measuring Acceleration [2]

- Substitute candidate solutions:

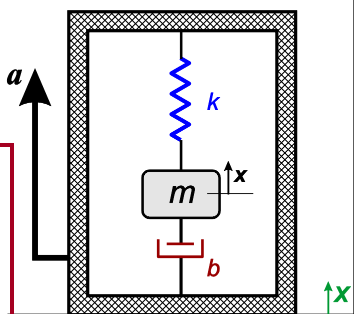
$$m\frac{d^2\left(X_0e^{i\omega t}\right)}{dt^2} = m\frac{d^2\left(Z_0e^{i\omega t}\right)}{dt^2} + k\left(Z_0e^{i\omega t}\right) + b\frac{d\left(Z_0e^{i\omega t}\right)}{dt}$$

$$-m\omega^2X_0e^{i\omega t} = -m\omega^2Z_0e^{i\omega t} + kZ_0e^{i\omega t} + \left(i\omega\right)bZ_0e^{i\omega t}$$

- Define Natural Frequency (ω_0) & Simplify for Z_0 (the spring displacement "magnitude"):

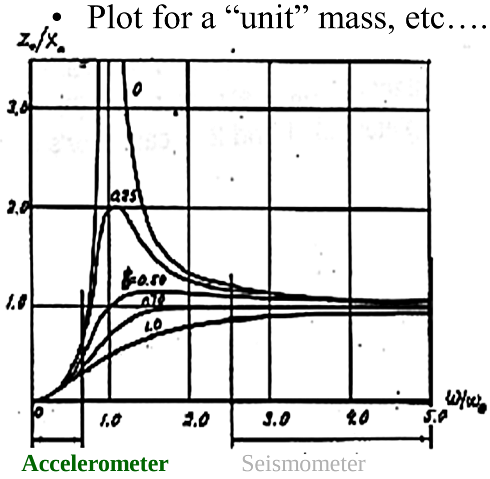
$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

$$Z_0 = \frac{m\omega^2X_0}{m\omega^2 - k - i\omega b} = \frac{X_0}{\sqrt{1 - \frac{\omega_0{}^2}{\omega^2} - \frac{b^2}{m^2\omega^2}}}$$



Acceleration: 2nd Order System

- Plot for a "unit" mass, etc….



- For $\omega \ll \omega_0$:

$$Z_0 \approx \frac{\omega^2 X_0}{\omega_0^2} = \frac{a}{\omega_0^2}$$

$$\rightarrow a = Z_0 \omega_0^2$$

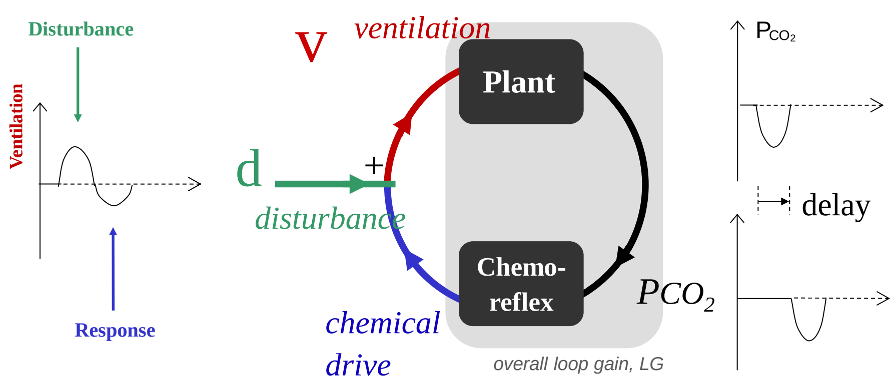
➔ **it's an Accelerometer**

- For $\omega \sim \omega_0$
  - As: b➔ 0, Z ➔ ∞
  - Sensitivity ↑
- For $\omega \gg \omega_0$:

$$Z_0 \approx X_0$$

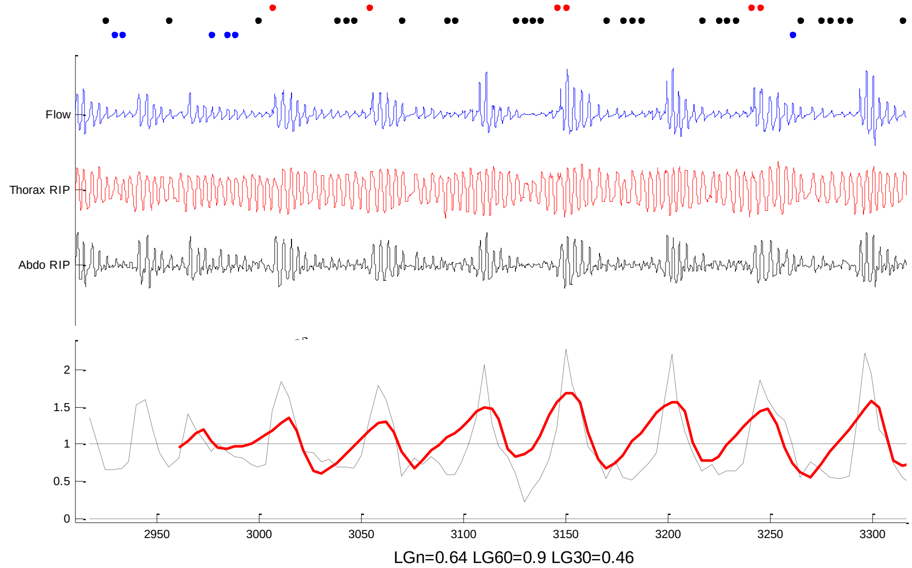
➔ **it's a Seismometer**

Ex3: "Loop Gain" to Quantify Ventilatory Stability:

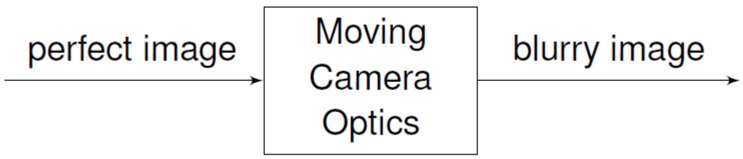


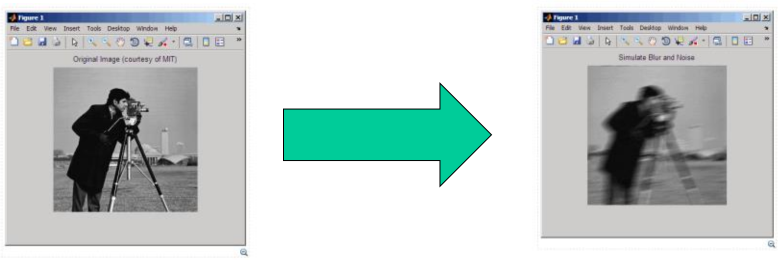
- Loop Gain $= \frac{Response}{Disturbance}$
- Loop Gain>1 implies an unstable control system
- Loop Gain<1 implies a stable control system
- Like EEG, disturbance can be characterised by frequency

Estimating LG from Clinical PSG:



Ex: Deblurring





- Matlab: **deconvwnr**

Next Time…

- We will talk about sampling
- Please complete the "practice assignment" **before** starting Problem Set 1
- Thank you!