http://elec3004.com

Signals as Vectors
Systems as Maps

ELEC 3004: **Systems**: Signals & Controls

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Lecture 3

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March 7, 2017

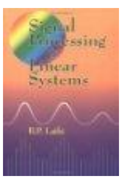
Tomorrow: UN International Women's Day 2017

- **Ada Lovelace**: English mathematician and writer
- Creator of the first algorithm and first computer program

Lecture Schedule:

Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	16-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	23-Mar	Convolution Review
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
5	4-Apr	Digital Filters (IIR)
	6-Apr	Digital Windows
6	11-Apr	Digital Filter (FIR)
	13-Apr	FFT
	18-Apr	**Holiday**
	20-Apr	
	25-Apr	
7	27-Apr	Active Filters & Estimation
8	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	11-May	Digitial Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response
	25-May	Applications in Industry
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 1:
  Introduction to Signals and Systems
  – § 1.7 Classification of Systems

- Chapter 3:
  Signal Representation By Fourier Series
  – § 3.1 Signals and Vectors
  – § 3.3 Signal Representation by Orthogonal Signal Set

System Terminology

System Classifications/Attributes

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Dynamical Systems…

- A system with a memory
  - Where past history (or derivative states) are **<u>relevant</u>** in determining the response
- Ex:
  - RC circuit: Dynamical
    - Clearly a function of the "capacitor's past" (initial state) and
    - Time! (charge / discharge)
  - R circuit: is memoryless ∵ the output of the system (recall V=IR) at some time **t** only depends on the input at time **t**

- Lumped/Distributed
  - Lumped: Parameter is constant through the process & can be treated as a "point" in space
- Distributed: System dimensions ≠ small over signal
  - Ex: waveguides, antennas, microwave tubes, etc.

Causality:
Causal (physical or nonanticipative) systems

- **Is one for which the output at any instant t_0 depends only on the value of the input x(t) for $t \leq t_0$** . Ex:

$u(t) = x(t-2) \Rightarrow \text{causal}$ $u(t) = x(t-2) + x(t+2) \Rightarrow \text{noncausal}$

- A "*real-time*" system must be causals
  - How can it respond to future inputs?
- A prophetic system: knows future inputs and acts on it (now)
  - The output would begin before t_0
- In some cases Noncausal maybe modelled as causal with delay
- Noncausal systems provide **an upper bound** on the performance of causal systems

Causality:
Looking at this from the output's perspective…

- **Causal** = The output *before* some time *t* does not depend on the input *after* time *t*.

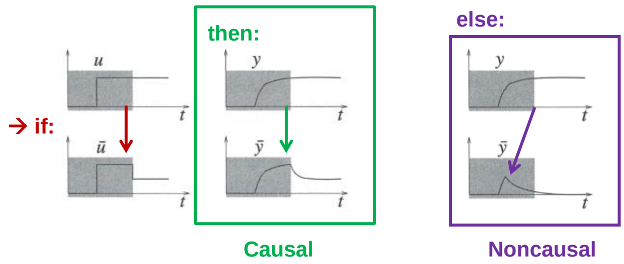
Given: $y(t) = F(u(t))$

For:

$\hat{u}(t) = u(t)\,,\ \forall 0 \le t < T \ or\ [0, T)$

Then for a T>0:

$$\rightarrow \hat{y}(t) = y(t)\,,\ \forall 0 \le t < T$$



Systems with Memory

- A system is said t have *memory* if the output at an arbitrary time $t = t_*$ depends on input values other than, or in addition to, $x(t_*)$

- Ex: Ohm's Law

$$V(t_o) = Ri(t_o)$$

- **Not** Ex: Capacitor

$$V(t_0) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

Time-Invariant Systems

- **Given** a shift (delay or advance) in the input signal
- **Then/Causes** simply a like shift in the output signal

- If $x(t)$ produces output $y(t)$
- Then $x(t - t_0)$ produces output $y(t - t_0)$

