Stability, Observability and Controllability

ELEC 3004: **Systems**: Signals & Controls
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Lecture 20

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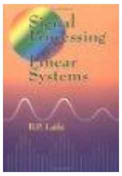
Lecture Schedule:

Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Systems: Linear Differential Systems
3	14-Mar	Sampling Theory & Data Acquisition
	16-Mar	Aliasing & Antialiasing
4	21-Mar	Discrete Time Analysis & Z-Transform
	23-Mar	Second Order LTID (& Convolution Review)
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
6	4-Apr	Digital Filters (IIR) & Filter Analysis
	6-Apr	Digital Filter (FIR)
7	11-Apr	Digital Windows
	13-Apr	FFT
	18-Apr	**Holiday**
	20-Apr	
	25-Apr	
8	27-Apr	Active Filters & Estimation
9	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	PID & State-Space
	11-May	State-Space Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response
	25-May	Applications in Industry
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

G. Franklin, J. Powell, M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990
[Available as UQ Ebook]

Today

➔ State-space 🡐

- FPW
  - Ch. 5: Transfer Functions: The Digital Filter
- Lathi Ch. 13
  - § 13.2 Systematic Procedure for Determining State Equations
  - § 13.3 Solution of State Equations

- FPW
  - Chapter 6 - Design of Digital Control Systems Using State-Space Methods

Next Time

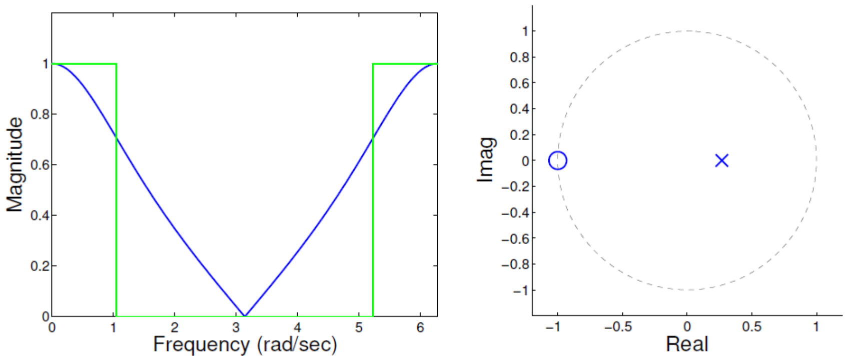
State Space

Can you use this for more than Control?

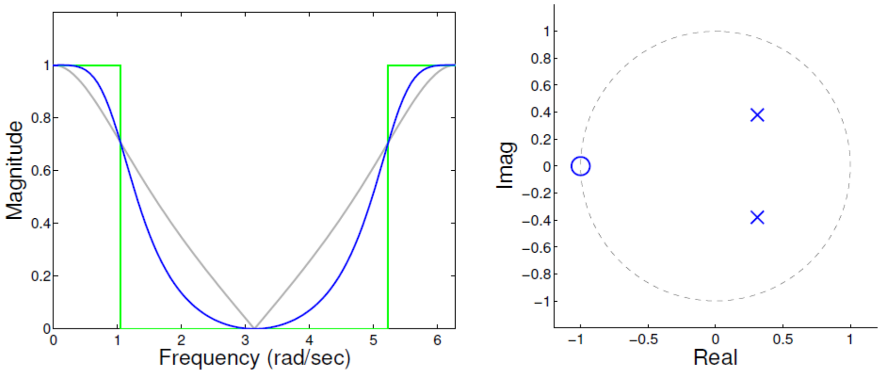
YES!

Discrete Time Butterworth Filters

"Maximally-flat filter". Sacrifice sharpness to have flat response in pass band and stop band.



"Maximally-flat filter". Sacrifice sharpness to have flat response in pass band and stop band.



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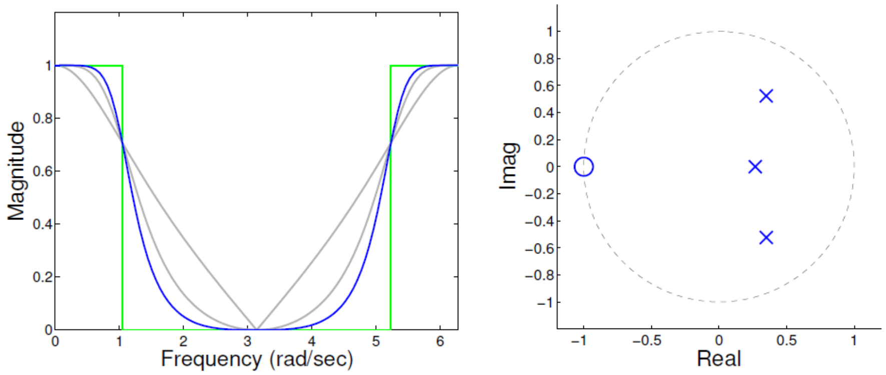

