http://elec3004.com

Digital Control Systems: State Space Control System Design

ELEC 3004: **Systems**: Signals & Controls

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Lecture 21

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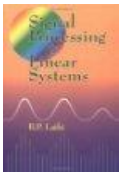
Lecture Schedule:

Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Systems: Linear Differential Systems
3	14-Mar	Sampling Theory & Data Acquisition
	16-Mar	Aliasing & Antialiasing
4	21-Mar	Discrete Time Analysis & Z-Transform
	23-Mar	Second Order LTID (& Convolution Review)
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
6	4-Apr	Digital Filters (IIR) & Filter Analysis
	6-Apr	Digital Filter (FIR)
7	11-Apr	Digital Windows
	13-Apr	FFT
	18-Apr	Holiday
	20-Apr	
	25-Apr	
8	27-Apr	Active Filters & Estimation
9	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	PID & State-Space
	11-May	State-Space Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	State Space Control System Design
	25-May	Shaping the Dynamic Response
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

G. Franklin,
J. Powell,
M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990
[Available as UQ Ebook]

Today

➔ **State-space** 🡐 **[A stately idea! ☺]**

- FPW
  - Chapter 6: Design of Digital Control Systems Using State-Space Methods
- Friedland
  - Chapter 6:
    Shaping The Dynamic Response

- Lathi Ch. 13
  - § 13.2 Systematic Procedure for Determining State Equations
  - § 13.3 Solution of State Equations

Next Time

More Online Reading Materials

CHAPTER SIX
SHAPING THE DYNAMIC RESPONSE

6.1 INTRODUCTION

CHAPTER THREE
DYNAMICS OF LINEAR SYSTEMS

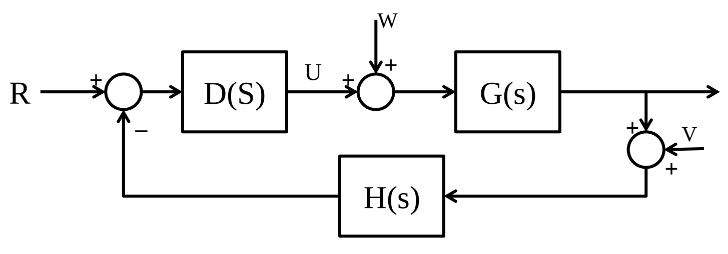
3.1 DIFFERENTIAL EQUATIONS REVISITED

- Friedland, Control System Design Ch. 6 and 3

➔ http://robotics.itee.uq.edu.au/~elec3004/tutes.html

Control Systems Design: `tf2ss`

Basic Closed-loop Block Diagram



$$TF(s) = \frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1 + D(s)G(s)H(s)} = \frac{DG}{1 + DGH}$$

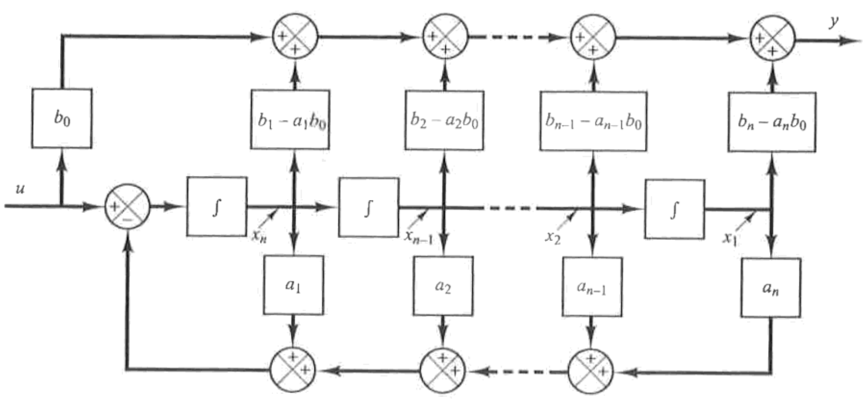
TF 2 SS – Control Canonical Form)

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Control Canonical Form as a Block Diagram



Modal Form

- CCF is not the only way to tf2ss
- Partial-fraction expansion of the system

➔ System poles appear as diagonals of Am

- Two issues:
  - The elements of matrix maybe complex if the poles are complex
  - It is non-diagonal with repeated poles