- Ex: Capacitor
- $V(t_0) = \frac{1}{C}\int_{-\infty}^{t} i(\tau - t_0)\, d\tau$

$$= \frac{1}{C}\int_{-\infty}^{t-t_0} i(\tau)\, d\tau$$

$$= V(t - t_0)$$

Time-Invariant Systems

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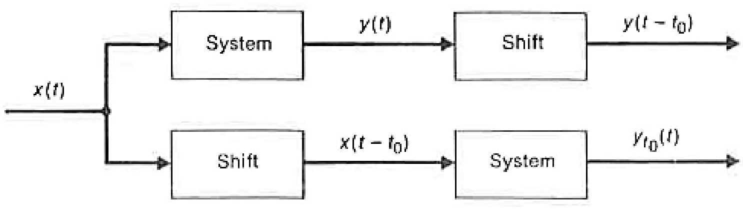
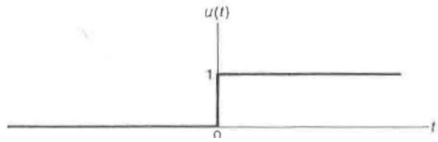


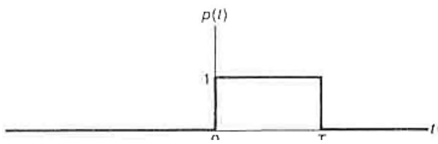
Unit Step Function

- $u(t) = \begin{cases} 0, t < 0 \\ 1, t > 0 \end{cases}$



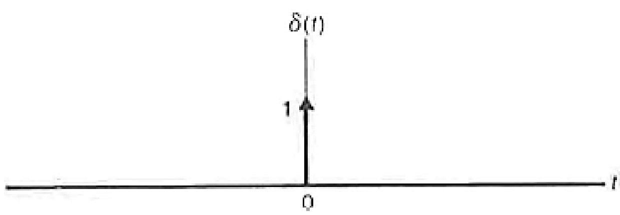
"Rectangular Pulse"

- $p(t) = u(t) - u(t - T)$



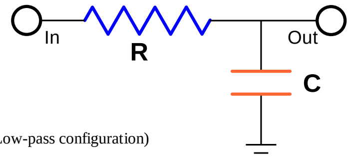
Unit-Impulse Function

1. $\delta(t) = 0$ for $t \neq 0$.
2. $\delta(t)$ undefined for $t = 0$.
3. $\int_{t_1}^{t_2} \delta(t)\, dt = \begin{cases} 1, & \text{if } t_1 < 0 < t_2 \\ 0, & \text{otherwise.} \end{cases}$



EXAMPLE: First Order RC Filter

- Passive, First-Order **R**esistor-**C**apacitor Design:



(Low-pass configuration)

$$T\left(s\right)=\frac{a_1 s+a_0}{s+\omega_0}$$

- 3dB (½ Signal Power):

$$\omega=2\pi f$$

$$f_c=\frac{1}{2\pi \mathbf{RC}}$$

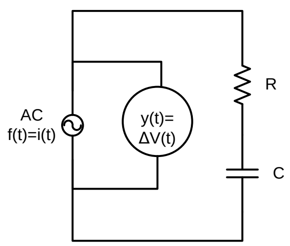
- Magnitude:

$$\left|V_{\mathrm{out}}\right|=\sqrt{\frac{1}{\left(\omega RC\right)^2}}\left|V_{\mathrm{in}}\right|$$

- Phase:

$$\phi=\tan^{-1}\left(-\omega RC\right)$$

Example 1: RC Circuits



$$y\left(t\right)=Rf\left(t\right)+\tfrac{1}{C}\int_{-\infty}^{t}f\left(\tau\right)d\tau$$

$$y\left(t\right)=Rf\left(t\right)+\tfrac{1}{C}\int_{-\infty}^{0}f\left(\tau\right)d\tau+\tfrac{1}{C}\int_{0}^{t}f\left(\tau\right)d\tau$$

$$y\left(t\right)=v_C\left(0\right)+Rf\left(t\right)+\tfrac{1}{C}\int_{0}^{t}f\left(\tau\right)d\tau$$

$$y\left(t\right)=v_C\left(t_0\right)+Rf\left(t\right)+\tfrac{1}{C}\int_{t_0}^{t}f\left(\tau\right)d\tau$$

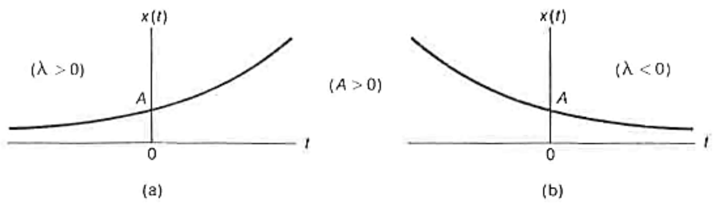
BREAK

Signals as Vectors

Complex Exponential Signals

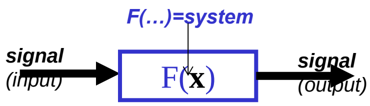
$$x(t) = Ae^{\lambda t}$$

- A and λ are generally complex numbers.
- If A and λ are, in fact, real-valued numbers, $x(t)$ is itself real-valued and is called a **real exponential**

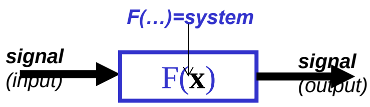


Signals as Vectors

- Back to the beginning!