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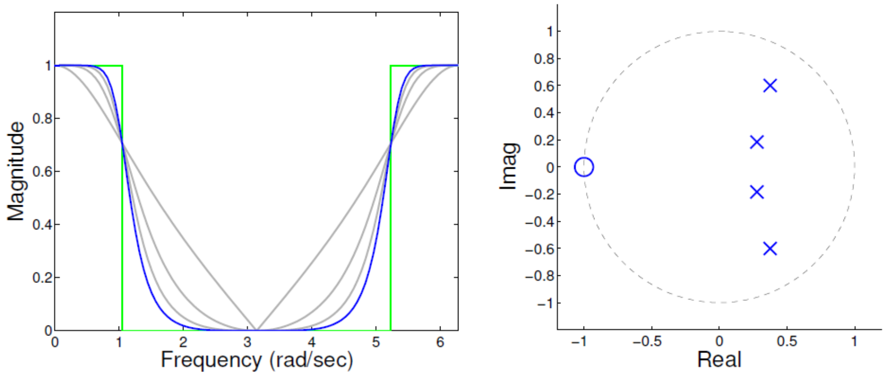

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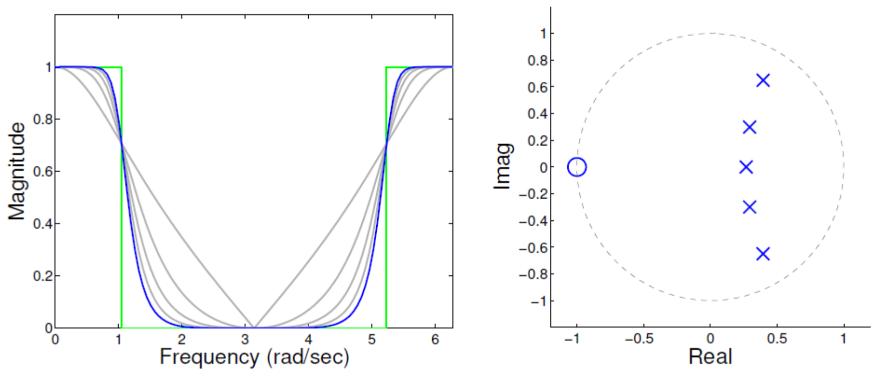

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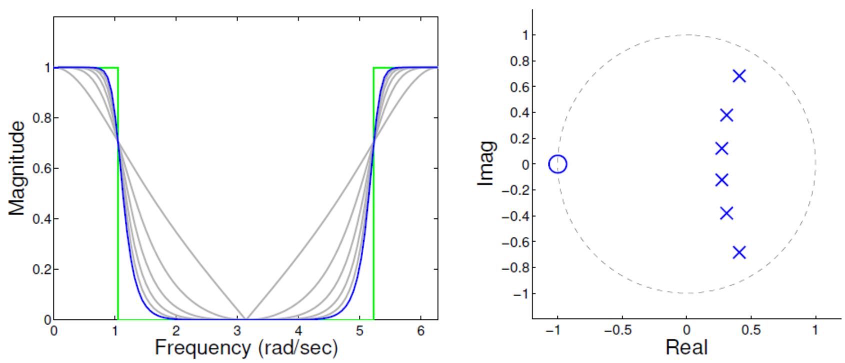

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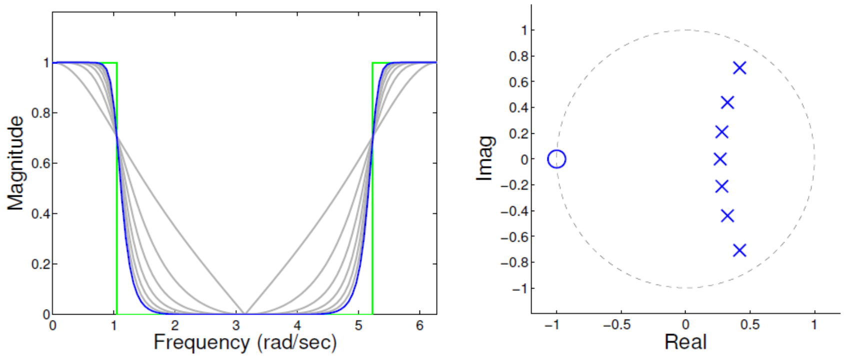


How?

