http://elec3004.com

Introduction to State-Space

ELEC 3004: **Systems**: Signals & Controls

Dr. Surya Singh

Lecture 18

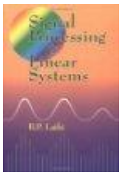
elec3004@itee.uq.edu.au
http://robotics.itee.uq.edu.au/~elec3004/

May 11, 2017

Lecture Schedule:

Week	Date	Lecture Title
1	28-Feb	Introduction
	2-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	9-Mar	Systems: Linear Differential Systems
3	14-Mar	Sampling Theory & Data Acquisition
	16-Mar	Aliasing & Antialiasing
4	21-Mar	Discrete Time Analysis & Z-Transform
	23-Mar	Second Order LTID (& Convolution Review)
5	28-Mar	Frequency Response
	30-Mar	Filter Analysis
6	4-Apr	Digital Filters (IIR) & Filter Analysis
	6-Apr	Digital Filter (FIR)
7	11-Apr	Digital Windows
	13-Apr	FFT
	18-Apr	Holiday
	20-Apr	
	25-Apr	
8	27-Apr	Active Filters & Estimation
9	2-May	Introduction to Feedback Control
	4-May	Servoregulation/PID
10	9-May	PID & State-Space
	11-May	State-Space Control
11	16-May	Digital Control Design
	18-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response
	25-May	Applications in Industry
13	30-May	System Identification & Information Theory
	1-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

G. Franklin,
J. Powell,
M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990
[Available as UQ Ebook]

Today

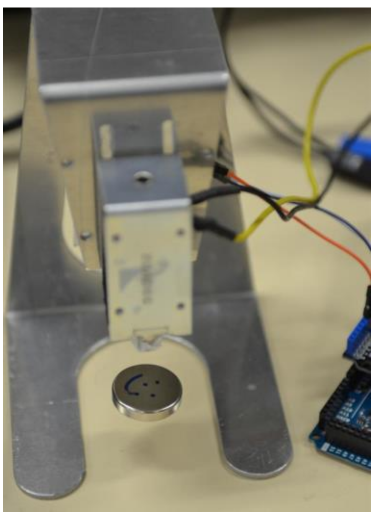
➔ **State-space** ➔ **[A stately idea! ☺]**

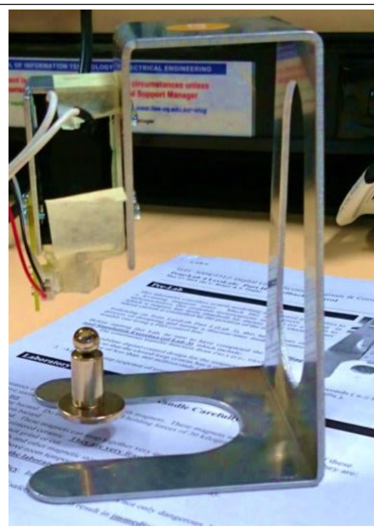
- FPW
  - Chapter 4: Discrete Equivalents to Continuous
  - Transfer Functions: The Digital Filter

- Lathi Ch. 13
  - § 13.2 Systematic Procedure for Determining State Equations
  - § 13.3 Solution of State Equations

Next Time

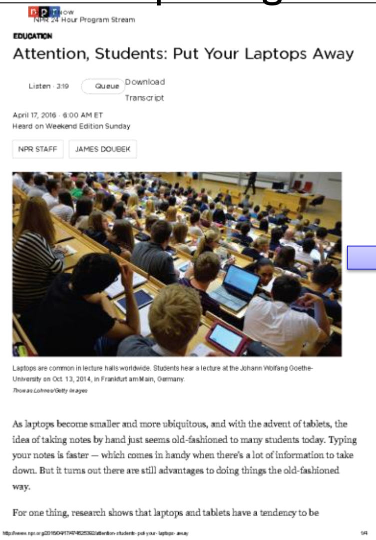
NEXT WEEK: Lab 4 – LeviLab II:





- AKA "Revenge of the TUNING!"

Final Exam Tip: Longhand Notes > Typed


EDUCATION

Attention, Students: Put Your Laptops Away

Listen · 3:19 Queue Download Transcript

April 17, 2016 · 6:00 AM ET
Heard on Weekend Edition Sunday

NPR STAFF JAMES DOUBEK

Laptops are common in lecture halls worldwide. Students hear a lecture at the Johann Wolfgang Goethe-University on Oct. 13, 2014, in Frankfurt am Main, Germany.
Thomas Lohnes/Getty Images

As laptops become smaller and more ubiquitous, and with the advent of tablets, the idea of taking notes by hand just seems old-fashioned to many students today. Typing your notes is faster — which comes in handy when there's a lot of information to take down. But it turns out there are still advantages to doing things the old-fashioned way.

For one thing, research shows that laptops and tablets have a tendency to be


Psychological Science OnlineFirst, published on May 22, 2014 as doi:10.1177/0956797614524581

Research Article

The Pen Is Mightier Than the Keyboard: Advantages of Longhand Over Laptop Note Taking

Pam A. Mueller[1] and Daniel M. Oppenheimer[2]
[1]Princeton University and [2]University of California, Los Angeles

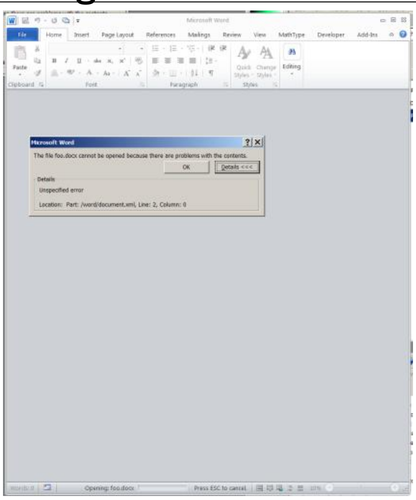
Abstract
Taking notes on laptops rather than in longhand is increasingly common. Many researchers have suggested that laptop note taking is less effective than longhand note taking for learning. Prior studies have primarily focused on students' capacity for multitasking and distraction when using laptops. The present research suggests that even when laptops are used solely to take notes, they may still be impairing learning because their use results in shallower processing. In three studies, we found that students who took notes on laptops performed worse on conceptual questions than students who took notes longhand. We show that whereas taking more notes can be beneficial, laptop note takers' tendency to transcribe lectures verbatim rather than processing information and reframing it in their own words is detrimental to learning.