Signals as Vectors



- There is a perfect analogy between signals and vectors …

Signals are vectors!

- A vector can be represented as a sum of its components in a variety of ways, depending upon the choice of coordinate system. A signal can also be represented as a sum of its components in a variety of ways.

Signals as Vectors

- Represent them as Column Vectors

$$x = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ \vdots \\ x[N] \end{bmatrix} .$$

Signals as Vectors

- Can represent phenomena of interest in terms of signals
- Natural vector space structure (addition/substraction/norms)
- Can use norms to describe and quantify properties of signals

Signals as vectors

Signals can take real or complex values.

In both cases, a natural vector space structure:

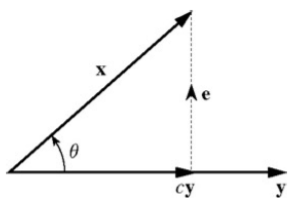
- Can add two signals: $x_1[n] + x_2[n]$
- Can multiply a signal by a scalar number: $C \cdot x[n]$
- Form linear combinations: $C_1 \cdot x_1[n] + C_2 \cdot x_2[n]$

Various Types

- Audio signal (sound pressure on microphone)
- B/W video signal (light intensity on
- photosensor)
- Voltage/current in a circuit (measure with
- multimeter)
- Car speed (from tachometer)
- Robot arm position (from rotary encoder)
- Daily prices of books / air tickets / stocks
- Hourly glucose level in blood (from glucose monitor)
- Heart rate (from heart rate sensor)

Vector Refresher

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos \theta \qquad (6.46)$$



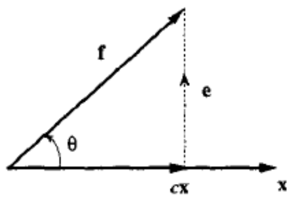
- Length: $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$
- Decomposition: $\mathbf{x} = c_1\mathbf{y} + \mathbf{e}_1 = c_2\mathbf{y} + \mathbf{e}_2$
- Dot Product of ⊥ is 0: $\mathbf{x} \cdot \mathbf{y} = 0$

Vectors [2]

- Magnitude and Direction

$$f \cdot x = |f||x| \cos(\theta)$$

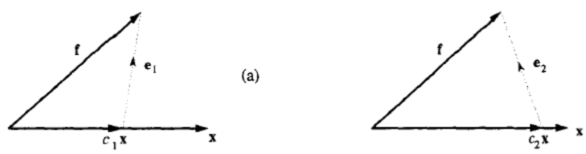
- Component (projection) of a vector along another vector



$\mathbf{f} = c\mathbf{x} + \mathbf{e}$ ← Error Vector

Vectors [3]

- ∞ bases given $\vec{\mathbf{x}}$



- Which is the best one?

$$\mathbf{f} \simeq c\mathbf{x}$$
$$c|\mathbf{x}| = |\mathbf{f}| \cos \theta$$
$$c|\mathbf{x}|^2 = |\mathbf{f}||\mathbf{x}| \cos \theta = \mathbf{f} \cdot \mathbf{x}$$
$$c = \frac{\mathbf{f} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{1}{|\mathbf{x}|^2} \mathbf{f} \cdot \mathbf{x}$$
$$\mathbf{f} \cdot \mathbf{x} = 0$$

- Can I allow more basis vectors than I have dimensions?

Signals **Are** Vectors

- A Vector / Signal can represent a sum of its components

Remember (Lecture 5, Slide 10):

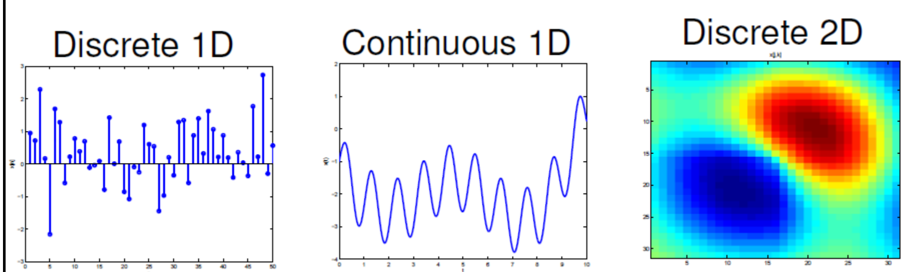
Total response = Zero-input response + Zero-state response