- Constrained Least-Squares …

One formulation: Given $x[0]$

$$\underset{u[0],u[1],\ldots,u[N]}{\text{minimize}} \quad ||\vec{u}||^2, \quad \text{where } \vec{u} = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N] \end{bmatrix}$$

$$\text{subject to} \quad x[N] = 0.$$

Note that

$$x[n] = A^n x[0] + \sum_{k=0}^{n-1} A^{(n-1-k)} B u[k],$$

so this problem can be written as

$$\underset{x_{ls}}{\text{minimize}} \, ||A_{ls} x_{ls} - b_{ls}||^2 \quad \text{subject to} \quad C_{ls} x_{ls} = D_{ls}.$$

Controllability

Controllability

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{y}$ = output vector (m-vector) $\quad (m \leq n)$

$\mathbf{A}$ = $n \times n$ matrix

$\mathbf{B}$ = $n \times r$ matrix

$\mathbf{C}$ = $m \times n$ matrix

is completely output controllable if and only if the composite $m \times nr$ matrix $\mathbf{P}$, where

$$\mathbf{P} = [\mathbf{CB} \mid \mathbf{CAB} \mid \mathbf{CA}^2\mathbf{B} \mid \cdots \mid \mathbf{CA}^{n-1}\mathbf{B}]$$

is of rank m. (Notice that complete state controllability is neither necessary nor sufficient for complete output controllability.)

Controllability matrix

- To convert an arbitrary state representation in **F**, **G**, **H** and *J* to control canonical form **A**, **B**, **C** and *D*, the "controllability matrix"

$$\mathcal{C} = [\mathbf{G} \quad \mathbf{FG} \quad \mathbf{F}^2\mathbf{G} \quad \cdots \quad \mathbf{F}^{n-1}\mathbf{G}]$$

must be nonsingular.

Why is it called the "controllability" matrix?

Controllability matrix

- If you can write it in CCF, then the system equations must be linearly independent.
- Transformation by any nonsingular matrix preserves the controllability of the system.
- Thus, a nonsingular controllability matrix means ***x*** can be driven to any value.

Controllability Example

- Is this fully controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- We see that vectors **B** and **AB** are **not linearly independent** and
- The rank of the matrix **[B | AB]** is 1 < **m** (m=2)
- ∴ **the system is not completely state controllable.**

- In fact, elimination of x_2 from the given problem yields:

$$\ddot{x}_1 + 1.5\dot{x}_1 - 2.5x_1 = \dot{u} + 2.5u \longrightarrow \frac{X_1(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$$

- Notice that cancellation of the factor (**s** + 2.5) occurs in the numerator and denominator of the transfer function. Because of this cancellation, this system is **not** completely state controllable and it's unstable system (s=1, RHP!). Remember that stability and controllability are quite different things. **There are many systems that are unstable, but are completely state controllable**.

Controllability Example II

- TF ➔ CCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.4 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = [0.8 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\longrightarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -0.4 \\ 1 & -1.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.8 \\ 1 \end{bmatrix} u \qquad y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution. Consider the system defined by Equations (9–120) and (9–121). The rank of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$$

is 2. Hence, the system is completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$$

is 1. Hence the system is not observable.

Next consider the system defined by Equations (9–122) and (9–123). The rank of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix} 0.8 & -0.4 \\ 1 & -0.5 \end{bmatrix}$$

is 1. Hence, the system is not completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix} 0 & 1 \\ 1 & -1.3 \end{bmatrix}$$

is 2. Hence, the system is observable.

The apparent difference in the controllability and observability of the same system is caused by the fact that the original system has a pole-zero cancellation in the transfer function. Referring to Equation (2–29), for $D = 0$ we have

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

If we use Equations (9–120) and (9–121), then

$$G(s) = [0.8 \quad 1] \begin{bmatrix} s & -1 \\ 0.4 & s + 1.3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2 + 1.3s + 0.4} [0.8 \quad 1] \begin{bmatrix} s + 1.3 & 1 \\ -0.4 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{s + 0.8}{(s + 0.8)(s + 0.5)}$$

[Note that the same transfer function can be obtained by using Equations (9–122) and (9–123).] Clearly, cancellation occurs in this transfer function.

Break ☺

Observability

Observability

- Observability is concerned with the issue of what can be said about the state when one is given measurements of the plant output.

- **Definition**: The state x0 ≠ 0 is said to be unobservable if, given x(0) = x0, and u[k] = 0 for k ≥ 0, then y[k] = 0 for k ≥ 0. The system is said to be completely observable if there exists no nonzero initial state that it is unobservable.