Keywords
academic achievement, cognitive processes, memory, educational psychology, open data, open materials

Received 5/11/13; Revision accepted 1/16/14

- http://www.npr.org/2016/04/17/474525392/attention-students-put-your-laptops-away
- doi: 10.1177/0956797614524581

Friendly computing tale…



- Please save (as) often ☺
- Use Platypus$_2$ (Cloud-based ☺)

PID Recap

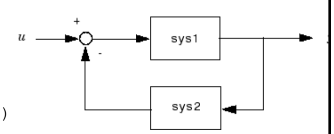
Effects of Increasing Gain

Effects of *increasing* a parameter independently

Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability[11]
K_p	Decrease	Increase	Small change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Minor change	Decrease	Decrease	No effect in theory	Improve if K_d small

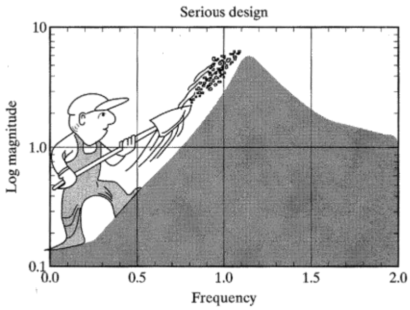
Matlab helps with PID tuning:

```
G_system=[???]; H=[1];
D_compensator = pidtune(G_system, 'PIDF')
CL_system = feedback(series(D_compensator,G_system), H)
step(CL_system)
```



Seeing PID – No Free Lunch

- The energy (and sensitivity) moves around (in this case in “frequency”)



- Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.

Source: Gunter Stein's interpretation of the water bed effect – G. Stein, *IEEE Control Systems Magazine*, 2003.

When Can PID Control Be Used?

When:

- “Industrial processes” such that the demands on the performance of the control are not too high.
  - Control authority/actuation
  - Fast (clean) sensing
- PI: Most common
  - All stable processes can be controlled by a PI law (modest performance)
  - First order dynamics

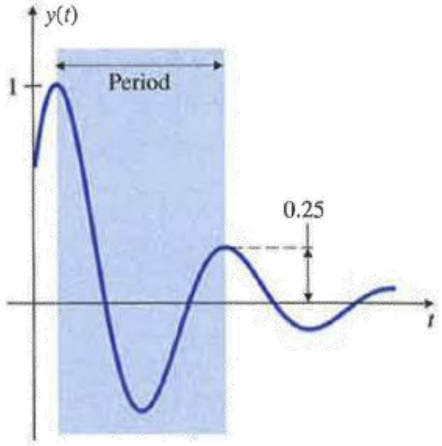
PID (PI + Derivative):

- Second order (A double integrator cannot be controlled by PI)
- Speed up response When time constants differ in magnitude (Thermal Systems)

Something More Sophisticated:

- Large time delays
- Oscillatory modes between inertia and compliances

Quarter decay ratio



Ziegler-Nichols Tuning – Reaction Rate

FPW § 5.8.5 [p.224]

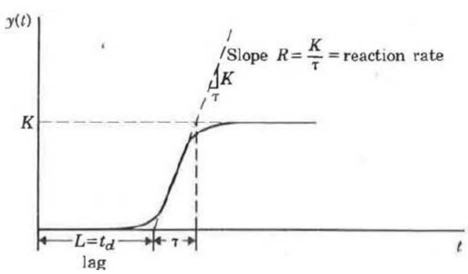


Table 5.2 Ziegler-Nichols tuning parameters using transient response.

	K_p	T_I	T_D
P	$1/RL$		
PI	$0.9/RL$	$3L$	
PID	$1.2/RL$	$2L$	$0.5L$

Ziegler-Nichols Tuning – Stability Limit Method

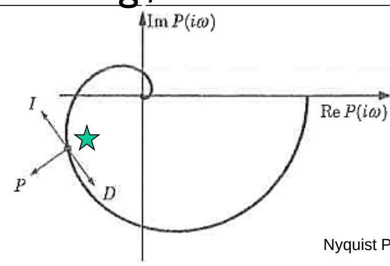
FPW § 5.8.5 [p.226]

- Increase K_P until the system has continuous oscillations
  - ≡ K_U : Oscillation Gain for "Ultimate stability"
  - ≡ P_U : Oscillation Period for "Ultimate stability"

Table 5.3 Ziegler-Nichols tuning parameters using stability limit.

	K_p	T_I	T_D
P	$0.5K_u$		
PI	$0.45K_u$	$P_u/1.2$	
PID	$0.6K_u$	$P_u/2$	$P_u/8$

Ziegler-Nichols Tuning / Intuition



Nyquist Plot

$$C(i\omega_u) = K\left(1 + i\left(\omega_u T_d - \frac{1}{\omega_u T_i}\right)\right) \approx 0.6K_u(1 + 0.467i)$$

- For a Given Point (★), the effect of increasing P,I and D in the "s-plane" are shown by the arrows above Nyquist plot

State-Space

Or more aptly…

Welcome to

State-Space!

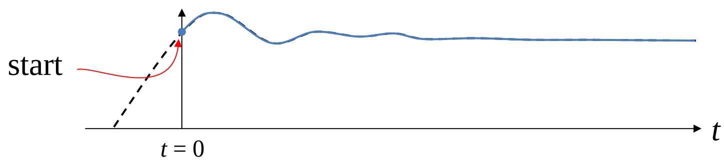
(It be stated -- Hallelujah !)