Zero-input response	Zero-state response
Initial conditions	**External Input**

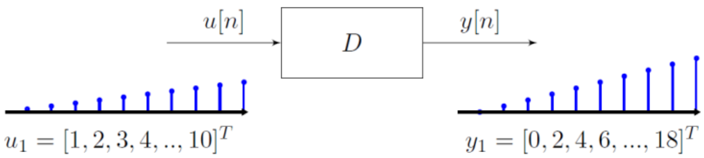
- Vectors are Linear
  - They have **additivity** and **homogeneity**

Vectors / Signals Can Be Multidimensional

- A signal is a quantity that varies as a function of an index set
- They can be multidimensional:
  - 1-dim, discrete index (time): x[n]
  - 1-dim, continuous index (time): x(t)
  - 2-dim, discrete (e.g., a B/W or RGB image): x[j; k]
  - 3-dim, video signal (e.g, video): x[j; k; n]



It's Just a Linear Map



- y[n]=2u[n-1] is a linear map
- BUT y[n]=2(u[n]-1) is **NOT Why?**

- **Because of homogeneity!**

    T(au)=aT(u)

Norms of signals

Can introduce a notion of signals being "nearby."

This is characterized by a metric (or distance function).

$$d(\mathbf{x}, \mathbf{y})$$

If compatible with the vector space structure, we have a norm.

$$\|\mathbf{x} - \mathbf{y}\|$$

Examples of Norms

Can use many different norms, depending on what we want to do.

The following are particularly important:

- ℓ_2 (Euclidean) norm:

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x[k]|^2 \right)^{\frac{1}{2}} \qquad \texttt{norm(x,2)}$$

- ℓ_1 norm:

$$\|x\|_1 = \sum_{k=1}^{n} |x[k]| \qquad \texttt{norm(x,1)}$$

- ℓ_∞ norm:

$$\|x\|_\infty = \max_k |x[k]| \qquad \texttt{norm(x,inf)}$$

What are the differences?

Properties of norms

For any norm $\|\cdot\|$, and any signal x, we have:

1. Linearity: if C is a scalar,

$$\|C \cdot \mathrm{x}\| = |C| \cdot \|\mathrm{x}\|$$

2. Subadditivity (triangle inequality):

$$\|\mathrm{x} + \mathrm{y}\| \leq \|\mathrm{x}\| + \|\mathrm{y}\|$$

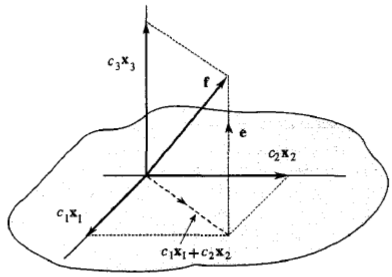
Can use norms:

- To detect whether a signal is (approximately) zero.
- To compare two signals, and determine if they are "close."

$$\|\mathrm{x} - \mathrm{y}\| \approx 0$$

Signal representation by Orthogonal Signal Set

- **Orthogonal Vector Space**



➔ A signal may be thought of as having components.

Component of a Signal

$$f(t) \simeq c\,x(t) \qquad t_1 \le t \le t_2$$

$$c = \frac{\int_{t_1}^{t_2} f(t)x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt} = \frac{1}{E_x}\int_{t_1}^{t_2} f(t)x(t)\,dt$$

$$\int_{t_1}^{t_2} f(t)x(t)\,dt = 0$$

- Let's take an example:

$$f(t) \simeq c \sin t \qquad 0 \le t \le 2\pi$$

$$x(t) = \sin t \qquad \text{and} \qquad E_x = \int_0^{2\pi} \sin^2(t)\,dt = \pi$$

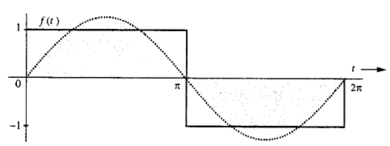


Fig. 3.3 Approximation of square signal in terms of a single sinusoid.