Observability and Detectability

- Consider again the state space model

$$
\begin{aligned}
\delta x[k] &= \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k] \\
y[k] &= \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]
\end{aligned}
$$

- In general, the dimension of the observed output, y, can be less than the dimension of the state, x. However, one might conjecture that, if one observed the output over some nonvanishing time interval, then this might tell us something about the state. The associated properties are called observability (or reconstructability). A related issue is that of detectability. We begin with observability.

2. Criteria for observability

Theorem 2-8 Dynamical equation

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u, \quad t \in [t_0, +\infty) \quad (2-1)$$

is observable at time t_0 if and only if there exists a finite $t_1 > t_0$, such that the n columns of matrix

$$C(t)\Phi(t, t_0)$$

is linearly independent over $[t_0, t_1]$.

Proof: Sufficiency:

1). Consider

$$y(t) = C(t)\Phi(t, t_0)x(t_0) + \int_{t_0}^{t} C(t)\Phi(t, \tau)B(\tau)u(\tau)d\tau \; (*)$$

2). Pre-multiplying both sides of the equation (*) with

$$[C(t)\Phi(t, t_0)]^* = \Phi^*(t, t_0)C^*(t)$$

we have

$$\Phi^*(t,t_0)C^*(t)C(t)\Phi(t,t_0)x(t_0) = \Phi^*(t,t_0)C^*(t)y_1(t)$$

$$y_1 := y(t) - \int_{t_0}^{t} C(t)\Phi(t,\tau)B(\tau)u(\tau)d\tau$$

3). Integrating both sides from t_0 to t_1, we have

$$V(t_0,t_1)x(t_0) = \int_{t_0}^{t_1} \Phi^*(\tau,t_0)C^*(\tau)y_1(\tau)d\tau$$

$$V(t_0,t_1) := \int_{t_0}^{t_1} \Phi^*(\tau,t_0)C^*(\tau)C(\tau)\Phi(\tau,t_0)d\tau$$

Form Theorem 2-1, it follows that $V(t_0, t_1)$ is nonsingular if and only if the columns of $C(t)\Phi(t, t_0)$ are linearly independent over $[t_0, t_1]$.

Necessity: the proof is by contradiction.

Assume that the system is observable but the columns of $C(t)\Phi(t, t_0)$ are linearly dependent for any $t_1>t_0$. Then, there exists a column vector $\alpha\neq0$, such that

$$C(t)\Phi(t,t_0)\alpha = 0, \quad \forall t \in [t_0,t_1]$$

If we choose $x(t_0)=\alpha$, then we have

$$y(t) = C(t)\Phi(t,t_0)\alpha = 0 \forall t > t_0$$

which means that $x(t_0)$ can not be determined by y.

Observability criteria for LTI systems

Theorem 2-11 For the n-dimensional linear time invariant dynamical equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (2\text{-}21)$$

the following statements are equivalent:
(1). (2–21) is observable for any t_0 in $[0, +\infty)$;
(2). All the columns of Ce^{At} are linearly independent on $[t_0, +\infty)$.

(3). The matrix

$$V(t_0, t) = \int_{t_0}^{t} e^{A*(\tau - t_0)} C * C e^{A(\tau - t_0)} d\tau$$

is nonsingular for any $t_0 \geq 0$ and $t > t_0$.

(4). The $nq \times n$ observability matrix

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

(5). All columns of $C(sI–A)^{-1}$ are linearly independent over $\mathbb{C}$.