- More general mathematical model
  - MIMO, time-varying, nonlinear
- Matrix notation (think LAPACK → MATLAB)
- Good for discrete systems
- More design tools!

Affairs of state

- Introductory brain-teaser:
  - If you have a dynamic system model with history (ie. integration) how do you represent the instantaneous state of the plant?

Eg. how would you setup a simulation of a step response, mid-step?



State-Space Control

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}$$

(That can not be all of it? There has to be more to it than this…)

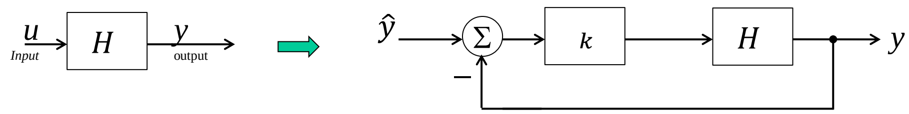
State-Space Control

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}$$

Benefits:

- Characterises the process by systems of coupled, first-order differential equations
- More general mathematical model
  - MIMO, time-varying, nonlinear
- Mathematically esoteric (who needs practical solutions)
- Yet, well suited for digital computer implementation
  - That is: based on vectors/matrices (think LAPACK ➔ MATLAB)

Difference Equations & Feedback



- Start with the Open-Loop:

$$y = kHu$$

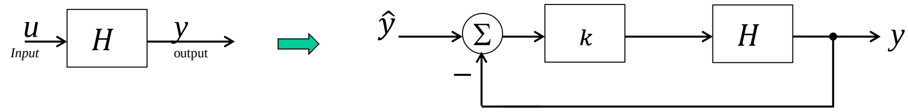
- Close the loop:

$$u = ke = k(\hat{y} - y) \rightarrow y = H[k(\hat{y} - y)]$$

$$\rightarrow y = \frac{Hk}{1+Hk}\hat{y}$$

- All easy! (yesa!)

Difference Equations & Feedback



- Now add delay (image the plant is a replica with a delay τ)

$$y(t) = u(t-\tau)$$

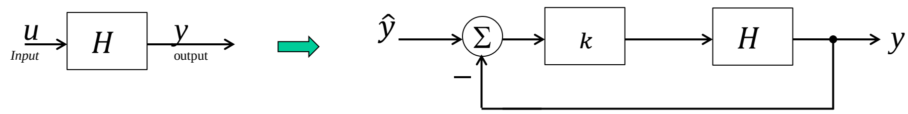
- Close the loop:

$$u(t-\tau) = ke(t-\tau) = k\,[\hat{y}(t-\tau) - \mathrm{y}(t-\tau)]$$

$$\Rightarrow \mathrm{y(t)} = k\,[\hat{y}(t-\tau) - \mathrm{y}(t-\tau)]$$

- Notice we have a difference equation!

Difference Equations & Feedback



- What happens with a single delay and a unit step?

$$u(t) = k \text{ for } 0<t<\tau$$

$$y(t) = u(t-\tau) \text{ for } \tau<t<2\tau$$

- Then with feedback we get:

$$u(t) = k(1-k) = k - k^2$$

$$y(t) = k - k^2 + k^3 + \cdots + (-1)^{n-1}k^{n-1}$$

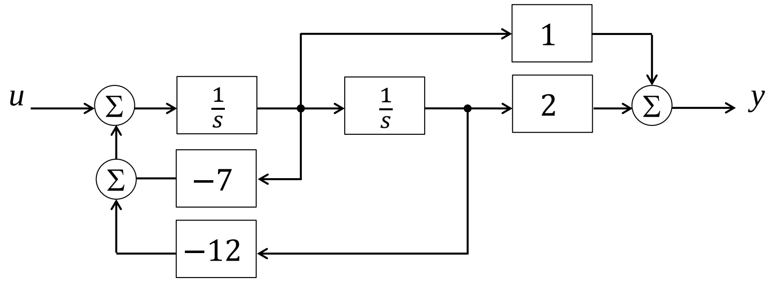
- If k<1: then:

$$\Rightarrow \lim \mathrm{y(t)} = \frac{k}{1+k}$$

Introduction to state-space

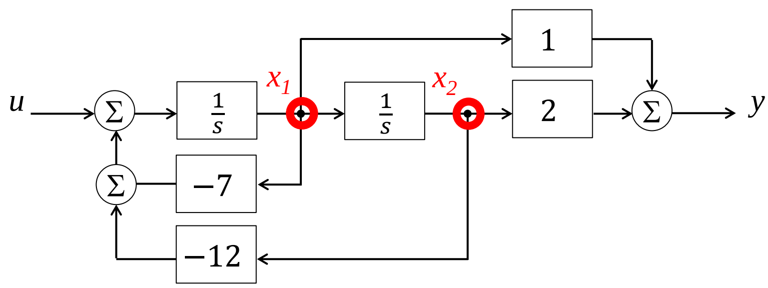
- Linear systems can be written as networks of simple dynamic elements:

$$H = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$$



Introduction to state-space

- We can identify the nodes in the system
  - These nodes contain the integrated time-history values of the system response
  - We call them "states"



Linear system equations

- We can represent the dynamic relationship between the states with a linear system:

$$
\begin{aligned}
\dot{x_1} &= -7x_1 - 12x_2 + u \\
\dot{x_2} &= x_1 + 0x_2 + 0u \\
y &= x_1 + 2x_2 + 0u
\end{aligned}
$$

State-space representation

- We can write linear systems in matrix form:

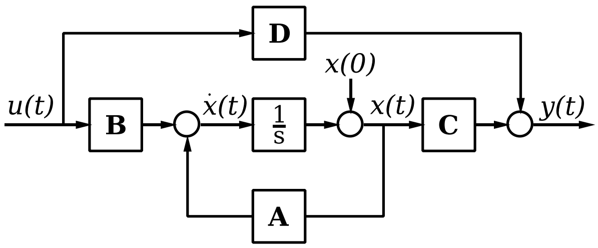
$$
\begin{aligned}
\dot{\boldsymbol{x}} &= \begin{bmatrix} -7 & 12 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\
\boldsymbol{y} &= \begin{bmatrix} 1 & 2 \end{bmatrix} \boldsymbol{x} + 0u
\end{aligned}
$$