Thus

$$f(t) \simeq \frac{4}{\pi}\sin t \qquad (3.14)$$

Basis Spaces of a Signal

$$\int_{t_1}^{t_2} x_m(t)x_n(t)\,dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases}$$

$$f(t) \simeq c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t)$$
$$= \sum_{n=1}^{N} c_nx_n(t)$$

$$e(t) = f(t) - \sum_{n=1}^{N} c_nx_n(t)$$

$$c_n = \frac{\int_{t_1}^{t_2} f(t)x_n(t)\,dt}{\int_{t_1}^{t_2} x_n{}^2(t)\,dt}$$
$$= \frac{1}{E_n}\int_{t_1}^{t_2} f(t)x_n(t)\,dt \qquad n = 1, 2, \ldots, N$$

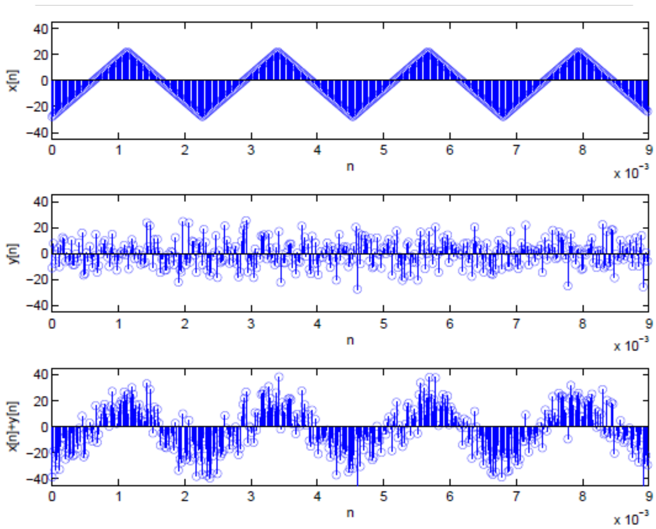
$$f(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + \cdots$$
$$= \sum_{n=1}^{\infty} c_nx_n(t) \qquad t_1 \leq t \leq t_2$$

Basis Spaces of a Signal

$$f(t) = c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + \cdots$$
$$= \sum_{n=1}^{\infty} c_nx_n(t) \qquad t_1 \leq t \leq t_2$$

- Observe that the error energy *Ee* generally decreases as *N*, the number of terms, is increased because the term *Ck* 2 *Ek* is nonnegative. Hence, it is possible that the error energy -> 0 as *N* -> 00. When this happens, the orthogonal signal set is said to be complete.
- In this case, it’s no more an approximation but an equality

Linear combinations of signals



Application Example: Active Noise Cancellation

A "noise" signal, that we want to get rid of.

- At subject location, signal is

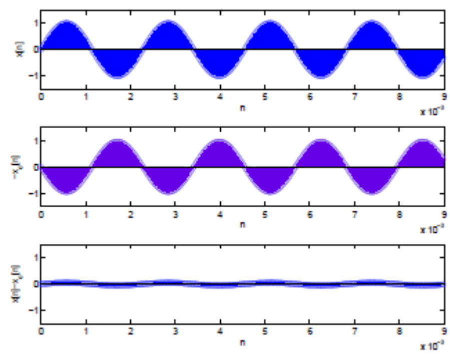
$$x[n]$$

- Microphone picks up signal

$$x_c[n]$$

- Subtract the two signals:

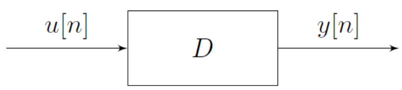
$$y(t) = x(t) - x_c(t)$$



Notice careful synchronization is needed!

Systems as Maps

Then a System is a **MATRIX**

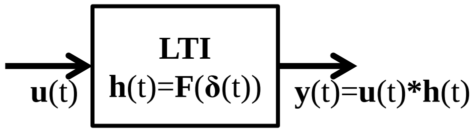


$$y = Du.$$

$$\begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[M] \end{bmatrix} = \begin{bmatrix} D_{11} & D12 & \cdots & D_{1N} \\ D_{21} & D_{22} & \cdots & D_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ D_{M1} & D_{M2} & \cdots & D_{MN} \end{bmatrix} \begin{bmatrix} u[1] \\ u[2] \\ \vdots \\ u[N] \end{bmatrix}.$$

$$y[i] = \sum_j D_{ij} u[j].$$

Linear Time Invariant



- Linear & Time-invariant (of course - tautology!)
- Impulse response: **h**(t)=**F**(**δ**(t))
- Why?
  - Since it is linear the output response (**y**) to any input (**x**) is:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$
$$y(t) = F\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] \overset{linear}{\rightarrow} \int_{-\infty}^{\infty} x(\tau)\,F\left[\delta(t-\tau)\right]d\tau$$
$$h(t-\tau) \overset{TI}{\equiv} F\left[\delta(t-\tau)\right]$$
$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

- The output of any continuous-time LTI system is the **convolution** of input **u**(t) with the impulse response **F**(**δ**(t)) of the system.

Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

Laplace:

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

- Total response = Zero-input response + Zero-state response

Initial conditions **External Input**

Linear Systems and ODE's

- Linear system described by differential equation

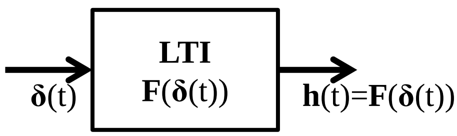
$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

- Which using Laplace Transforms can be written as

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s) Y(s) = B(s) X(s)$$

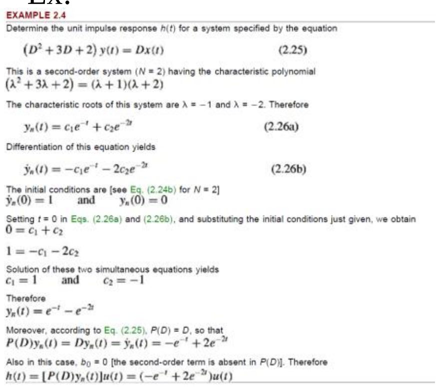
where *A(s)* and *B(s)* are polynomials in *s*

Unit Impulse Response



- **δ**(t): Impulsive excitation
- h(t): characteristic mode terms

Ex:



EXAMPLE 2.4

Determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)\, y(t) = Dx(t) \quad (2.25)$$

This is a second-order system ($N = 2$) having the characteristic polynomial
$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2)$

The characteristic roots of this system are $\lambda = -1$ and $\lambda = -2$. Therefore

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t} \quad (2.26a)$$

Differentiation of this equation yields

$$\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \quad (2.26b)$$

The initial conditions are [see Eq. (2.24b) for $N = 2$]
$\dot{y}_n(0) = 1$ and $y_n(0) = 0$

Setting $t = 0$ in Eqs. (2.26a) and (2.26b), and substituting the initial conditions just given, we obtain
$0 = c_1 + c_2$

$1 = -c_1 - 2c_2$

Solution of these two simultaneous equations yields
$c_1 = 1$ and $c_2 = -1$

Therefore
$y_n(t) = e^{-t} - e^{-2t}$

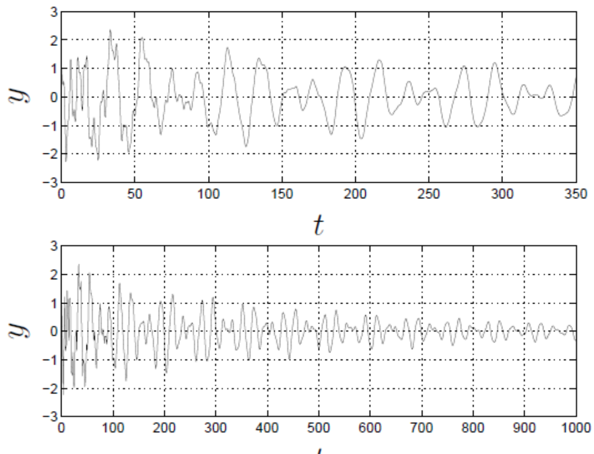
Moreover, according to Eq. (2.25), $P(D) = D$, so that
$P(D)y_n(t) = Dy_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}$

Also in this case, $b_0 = 0$ [the second-order term is absent in $P(D)$]. Therefore
$h(t) = [P(D)y_n(t)]u(t) = (-e^{-t} + 2e^{-2t})u(t)$

Where are we going with this?

This can help simplify matters…
An Example

Consider the following system:

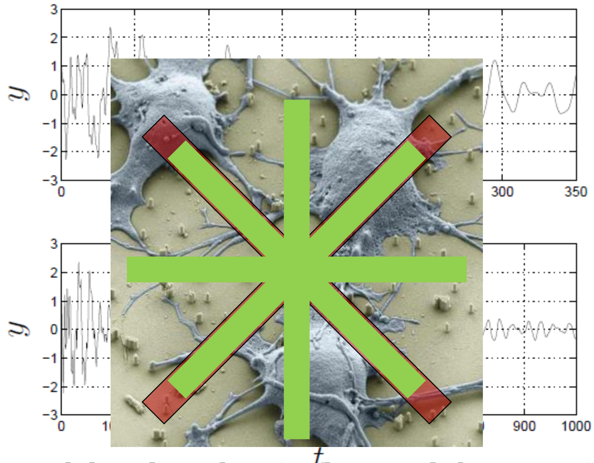


- How to model and predict (and control the output)?