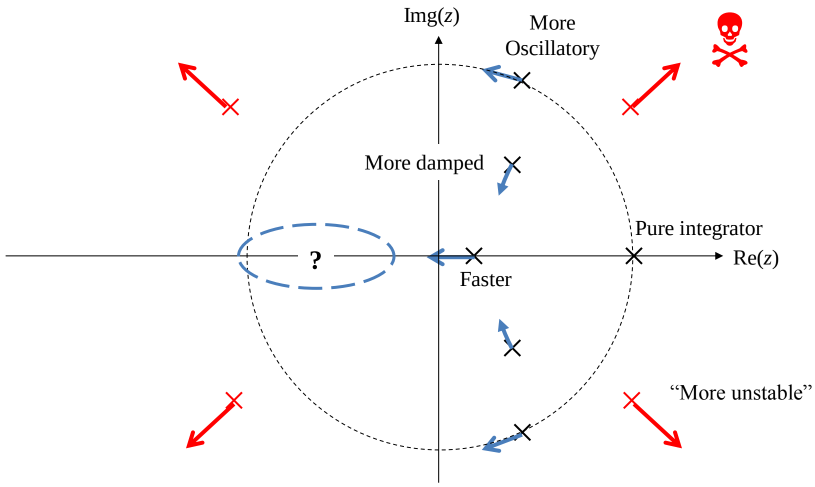
(6) For every eigenvalue λ_i of A,

$$rank\begin{bmatrix} A-\lambda_i I \\ C \end{bmatrix} = n \qquad (2-15)$$

Stability

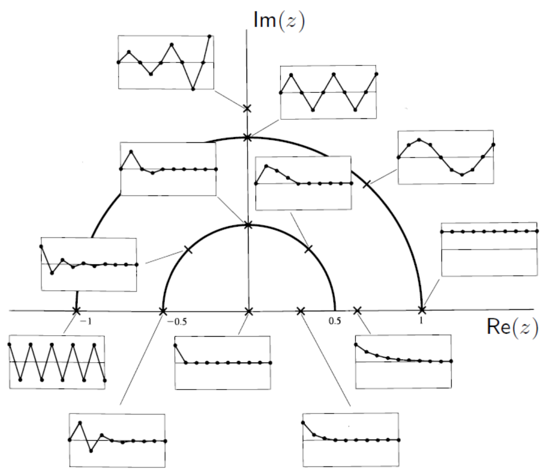
Recall dynamic responses

- Ditto the z-plane:

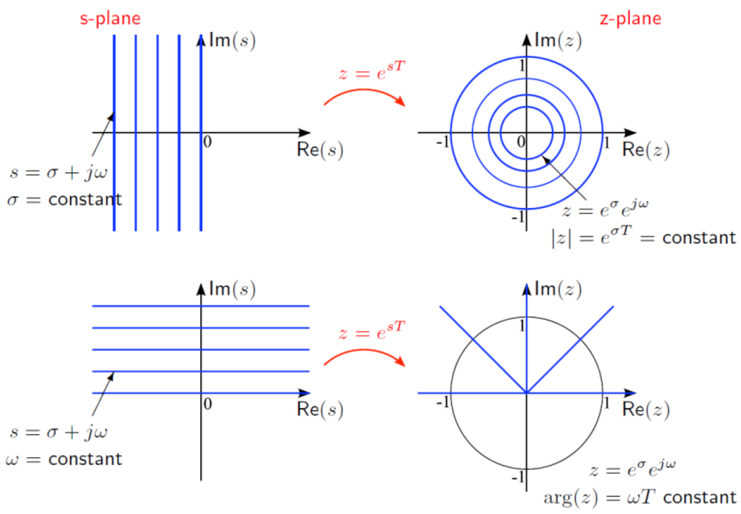


Pole positions in the z-plane

- Poles inside the unit circle are **stable**
- Poles outside the unit circle **unstable**
- Poles on the unit circle are oscillatory
- Real poles at 0 < z < 1 give exponential response
- Higher frequency of oscillation for larger
- Lower apparent damping for larer and r



S-Plane to z-Plane [1/2]



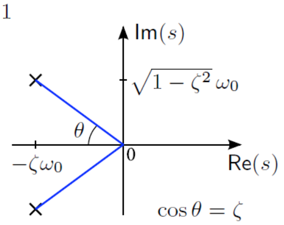
S-Plane to z-Plane [2/2]

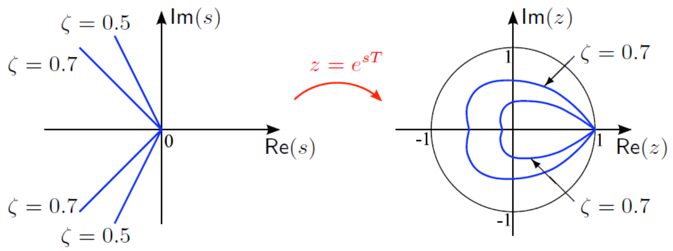
Pole locations for constant damping ratio $\zeta < 1$

$$s^2 + \zeta\omega_0 s + \omega_0^2 = 0$$

$$\Downarrow$$

$$s = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\,\omega_0$$

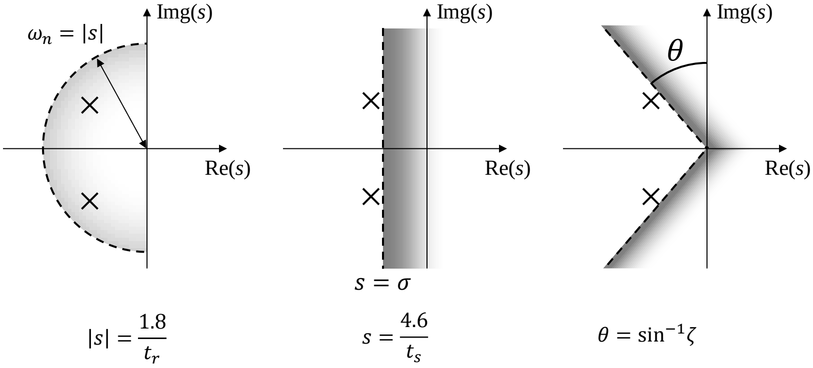




$s = -\zeta\omega_0 + j\sqrt{1-\zeta^2}\,\omega_0$: $\zeta =$ constant $\qquad z = e^{-\zeta\omega_0 T} e^{-j\sqrt{1-\zeta^2}\omega_0 T}$

