ELEC 3004/7312: Digital Linear Systems: Signals & Control!

Tutorial 3 (Week 6): The Discrete Time Fourier Transform

By: Tim Sherry

Review: A bit of linear algebra and a discussion on the basis

A basis is any set of linearly independent vectors. It is useful because for a basis of R^n space, any n-dimensional vector can be exactly reproduced by a linear combination of the basis.

Are the following vector sets a basis?

$$\begin{Bmatrix}1\\1\\0\end{Bmatrix},\begin{Bmatrix}1\\-1\\0\end{Bmatrix},\begin{Bmatrix}0\\0\\1\end{Bmatrix}$$

$$\begin{Bmatrix}0\\1\\0\end{Bmatrix},\begin{Bmatrix}1\\0\\0\end{Bmatrix},\begin{Bmatrix}0\\0\\1\end{Bmatrix}$$

$$\begin{Bmatrix}1\\0\\0\end{Bmatrix},\begin{Bmatrix}1\\0\\1\end{Bmatrix},\begin{Bmatrix}0\\0\\1\end{Bmatrix}$$

$$\begin{Bmatrix}3\\2\\1\end{Bmatrix},\begin{Bmatrix}0\\3\\0\end{Bmatrix},\begin{Bmatrix}0\\0\\4\end{Bmatrix}$$

Useful basis traits:

Orthogonal basis : every vector in the basis is orthogonal to every other vector. IE $< \bar{v}_i, \bar{v}_j > = 0 \ ; \forall \ i \neq j$

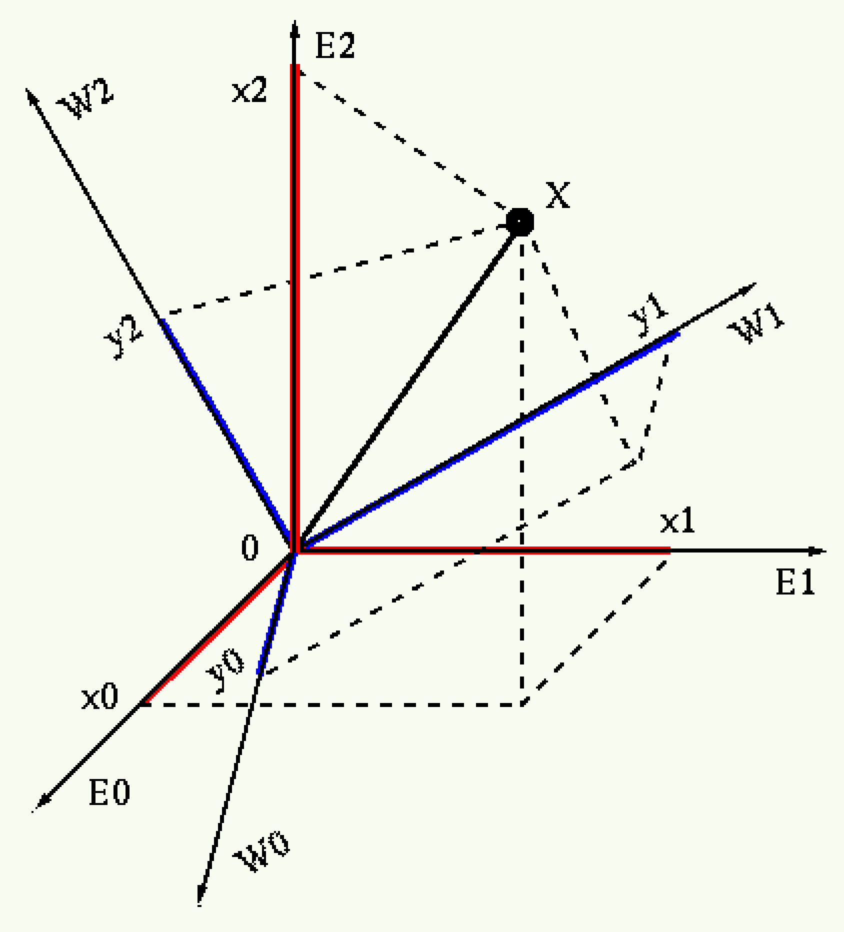
we like this property because the inverse of the matrix formed by an orthogonal basis is of the form $B^T B = D$ where D is a diagonal matrix. So $B^T/D = B^{-1}$