Or, more generally:

$$
\left.
\begin{aligned}
\dot{\boldsymbol{x}} &= \mathbf{A}\boldsymbol{x} + \mathbf{B}u \\
y &= \mathbf{C}\boldsymbol{x} + Du
\end{aligned}
\right\}
$$

"State-space equations"

State-Space Terminology



$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

LTI State-Space

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- **If** the system is **linear and time invariant**, **then** A,B,C,D are constant coefficient

$$\rightarrow \dot{x} = Ax + Bu$$
$$\rightarrow y = Cx + Du$$

Discrete Time State-Space

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

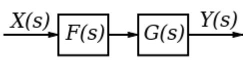
- **If** the system is **discrete**,
  then x and u are given by difference equations

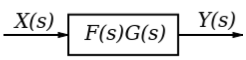
$$\rightarrow x[k+1] = A[k]\,x[k] + B[k]\,u[k]$$
$$y[k] = C[k]\,x[k] + D[k]\,u[k]$$

$$\rightarrow x^{+} = Ax + Bu$$
$$y = Cx + Du$$

Block Diagram Algebra in State Space

- Series:





System 1:

$$x'_F = A_F x_F + B_F u$$
$$y_F = C_F x_F + D_F u$$

System 2:

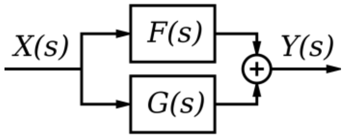
$$x'_G = A_G x_G + B_G y_F$$
$$y_G = C_G x_G + D_G y_F$$

$$\begin{bmatrix} x'_G \\ x'_F \end{bmatrix} = \begin{bmatrix} A_G & B_G C_F \\ 0 & A_F \end{bmatrix} \begin{bmatrix} x_G \\ x_F \end{bmatrix} + \begin{bmatrix} B_G D_F \\ B_F \end{bmatrix} u$$

$$\begin{bmatrix} y_G \\ y_F \end{bmatrix} = \begin{bmatrix} C_G & D_G C_F \\ 0 & C_F \end{bmatrix} \begin{bmatrix} x_G \\ x_F \end{bmatrix} + \begin{bmatrix} D_G D_F \\ D_F \end{bmatrix} u$$

Block Diagram Algebra in State Space

- Parallel:



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_1 + D_2) u$$

State-space representation

- State-space matrices are not necessarily a unique representation of a system
  - There are two common forms
- <u>Control canonical form</u>
  - Each node – each entry in ***x*** – represents a state of the system (each order of *s* maps to a state)
- <u>Modal form</u>
  - Diagonals of the state matrix **A** are the poles ("modes") of the transfer function

Why is this "Kind of awesome"?

- The controllability of a system depends on the particular set of states you chose
- You can't tell just from a transfer function whether all the states of $\boldsymbol{x}$ are controllable
- The poles of the system are the Eigenvalues of $\mathbf{F}$, (p_i).

State evolution

- Consider the system matrix relation:

$$\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$$
$$y = \mathbf{H}\boldsymbol{x} + Ju$$

The time solution of this system is:

$$\boldsymbol{x}(t) = e^{\mathbf{F}(t-t_0)}\,\boldsymbol{x}(t_0) + \int_{t_0}^{t} = e^{\mathbf{F}(t-\tau)}\,\mathbf{G}u(\tau)d\tau$$

If you didn't know, the matrix exponential is:

$$e^{\mathbf{K}t} = \mathbf{I} + \mathbf{K}t + \frac{1}{2!}\mathbf{K}^2t^2 + \frac{1}{3!}\mathbf{K}^3t^3 + \cdots$$

Break ☺

Solving State Space

Great, so how about control?

- Given $\dot{\boldsymbol{x}} = \mathbf{F}\boldsymbol{x} + \mathbf{G}u$, if we know $\mathbf{F}$ and $\mathbf{G}$, we can design a controller $u = -\mathbf{K}\boldsymbol{x}$ such that

$$\mathrm{eig}(\mathbf{F} - \mathbf{GK}) < 0$$

- In fact, if we have full measurement and control of the states of $\boldsymbol{x}$, we can position the poles of the system in arbitrary locations!

(Of course, that never happens in reality.)

Solving State Space…

- Recall:

$$\dot{x} = f(x, u, t)$$

- For Linear Systems:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$
$$y(t) = C(t)x(t) + D(t)u(t)$$

- For LTI:

$$\rightarrow \dot{x} = Ax + Bu$$
$$\rightarrow y = Cx + Du$$

➔ Solutions to State Equations

$$\dot{x} = Ax + Bu$$
$$sX(s) - x(0) = AX(s) + BU(s)$$
$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

$$X(s) = \mathcal{L}[e^{At}]x(0) + \mathcal{L}[e^{At}]BU(s)$$

$$x(t) = \int_0^t e^{At}\,Bu(\tau)d\tau$$

$$\Rightarrow\ e^{At}$$

➔ State-Transition Matrix Φ

- $\Phi(t) = e^{At}\ = \mathcal{L}^{-1}[(sI - A)^{-1}]$
- It contains all the information about the free motions of the system described by $\dot{x} = Ax$

<u>LTI</u> Properties:

- $\Phi(0) = e^{0t} = I$
- $\Phi^{-1}(t) = \Phi(-t)$
- $\Phi(t_1 + t_2) = \Phi(t_1)\Phi(t_2) = \Phi(t_2)\Phi(t_1)$
- $[\Phi(t)]^n = \Phi(nt)$

➔ The closed-loop poles are the eignvalues of the system matrix

Digital State Space:

- Difference equations in state-space form:

$$x[n+1] = Ax[n] + Bu[n]$$
$$y[n] = Cx[n] + Du[n]$$

- Where:
  - u[n], y[n]: input & output (scalars)
  - x[n]: state vector

Digital Control Law Design

In Chapter 2, we saw that the state-space description of a continuous system is given by (2.43),

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u, \qquad (6.1)$$

and (2.44),

$$y = \mathbf{H}\mathbf{x}. \qquad (6.2)$$

We assume the control is applied from the computer by a ZOH as shown in Fig. 1.1. Therefore, (6.1) and (6.2) have an exact discrete representation as given by (2.57),

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k),$$
$$y(k) = \mathbf{H}\mathbf{x}(k), \qquad (6.3)$$

where

$$\mathbf{\Phi} = e^{\mathbf{F}T}, \qquad (6.4a)$$
$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G}, \qquad (6.4b)$$

A Systematic Procedure for Determining State Eqs.