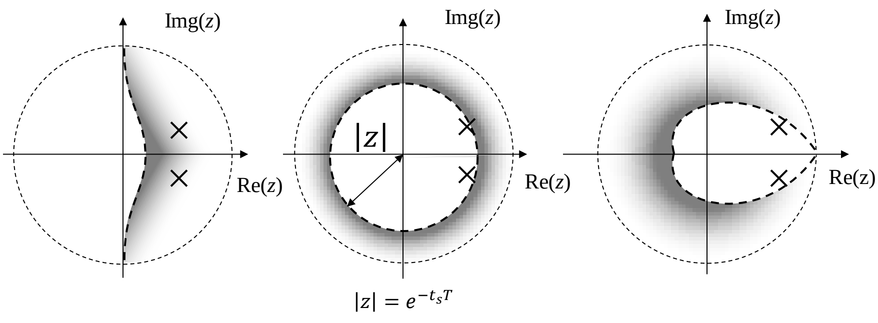
Specification bounds

- Recall in the continuous domain, response performance metrics map to the s-plane:



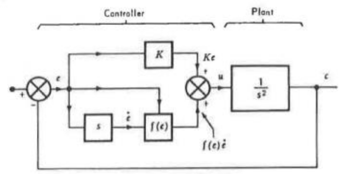
Discrete bounds

- These map to the discrete domain:



In practice, you'd use Matlab to plot these, and check that the spec is satisfied

Stability of a 2nd order regulator



$$u = Ke + f(e)\dot{e}$$

state equations let $e = x_1$ and $\dot{e} = x_2$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -Kx_1 - f(x_1)\ x_2$$

assume for simplicity that K = 1.

$$0 = x_2^0$$

$$0 = -x_1^0 - f(x_1^0)x_2^0$$

The Jacobian matrix is

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -f(0) \end{bmatrix}$$

- The linear behavior of the system in the close neighborhood of the origin is described by

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - f(0)x_2$$

- AND, the characteristic equation is:

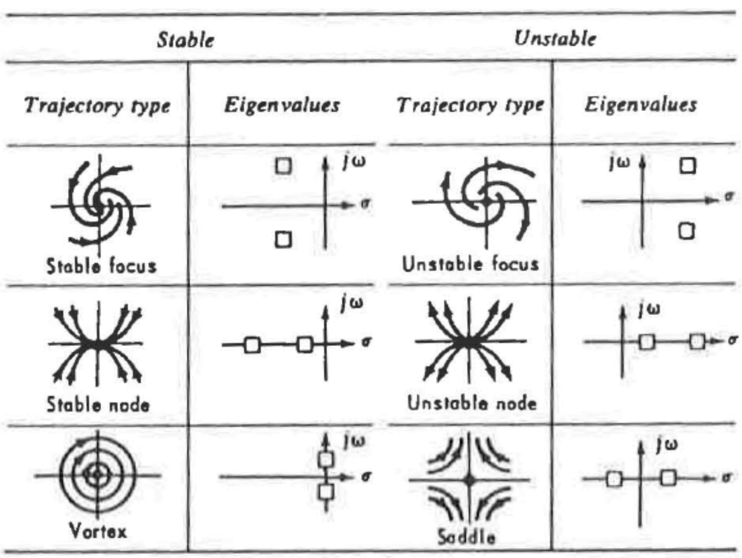
$$s[s + f(0)] + 1 = 0$$

with the eigenvalues

$$\lambda_1 = -\frac{1}{2}f(0) + \sqrt{\frac{1}{4}f^2(0) - 1}$$

$$\lambda_2 = -\frac{1}{2}f(0) - \sqrt{\frac{1}{4}f^2(0) - 1}$$

Various Types of Singularities (2nd order systems)



Linear Transformation of State Vectors

Discretization FPW!

- We can use the time-domain representation to produce difference equations!

$$\boldsymbol{x}(kT+T) = e^{\mathbf{F}T}\ \boldsymbol{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

Notice $\boldsymbol{u}(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series.