Modal Form

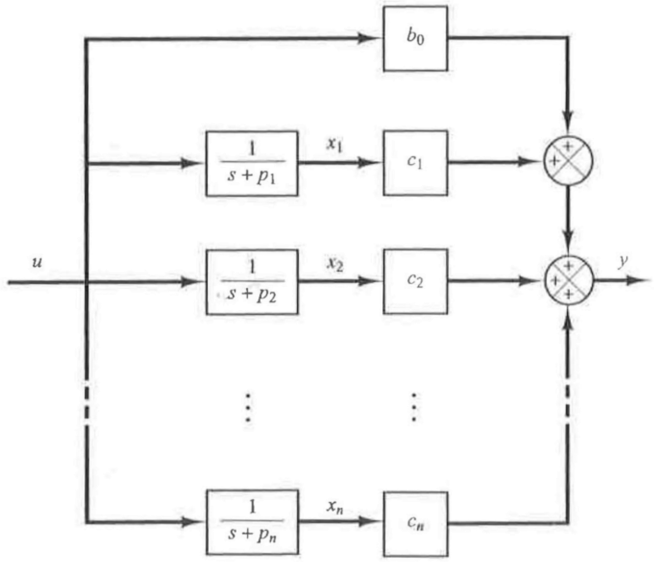
$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

$$= b_0 + \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Modal Form Block Diagram



Matlab's tf2ss

- Given: $\frac{Y(s)}{U(s)} = \frac{25.04s+5.008}{s^3+5.03247s^2+25.1026s+5.008}$
  Get a state space representation of this system
- Matlab:

```
num = [25.04 5.008];
den = [1 5.03247 25.1026 5.008];
[A,B,C,D] = tf2ss(num/den);
```

- Answer:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5.0325 & -25.1026 & -5.008 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = [0 \quad 25.04 \quad 5.008] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Control System Design: Obtaining a Time Response

From SS to Time Response — Impulse Functions

- Given: $\dot{x} = Ax + Bu$
- Solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Substituting $t_0 = 0$ into this:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Write the impulse as: $\mathbf{u}(t) = \delta(t)\mathbf{w}$
  - where **w** is a vector whose components are the magnitudes of **r** impulse functions applied at t=0

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)\mathbf{w}\,d\tau$$

$$= e^{\mathbf{A}t}\mathbf{x}(0-) + e^{\mathbf{A}t}\mathbf{B}\mathbf{w}$$

From SS to Time Response — Step Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = \boldsymbol{k}$
  Where **k** is a vector whose components are the magnitudes of *r* step functions applied at t=0.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{Bk}\,d\tau$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[\int_0^t\left(\mathbf{I} - \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} - \cdots\right)d\tau\right]\mathbf{Bk}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{\mathbf{A}^2t^3}{3!} - \cdots\right)\mathbf{Bk}$$

  – Assume A is non-singular

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[-(\mathbf{A}^{-1})(e^{-\mathbf{A}t} - \mathbf{I})\right]\mathbf{Bk}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{Bk}$$

From SS to Time Response — Ramp Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = t\boldsymbol{v}$
  Where **v** is a vector whose components are magnitudes of ramp functions applied at $t = 0$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\tau\mathbf{v}\,d\tau$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\tau\,d\tau\,\mathbf{Bv}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\frac{\mathbf{I}}{2}t^2 - \frac{2\mathbf{A}}{3!}t^3 + \frac{3\mathbf{A}^2}{4!}t^4 - \frac{4\mathbf{A}^3}{5!}t^5 + \cdots\right)\mathbf{Bv}$$

  – Assume A is non-singular

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + (\mathbf{A}^{-2})(e^{\mathbf{A}t} - \mathbf{I} - \mathbf{A}t)\mathbf{Bv}$$
$$= e^{\mathbf{A}t}\mathbf{x}(0) + \left[\mathbf{A}^{-2}(e^{\mathbf{A}t} - \mathbf{I}) - \mathbf{A}^{-1}t\right]\mathbf{Bv}$$

Example: Obtain the Step Response

- Given: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u(t) = 1(t)$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+1 & 0.5 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2 + s + 0.5} \begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s + 0.5 - 0.5}{(s+0.5)^2 + 0.5^2} & \dfrac{-0.5}{(s+0.5)^2 + 0.5^2} \\ \dfrac{1}{(s+0.5)^2 + 0.5^2} & \dfrac{s + 0.5 + 0.5}{(s+0.5)^2 + 0.5^2} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t) & -e^{-0.5t} \sin 0.5t \\ 2e^{-0.5t} \sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t) \end{bmatrix}$$

  - Set k=1, x(0)=0:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}k$$

$$= \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} 0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t} \sin 0.5t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-0.5t} \sin 0.5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1 \end{bmatrix}$$

$$y(t) = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 = e^{-0.5t} \sin 0.5t$$