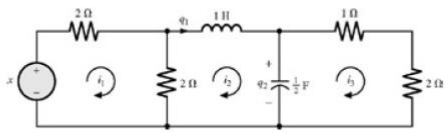
1. Choose all independent capacitor voltages and inductor currents to be the state variables.

2. Choose a set of loop currents; express the state variables and their first derivatives in terms of these loop currents.

3. Write the loop equations and eliminate all variables other than state variables (and their first derivatives) from the equations derived in Steps 2 and 3.

See also: Lathi § 13.2-1 (p. 788)

A Quick Example



1. The inductor current q_1 and the capacitor voltage $q2$ as the state variables.

2. $q_1 = i_2$
   $\frac{1}{2}\dot{q}_2 = i_2 - i_3$

3. $4i_1 - 2i_2 = x$
   $2(i_2 - i_1) + \dot{q}_1 + q_2 = 0$
   $-q_2 + 3i_3 = 0$

$$\dot{q}_1 = 2(i_1 - i_2) - q_2$$

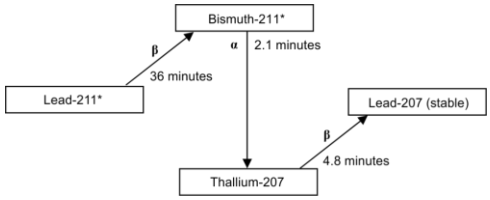
$$\dot{q}_1 = -q_1 - q_2 + \frac{1}{2}x$$

$$\dot{q}_2 = 2q_1 - \frac{2}{3}q_2$$

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} x$$

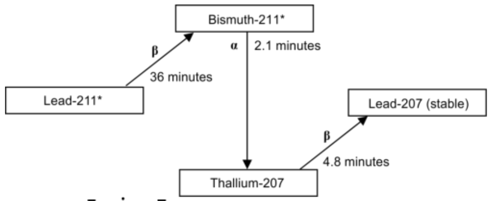
See also: Fig. 13.2, Lathi p. 789

Another Example



- $\frac{dN1(t)}{dt} = -\lambda_1 N1(t)$
- $\frac{dN2(t)}{dt} = -\lambda_2 N2(t) + \lambda_1 N1(t)$
- $\frac{dN3(t)}{dt} = -\lambda_3 N3(t) + \lambda_2 N2(t)$
- $\frac{dN4(t)}{dt} = \lambda_3 N3(t)$

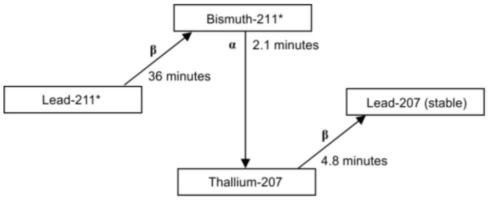
Another Example



$$X = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \rightarrow \dot{X} = \begin{bmatrix} \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}_3 \\ \dot{N}_4 \end{bmatrix}$$

$$\dot{X} = FX \rightarrow \begin{bmatrix} \dot{N}_1 \\ \dot{N}_2 \\ \dot{N}_3 \\ \dot{N}_4 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 & 0 \\ 0 & \lambda_2 & -\lambda_3 & 0 \\ 0 & 0 & \lambda_3 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$$

Another Example



- $N_1(t)=N_1(0)\exp(-\lambda_1 t)$
- $N2(t) = N2(0)exp(-\lambda_2 t) - N1(0)\frac{\lambda_1}{\lambda_2-\lambda_1}(exp(-\lambda_2 t) - exp(-\lambda_1 t))$
- $N3(t) = \lambda_1\lambda_2 N1(0)\left[\frac{exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)} + \frac{exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)} + \frac{exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right]$
- $N4(t) = \lambda_1\lambda_2\lambda_3 N1(0)\left[\frac{exp(-\lambda_1 t)}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(-\lambda_1)} + \frac{exp(-\lambda_2 t)}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(-\lambda_2)} + \frac{exp(-\lambda_3 t)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(-\lambda_3)} + \frac{1}{(\lambda_1\lambda_2\lambda_3)}\right]$

Discretisation (FPW!)

- We can use the time-domain representation to produce difference equations!

$$\boldsymbol{x}(kT+T) = e^{\mathbf{F}T}\,\boldsymbol{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

Notice $\boldsymbol{u}(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series.

We can structure this by using the form:

$$u(\tau) = u(kT), \qquad kT \le \tau \le kT+T$$

State-space z-transform

We can apply the z-transform to our system:

$$(z\mathbf{I} - \boldsymbol{\Phi})\boldsymbol{X}(z) = \boldsymbol{\Gamma} U(k)$$
$$Y(z) = \mathbf{H}\boldsymbol{X}(z)$$

which yields the transfer function:

$$\frac{Y(z)}{\boldsymbol{X}(z)} = G(z) = \mathbf{H}(z\mathbf{I} - \boldsymbol{\Phi})^{-\mathbf{1}}\boldsymbol{\Gamma}$$

State-space control design ¿¿¿Que pasa????