We can structure this by using the form:

$$u(\tau) = u(kT), \qquad kT \le \tau \le kT + T$$

Similarity Transformations

It is readily seen that the definition of the state of a system is nonunique. Consider, for example, a linear transformation of *x*(*t*) to $\overline{x}(t)$ defined as

$$\overline{x}(t) = \mathbf{T}^{-1}x(t) \qquad x(t) = \mathbf{T}\,\overline{x}(t)$$

where $\mathbf{T}$ is any nonsingular matrix, called a similarity transformation.

Similarity Transformations

The following alternative state description is obtained

$$\overline{\mathbf{A}} \triangleq \mathbf{T}^{-1}\mathbf{A}\mathbf{T} \quad \overline{\mathbf{B}} \triangleq \mathbf{T}^{-1}\mathbf{B} \quad \overline{\mathbf{C}} \triangleq \mathbf{C}\mathbf{T} \quad \overline{\mathbf{D}} \triangleq \mathbf{D}$$

Where

$$\begin{aligned}\dot{\overline{x}}(t) &= \overline{\mathbf{A}}\,\overline{x}(t) + \overline{\mathbf{B}}\,u(t) \\ y(t) &= \overline{\mathbf{C}}\,\overline{x}(t) + \overline{\mathbf{D}}\,u(t)\end{aligned} \qquad \overline{x}(t_o) = \mathbf{T}^{-1}x_o$$

The above model is an equally valid description of the system.

Similarity Transformations

An illustration, say that the matrix **A** can be diagonalized by a similarity transformation **T**; then

$$\overline{\mathbf{A}} = \mathbf{\Lambda} \triangleq \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

where if $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of **A**, then

$$\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots \lambda_n)$$

Similarity Transformations

Because **Λ** is diagonal, we have

$$\overline{x}_i(t) = e^{\lambda_i(t-t_o)}\overline{x}_o + \int_{t_o}^{t} e^{\lambda_i(t-\tau)}\overline{b}_i u(\tau)d\tau$$

where the subscript i denotes the i^{th} component of the state vector.

Similarity Transformations: Example

$$\mathbf{A} = \begin{bmatrix} -4 & -1 & 1 \\ 0 & -3 & 1 \\ 1 & 1 & -3 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} -1 & -1 & 0 \end{bmatrix} \quad \mathbf{D} = 0$$

The matrix **T** can also be obtained by using the MATLAB command **eig**, which yields

$$\mathbf{T} = \begin{bmatrix} 0.8018 & 0.7071 & 0.0000 \\ 0.2673 & -0.7071 & 0.7071 \\ -0.5345 & -0.0000 & 0.7071 \end{bmatrix}$$

Similarity Transformations: Example

We obtain the similar state space description given by

$$\overline{\mathbf{A}} = \mathbf{\Lambda} = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad \overline{\mathbf{B}} = \begin{bmatrix} 0.0 \\ -1.414 \\ 0.0 \end{bmatrix};$$

$$\overline{\mathbf{C}} = \begin{bmatrix} -0.5345 & -1.4142 & 0.7071 \end{bmatrix} \quad \overline{\mathbf{D}} = 0$$

Transfer Functions Revisited

The solution to the state equation model can be obtained via

$$\begin{aligned} Y(s) &= [\overline{\mathbf{C}}(s\mathbf{I}-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}+\overline{\mathbf{D}}]U(s)+\overline{\mathbf{C}}(s\mathbf{I}-\overline{\mathbf{A}})^{-1}\overline{x}(0) \\ &= [\mathbf{CT}(s\mathbf{I}-\mathbf{T}^{-1}\mathbf{AT})^{-1}\mathbf{T}^{-1}\mathbf{B}+\mathbf{D}]U(s)+\mathbf{CT}(s\mathbf{I}-\mathbf{T}^{-1}\mathbf{AT})^{-1}\mathbf{T}^{-1}x(0) \\ &= [\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}]U(s)+\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}x(0) \end{aligned}$$

Transfer Functions Revisited

We thus see that different choices of state variables lead to different internal descriptions of the model, but to the same input-output model, because the system transfer function can be expressed in either of the two equivalent fashions.

$$\overline{\mathbf{C}}(s\mathbf{I}-\overline{\mathbf{A}})^{-1}\overline{\mathbf{B}}+\overline{\mathbf{D}} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$$

for any nonsingular $\mathbf{T}$.