Source: EE263 (s.1-13)

This can help simplify matters…
An Example

Consider the following system:



- How to model and predict (and control the output)?

Source: EE263 (s.1-13)

This can help simplify matters…
An Example

- Consider the following system:

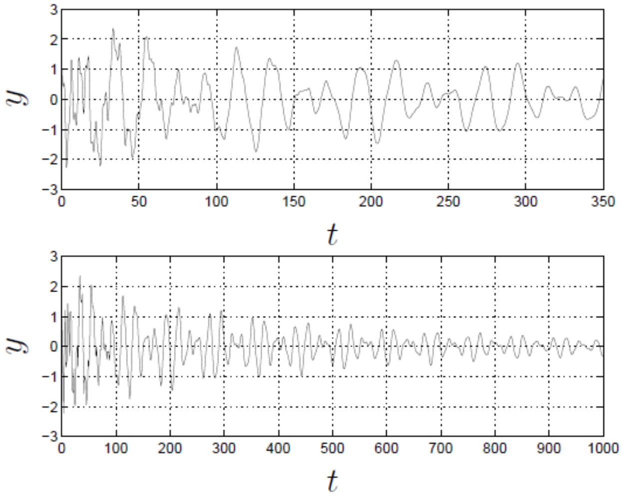
$$\dot{x} = Ax, \qquad y = Cx$$

- $x(t) \in \mathbb{R}^8$, $y(t) \in \mathbb{R}^1$ → 8-state, single-output system
- Autonomous: No input yet! ($u(t) = 0$)

Source: EE263 (s.1-13)

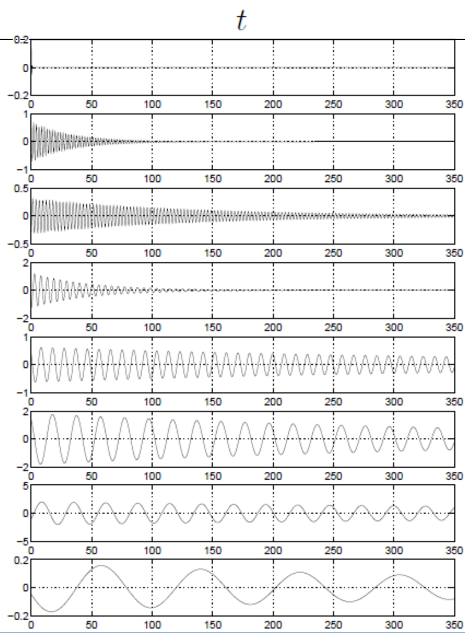
This can help simplify matters… An Example

- Consider the following system:



Source: EE263 (s.1-13)

This can help simplify matters… An Example



Source: EE263 (s.1-13)

Example: Let's consider the control…

Expand the system to have a control input…

- $B \in \mathbb{R}^{8\times 2}$, $C \in \mathbb{R}^{2\times 8}$ (note: the 2nd dimension of C)

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = 0$$

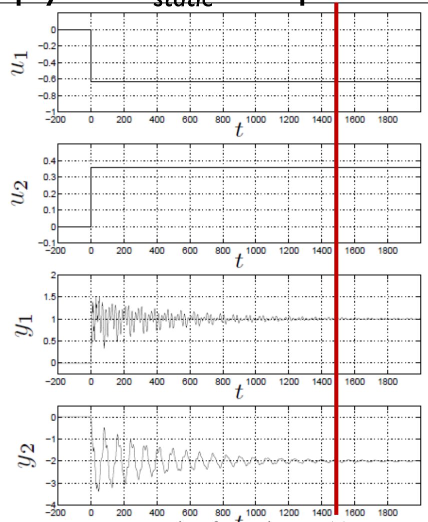
- Problem: Find **u** <u>such that</u> $y_{des}(t)=(1,-2)$
- A simple (and rational) approach:
  - solve the above equation!
  - Assume: static conditions (u, x, y constant)

$$\dot{x} = 0 = Ax + Bu_{\text{static}}, \quad y = y_{\text{des}} = Cx$$

➔ Solve for u:

$$u_{\text{static}} = \left(-CA^{-1}B\right)^{-1} y_{\text{des}} = \begin{bmatrix} -0.63 \\ 0.36 \end{bmatrix}$$

Example: Apply $u=u_{static}$ and presto!