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:

$$u = -\mathbf{K}\boldsymbol{x}$$

such that $\det(z\mathbf{I} - \boldsymbol{\Phi} + \boldsymbol{\Gamma}\mathbf{K}) = \alpha_c(z)$

where $\alpha_c(z)$ is the desired control characteristic equation

Predictably, this requires the system controllability matrix

$\boldsymbol{\mathcal{C}} = [\boldsymbol{\Gamma} \quad \boldsymbol{\Phi\Gamma} \quad \boldsymbol{\Phi}^2\boldsymbol{\Gamma} \quad \cdots \quad \boldsymbol{\Phi}^{n-1}\boldsymbol{\Gamma}]$ to be full-rank.

Example: PID control

- Consider a system parameterised by three states:
  - x_1, x_2, x_3
  - where $x_2 = \dot{x}_1$ and $x_3 = \dot{x}_2$

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix} \boldsymbol{x} - \mathbf{K}u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \boldsymbol{x} + 0u$$

x_2 is the output state of the system;
x_1 is the value of the integral;
x_3 is the velocity.

Example: PID control [2]

- We can choose **K** to move the eigenvalues of the system as desired:

$$\det \begin{bmatrix} 1 - K_1 & & \\ & 1 - K_2 & \\ & & -2 - K_3 \end{bmatrix} = \mathbf{0}$$

All of these eigenvalues must be positive.

It's straightforward to see how adding derivative gain K_3 can stabilise the system.

Can you use this for more than Control?

YES!

Frequency Response in State Space

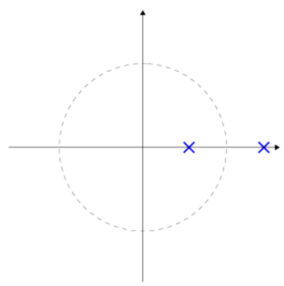
$$H(z) = C(zI - A)^{-1}B + D = \frac{1}{100z^2 - 200z + 80}$$

Poles at $\approx 0.55, 1.45$.

Eigenvalues of A:

$1, 1, 1.45, .55$

What are the (physical) implications?

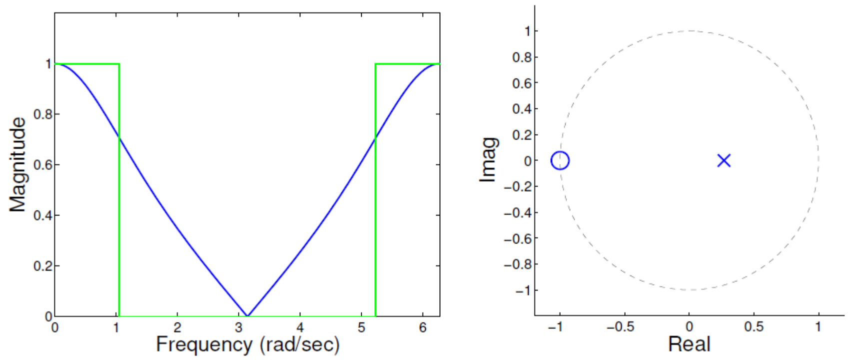


The Approach:

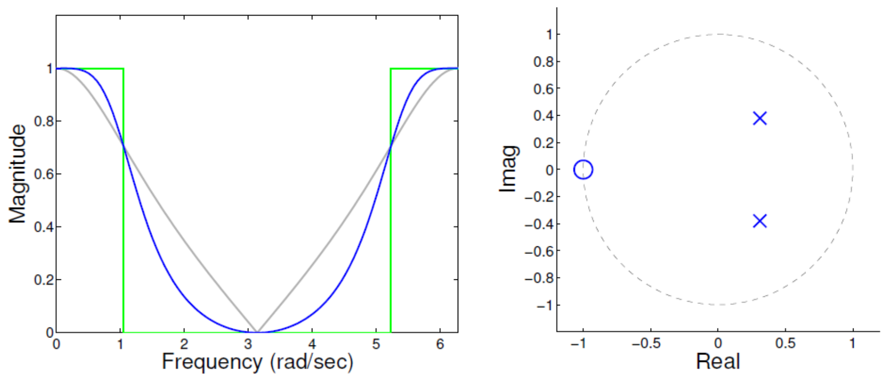
- Formulate the goal of control as an **optimization** (e.g. minimal impulse response, minimal effort, ...).
- You've already seen some examples of optimization-based design:
  - Used least-squares to obtain an FIR system which matched (in the least-squares sense) the desired frequency response.
  - Poles/zeros lecture: Butterworth filter

Discrete Time Butterworth Filters

"Maximally-flat filter". Sacrifice sharpness to have flat response in pass band and stop band.



"Maximally-flat filter". Sacrifice sharpness to have flat response in pass band and stop band.



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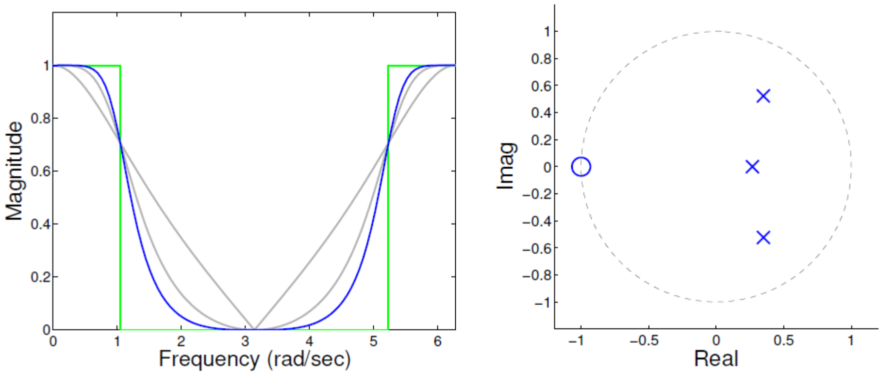