Example II: Obtain the Step Response

- Given:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad u(t) = 1(t)$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [1]\, d\tau$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

  - Assume x(0)=0:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Break ☺

The Direct Method of Digital Controls –

NOT to be confused with
Controller Emulation
(e.g., Tustin's Method)

Direct Design Method Of Ragazzini (See also: FPW 5.7 pp.216-222)

Start with 3 Discrete Transfer Functions:

- **G(z)**: TF[1] of a **plant + a hold** (e.g., from a ZOH)
- **D(z)**: A **controller** TF to do the job (what we want here)
- **H(z)**: The final desired TF between **R** (reference) and **Y** (output)
- Thus[2]:

$$H(z) = \frac{DG}{1+DG}$$
$$\rightarrow D(z) = \frac{1}{G}\frac{H}{1-H}$$

- This calls for a **D(z)** that will **cancel the plant effects** and that will **add whatever is necessary to give the desired result**. The problem is to discover and implement constraints on **H(z)** so that we do not ask for the impossible.
  - This implies that we **need some constraints** on both **H(z)** and **D(z)**

1: Transfer Function

2: Mental Quiz: What does **1+DG** say about the sign of the feedback (positive or negative)? That is, what is the characteristic equation for a system with positive feedback?

Direct Design Method Of Ragazzini [2]: Design Constraints: I. Causality

- Remember/Recall an Interesting Point:
  - From z-transform theory we know that **if D(z)** is causal, **then** as $z \rightarrow \infty$ its transfer function is well behaved & it **does not** have a pole at infinity.

- $D(z) = \frac{1}{G}\frac{H}{1-H}$ implies that **if** G**(z)** = 0 (at ∞), **then D(z)** would have a pole (at ∞) **unless H(z)** cancels it.

∴

H(z) must have a zero (at ∞) of the same order as G**(z)**'s 0s (at ∞)

→ Which means: **If** there is a lag in the plant (**G(z) starts with z^{-l}**) **then** causality requires that the delay of **H(z)** is that the closed-loop system must be at least as long a delay of the plant.

(Whoa! It might sound deep, but it's rather intuitive ☺)

Direct Design Method Of Ragazzini [3]: Design Constraints: II. Stability

- The characteristic equation and the closed loop roots:

$$1 + D(z)G(z) = 0$$

- Define[3] $D = \frac{c}{d}$ and G= $\frac{b}{a}$ ➔ $ad + bc = 0$
- Define $z - \alpha$ as a pole of **G(z)** and a common factor in DG that represents **D(z)** cancelling a pole/zero of **G(z)**.
- **Then** this common factor *remains a factor of the characteristic polynomial.*
- **If** this factor is outside the unit circle, **then** the system is unstable!

∴

1-H(z) must contain as zeros
all the poles of **G(z)** that are outside the unit circle &
H(z) must contain as zeros
all the zeros of G(z) that are outside the unit circle

3: Note the switching of the "alphabetical-ness" of these two fractions

Direct Design Method Of Ragazzini [4]: Design Constraints: III. Steady State Accuracy

- The error from **H(z)** is given by:

$$\mathrm{E(z)} = \mathrm{R(z)}\big(1 - \mathrm{H(z)}\big)$$

- **If** the system is "Type 1" (with a constant velocity/first derivative (K_v)
  - **Then**[4] $E_{ss}^{Step} = 0$ and $E_{ss}^{Ramp} = {}^{1}/_{K_v}$

∴

$$\mathrm{H(z)} = 1$$

&

$$-\mathrm{T_s} \frac{dH(z)}{d(z)}\bigg|_{z=1} = \frac{1}{K_v}\mathrm{H(z)} = 1$$

4: E_{ss}: steady-state error

Direct Design Method Of Ragazzini [5]: An Example

- Consider the plant: $s^2 + s + 1 = 0$

With T_s=1 → z-Transform: $z^2 + 0.786z + 0.368 = 0$

- Let's design this **system** such that
  - $K_v = 1$
  - Poles at the roots of the plant equation & additional poles as needed

$$\rightarrow H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots}{1 - 0.786 z^{-1} + 0.368\, z^{-2}}$$

I. Causality: $H(z)|_{z=\infty} = 0 \rightarrow b_0 = 0$

II. Stability: All poles/zeros of G(z) are in the unit circle
  - except for b_0, which is taken care of by $b_0 = [Const] = 0$