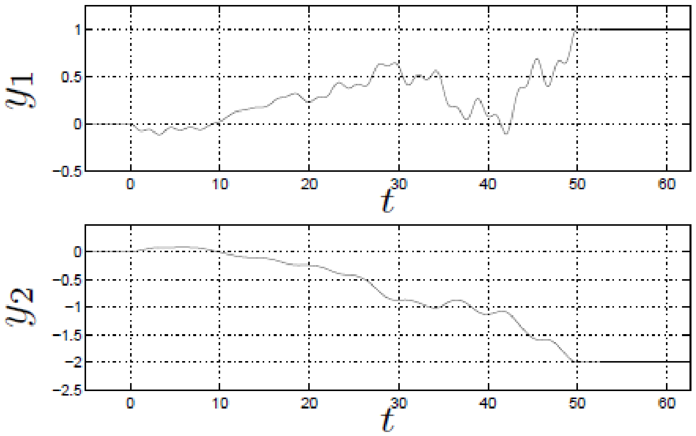


- Note: It takes 1500 seconds for the y(t) to converge … but that's natural … can we do better?

Source: EE263 (s.1-13)

Example: Yes we can!

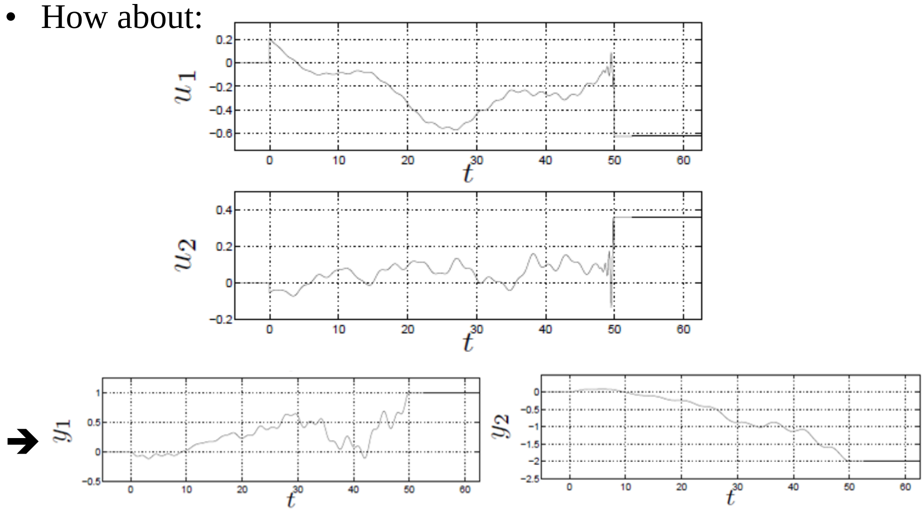
- How about:



Source: EE263 (s.1-13)

Example: **How?** How about a more clever input?

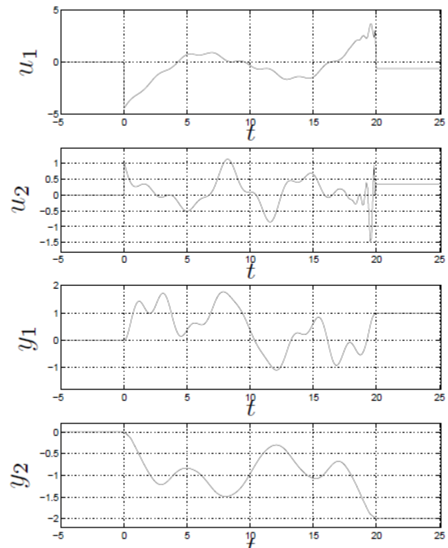
- How about:



- Converges in 50 seconds (3.3% of the time ☺)

Source: EE263 (s.1-13)

Example: Can we beat it? Larger inputs & LDS



Source: EE263 (s.1-13)

- Converges in 20 seconds (1.3% of the time ☺)

Next Time…

- We'll talk about Other System Properties ☺
- We will introduce this via the lens of:
  "Systems as Maps. Signals as Vectors"
- Review:
  - Phasers, complex numbers, polar to rectangular, and general functional forms.
  - Chapter B and Chapter 1 of Lathi
    (particularly the first sections on signals & classification thereof)
- Register on Platypus
- Try the practise assignment