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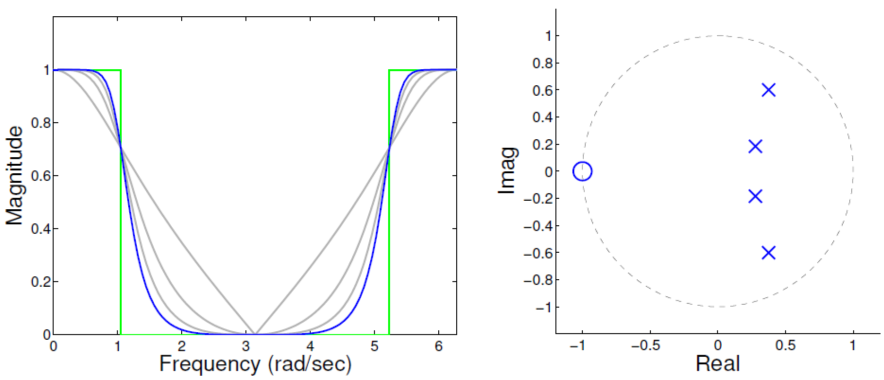

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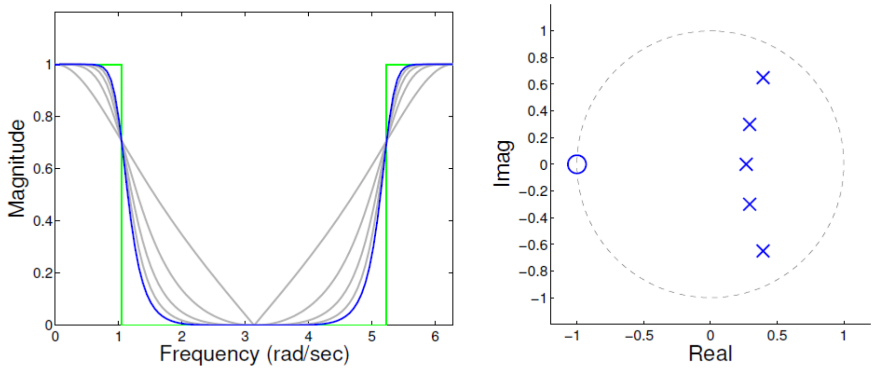

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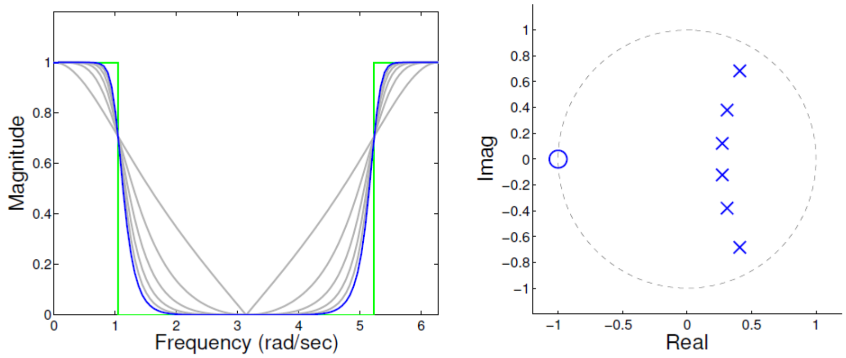

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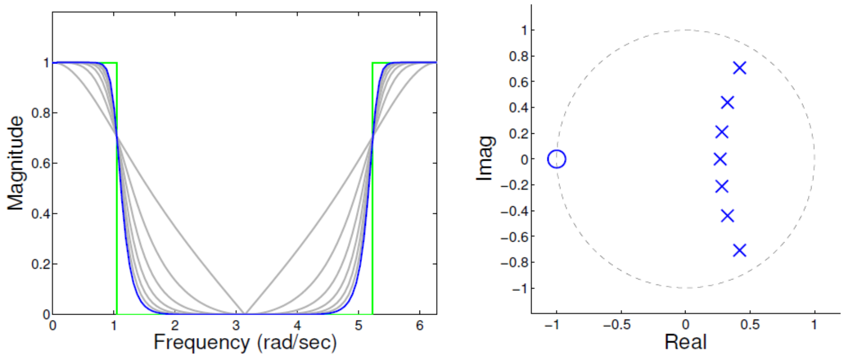

How?

- Constrained Least-Squares …

One formulation: Given $x[0]$

$$\underset{u[0],u[1],\ldots,u[N]}{\text{minimize}} \quad ||\vec{u}||^2, \quad \text{where } \vec{u} = \begin{bmatrix} u[0] \\ u[1] \\ \vdots \\ u[N] \end{bmatrix}$$

$$\text{subject to} \quad x[N] = 0.$$

Note that

$$x[n] = A^n x[0] + \sum_{k=0}^{n-1} A^{(n-1-k)} B u[k],$$

so this problem can be written as

$$\underset{x_{ls}}{\text{minimize}} \, ||A_{ls}x_{ls} - b_{ls}||^2 \quad \text{subject to} \quad C_{ls}x_{ls} = D_{ls}.$$

Extension!

Solving State Space (Extended Version)

Solving State Space (Extended Version)…

- Recall:

$$\dot{x} = f\left(x, u, t\right)$$

- For Linear Systems:

$$\dot{x}\left(t\right) = A\left(t\right)x\left(t\right) + B\left(t\right)u\left(t\right)$$

$$y\left(t\right) = C\left(t\right)x\left(t\right) + D\left(t\right)u\left(t\right)$$

- For LTI:

$$\rightarrow \dot{x} = Ax + Bu$$

$$\rightarrow y = Cx + Du$$

Solving State Space

- In the conventional, frequency-domain approach the differential equations are converted to transfer functions as soon as possible
  - The dynamics of a system comprising several subsystems is obtained by combining the transfer functions!

- With the state-space methods, on the other hand, the description of the system dynamics in the form of differential equations is retained throughout the analysis and design.