III. Tracking:
  - $H(1) = b_1 + b_2 + b_3 + \cdots = 1 \cdot (1 - 0.786 + 0.368)$ &
  - $-\{1\} \left.\frac{dH(z)}{d(z^{-1})}\right|_{z=1} = \frac{1}{\{1\}} \rightarrow \frac{b_1 + 2b_2 + 3b_3 + \cdots \; -[-.05014]}{(1 - 0.786 + 0.368)}$ (note the z^{-1})
  - Truncate the number of unknowns to 2 "zeros" … thus solve for b_1 and b_2 (& set $b_3, b_4, \ldots = 0$)

$$\therefore H(z) = \frac{b_1 z + b_2}{z^2 - 0.786z + 0.368} \rightarrow$$

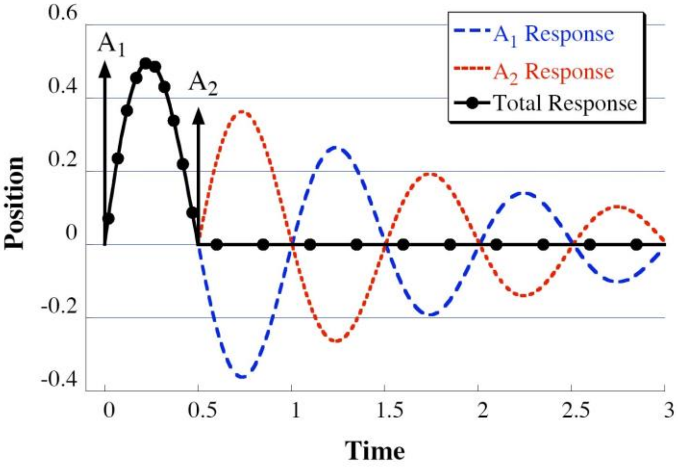
$$D(z) = \frac{(z-1)(z-0.9048)(0.6321)}{(0.04837)(z+0.9672)} \frac{(z-0.07932)}{(z-1)(z-0.4180)}$$

$$= 13.07 \frac{(z-0.9048)}{(z+0.9672)} \frac{(z-0.07932)}{(z-0.4180)}.$$

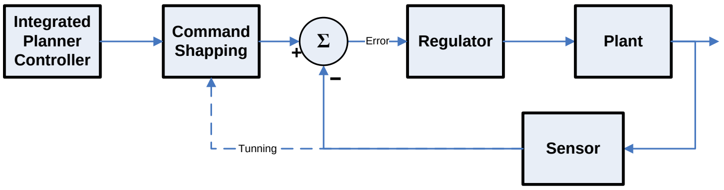
Application Example 1: Command Shaping

Experiments: Scanning Over Obstacle

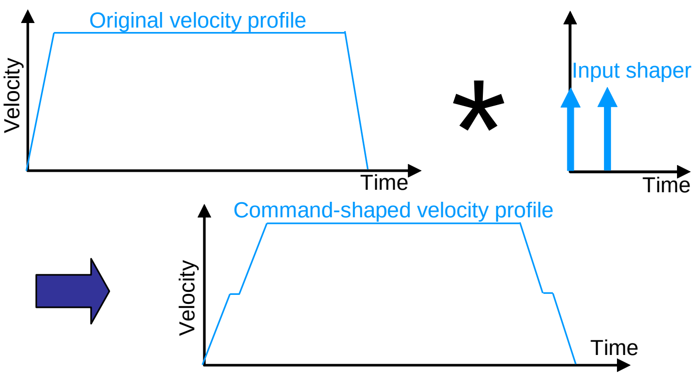
Command Shaping



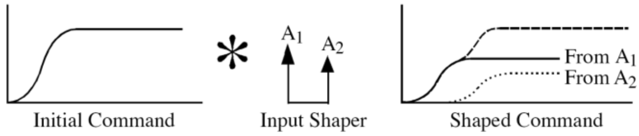
Robust Control: Command Shaping for Vibration Reduction



Command Shaping



Command Shaping



- Zero Vibration (ZV)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{1+K} & \frac{K}{1+K} \\ 0 & \frac{T_d}{2} \end{bmatrix} \qquad K = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

- Zero Vibration and Derivative (ZVD)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{(1+K)^2} & \frac{2K}{(1+K)^2} & \frac{K^2}{(1+K)^2} \\ 0 & \frac{T_d}{2} & T_d \end{bmatrix}$$

Experiments: Command Shaping

Next Time…

- **Digital Feedback Control**

- Review:
  - Chapter 2 of FPW

- More Pondering??