From Transfer Function to State Space Representation

Consider a transfer function $G(s) = {}^{B(s)}/_{A(s)}$. We can then write

$$Y(s) = \sum_{i=1}^{n} b_{i-1} V_i(s) \qquad \text{where} \qquad V_i(s) = \frac{s^{i-1}}{A(s)} U(s)$$

We note from the above definitions that

$$v_i(t) = \mathcal{L}^{-1}[V(s)] = \frac{dv_{i-1}(t)}{dt} \qquad \text{for} \qquad i = 1, 2, \ldots, n$$

From Transfer Function to State Space Representation

- We can then choose, as state variables, xi(t) = vi(t), which lead to the following state space model for the system.

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-1} \end{bmatrix} \qquad \mathbf{D} = 0$$

- The above model has a special form. Any *completely controllable* system can be expressed in this way.

Extension!

Additional Notes on Calculating Φ and Γ for Discrete Control

Discretization FPW!

- We can use the time-domain representation to produce difference equations!

$$\boldsymbol{x}(kT+T) = e^{\mathbf{F}T}\,\boldsymbol{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

Notice $\boldsymbol{u}(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series.

We can structure this by using the form:

$$u(\tau) = u(kT), \qquad kT \le \tau \le kT+T$$

Discretization FPW!

- Put this in the form of a new variable:

$$\eta = kT + T - \tau$$

Then:

$$\boldsymbol{x}(kT+T) = e^{\boldsymbol{F}T}\boldsymbol{x}(kT) + \left(\int_{kT}^{kT+T} e^{\boldsymbol{F}\eta} d\eta\right)\boldsymbol{G}u(kT)$$

Let's rename $\boldsymbol{\Phi} = e^{\boldsymbol{F}T}$ and $\boldsymbol{\Gamma} = \left(\int_{kT}^{kT+T} e^{\boldsymbol{F}\eta} d\eta\right)\boldsymbol{G}$

Discrete state matrices

So,

$$\boldsymbol{x}(k+1) = \boldsymbol{\Phi}\boldsymbol{x}(k) + \boldsymbol{\Gamma}u(k)$$
$$y(k) = \mathbf{H}\boldsymbol{x}(k) + \mathbf{J}\boldsymbol{u}(k)$$

Again, $\boldsymbol{x}(k+1)$ is shorthand for $\boldsymbol{x}(kT+T)$

Note that we can also write $\boldsymbol{\Phi}$ as:

$$\boldsymbol{\Phi} = \mathbf{I} + \mathbf{F}T\boldsymbol{\Psi}$$

where

$$\boldsymbol{\Psi} = \mathbf{I} + \frac{\mathbf{F}T}{2!} + \frac{\mathbf{F}^2T^2}{3!} + \cdots$$

Simplifying calculation

- We can also use $\boldsymbol{\Psi}$ to calculate $\boldsymbol{\Gamma}$
  - Note that:

$$\Gamma = \sum_{k=0}^{\infty} \frac{\mathbf{F}^k T^k}{(k+1)!} T\mathbf{G}$$
$$= \boldsymbol{\Psi} T \mathbf{G}$$

$\boldsymbol{\Psi}$ itself can be evaluated with the series:

$$\boldsymbol{\Psi} \cong \mathbf{I} + \frac{\mathbf{F}T}{2}\left\{\mathbf{I} + \frac{\mathbf{F}T}{3}\left[\mathbf{I} + \cdots \frac{\mathbf{F}T}{n-1}\left(\mathbf{I} + \frac{\mathbf{F}T}{n}\right)\right]\right\}$$

State-space z-transform

We can apply the z-transform to our system:

$$(z\mathbf{I} - \boldsymbol{\Phi})\boldsymbol{X}(z) = \boldsymbol{\Gamma} U(k)$$
$$Y(z) = \mathbf{H}\boldsymbol{X}(z)$$

which yields the transfer function:

$$\frac{Y(z)}{\boldsymbol{X}(z)} = G(z) = \mathbf{H}(z\mathbf{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{\Gamma}$$

∴ State-space Control Design

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:

$$u = -\mathbf{K}\boldsymbol{x}$$

such that $\det(z\mathbf{I} - \boldsymbol{\Phi} + \boldsymbol{\Gamma}\mathbf{K}) = \alpha_c(z)$

where $\alpha_c(z)$ is the desired control characteristic equation

Predictably, this requires the system controllability matrix $\boldsymbol{\mathcal{C}} = [\boldsymbol{\Gamma} \quad \boldsymbol{\Phi\Gamma} \quad \boldsymbol{\Phi}^2\boldsymbol{\Gamma} \quad \cdots \quad \boldsymbol{\Phi}^{n-1}\boldsymbol{\Gamma}]$ to be full-rank.

Next Time…

- **Digital Control via Emulation!**

- Review:
  - Chapter 5 of FPW

- Deeper Pondering??