State-transition matrix Φ(t)

- Describes how the state *x(t)* of the system at some time *t* evolves into (or from) the state x(τ*)* at some other time *T*.

$$x(t) = \Phi\left(t, \tau\right) x\left(\tau\right)$$

Solving State Space…

Time-invariant dynamics The simplest form of the general differential equation of the form (3.1) is the "homogeneous," i.e., unforced equation

$$\dot{x} = Ax \tag{3.2}$$

where A is a constant k by k matrix. The solution to (3.2) can be expressed as

$$x(t) = e^{At}c \tag{3.3}$$

where e^{At} is the matrix exponential function

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + \cdots \tag{3.4}$$

and c is a suitably chosen constant vector. To verify (3.3) calculate the derivative of $x(t)$

$$\frac{dx(t)}{dt} = \frac{d}{dt}(e^{At})c \tag{3.5}$$

and, from the defining series (3.4),

$$\frac{d}{dt}(e^{At}) = A + A^2t + A^3\frac{t^2}{2!} + \cdots = A\left(I + At + A^2\frac{t^2}{2!} + \cdots\right) = A\,e^{At}$$

Thus (3.5) becomes

$$\frac{dx(t)}{dt} = Ae^{At}c = Ax(t)$$

Solving State Space

which was to be shown. To evaluate the constant c suppose that at some time τ the state $x(\tau)$ is given. Then, from (3.3),

$$x(\tau) = e^{A\tau}c \tag{3.6}$$

Multiplying both sides of (3.6) by the inverse of $e^{A\tau}$ we find that

$$c = (e^{A\tau})^{-1}x(\tau)$$

Thus the general solution to (3.2) for the state $x(t)$ at time t, given the state $x(\tau)$ at time τ, is

$$x(t) = e^{At}(e^{A\tau})^{-1}x(\tau) \tag{3.7}$$

The following property of the matrix exponential can readily be established by a variety of methods—the easiest perhaps being the use of the series definition (3.4)—

$$e^{A(t_1+t_2)} = e^{At_1}e^{At_2} \tag{3.8}$$

for any t_1 and t_2. From this property it follows that

$$(e^{A\tau})^{-1} = e^{-A\tau} \tag{3.9}$$

and hence that (3.7) can be written

$$x(t) = e^{A(t-\tau)}x(\tau) \tag{3.10}$$

Solving State Space

The matrix $e^{A(t-\tau)}$ is a special form of the *state-transition matrix* to be discussed subsequently.

We now turn to the problem of finding a "particular" solution to the nonhomogeneous, or "forced," differential equation (3.1) with A and B being constant matrices. Using the "method of the variation of the constant,"[1] we seek a solution to (3.1) of the form

$$x(t) = e^{At}c(t) \tag{3.11}$$

where $c(t)$ is a function of time to be determined. Take the time derivative of $x(t)$ given by (3.11) and substitute it into (3.1) to obtain:

$$Ae^{At}c(t) + e^{At}\dot{c}(t) = Ae^{At}c(t) + Bu(t)$$

or, upon cancelling the terms $A e^{At}c(t)$ and premultiplying the remainder by e^{-At},

$$\dot{c}(t) = e^{-At}Bu(t) \tag{3.12}$$

Thus the desired function $c(t)$ can be obtained by simple integration (the mathematician would say "by a quadrature")

$$c(t) = \int_T^t e^{-A\lambda}Bu(\lambda)\,d\lambda$$

The lower limit T on this integral cannot as yet be specified, because we will need to put the particular solution together with the solution to the

Solving State Space

homogeneous equation to obtain the complete (general) solution. For the present, let T be undefined. Then the particular solution, by (3.11), is

$$x(t) = e^{At}\int_T^t e^{-A\lambda}Bu(\lambda)\,d\lambda = \int_T^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.13}$$

In obtaining the second integral in (3.13), the exponential e^{At}, which does not depend on the variable of integration λ, was moved under the integral, and property (3.8) was invoked to write $e^{At}e^{-A\lambda} = e^{A(t-\lambda)}$.

The complete solution to (3.1) is obtained by adding the "complementary solution" (3.10) to the particular solution (3.13). The result is

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_T^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.14}$$

We can now determine the proper value for lower limit T on the integral. At $t = \tau$ (3.14) becomes

$$x(\tau) = x(\tau) + \int_T^\tau e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.15}$$

Thus, the integral in (3.15) must be zero for any $u(t)$, and this is possible only if $T = \tau$. Thus, finally we have the complete solution to (3.1) when A and B are constant matrices

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.16}$$

Solving State Space

This important relation will be used many times in the remainder of the book. It is worthwhile dwelling upon it. We note, first of all, that the solution is the sum of two terms: the first is due to the "initial" state $x(\tau)$ and the second—the integral—is due to the input $u(\tau)$ in the time interval $\tau \leqq \lambda \leqq t$ between the "initial" time τ and the "present" time t. The terms initial and present are enclosed in quotes to denote the fact that these are simply convenient definitions. There is no requirement that $t \geqq \tau$. The relationship is perfectly valid even when $t \leqq \tau$.

Another fact worth noting is that the integral term, due to the input, is a "convolution integral": the contribution to the state $x(t)$ due to the input u is the convolution of u with $e^{At}B$. Thus the function $e^{At}B$ has the role of the impulse response[1] of the system whose output is $x(t)$ and whose input is $u(t)$.

If the output y of the system is not the state x itself but is defined by the observation equation

$$y = Cx$$

then this output is expressed by

$$y(t) = Ce^{A(t-\tau)}x(t) + \int_{\tau}^{t} Ce^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.17}$$

Solving State Space

and the impulse response of the system with y regarded as the output is $Ce^{A(t-\lambda)}B$.

The development leading to (3.16) and (3.17) did not really require that B and C be constant matrices. By retracing the steps in the development it is readily seen that when B and C are time-varying, (3.16) and (3.17) generalize to

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} e^{A(t-\lambda)}B(\lambda)u(\lambda)\,d\lambda \tag{3.18}$$

and

$$y(t) = C(t)\,e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} C(t)\,e^{A(t-\lambda)}B(\lambda)u(\lambda)\,d\lambda \tag{3.19}$$

Next Time…

- **Digital Feedback Control**

- Review:
  - Chapter 2 of FPW

- More Pondering??