Orthonormal basis: an orthogonal basis where each vector is of unit length, IE $||\bar{v}|| = 1$. In this case $B^T B = I$, IE $B^T = B^{-1}$

➥ *For the bases given above, which are orthogonal? Which orthonormal?*

Why is an orthonormal basis useful?

For transformation between two orthonormal bases, length and angles are maintained.



From a signal processing perspective, length is equivalent to signal energy, so for an orthonormal basis, signal energy is maintained. This is the basis of Parsevals theorem, which will be covered soon in lectures.

Estimation of vectors.

Let's consider the first basis vector, which I will call $\bar{v}$ here, that we saw today :

$$\bar{v} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

We want to try and approximate another vector, $\bar{x}$, by linearly scaling $\bar{v}$ by some constant. Formally we aim to optimize the following

$$\min_{k \in R}(||\bar{x} - k\bar{v}||)$$

Solving for k is straight forward, we have seen it briefly in lecture 3. Refer to lecture 3 pages 15 and 16.

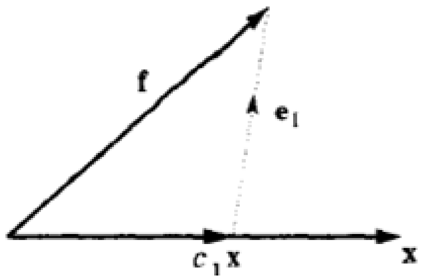
➥ *What is the vector,* $k\bar{v}$ *, that best approximates* $\bar{x} = \begin{matrix} 4 \\ 5 \\ 1 \end{matrix}$ *?*

➥ *What is the value of k?*

➥ *What is the mean squared error (note this is* $\frac{||\bar{x} - k\bar{v}||}{N}$*)?*

Vector Review (Lecture 3 pages 15 and 16):

∞ bases given $\vec{x}$



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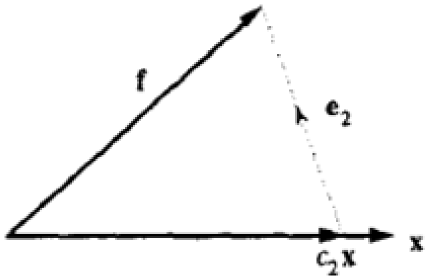

Which is the best one?

$$\mathbf{f} \simeq c\mathbf{x}$$

$$c|\mathbf{x}| = |\mathbf{f}| \cos \theta$$

$$c|\mathbf{x}|^2 = |\mathbf{f}||\mathbf{x}| \cos \theta = \mathbf{f} \cdot \mathbf{x}$$

$$c = \frac{\mathbf{f} \cdot \mathbf{x}}{\mathbf{x} \cdot \mathbf{x}} = \frac{1}{|\mathbf{x}|^2} \mathbf{f} \cdot \mathbf{x}$$

$$\mathbf{f} \cdot \mathbf{x} = 0$$

Can I allow more basis vectors than I have dimensions?

The key point to this is that for an orthogonal basis our new coefficients are found by solving this problem for each basis vector. If we expand the equation

$$\bar{k} = \frac{B^T}{D} \bar{x}$$

We find that $k = \frac{<\bar{x},\bar{v}>}{||\bar{v}||}$ will fall out, and that the elements of $D_{i,i}$ are the energy of each basis vector $||\bar{v}_i||$.

For the orthonormal case, the energy $||\bar{v}||$ = 1, and so the result simplifies to $k = < \bar{x}, \bar{v} >$. Fourier referred to this process as decomposing our vector $\bar{x}$ into a sum of linearly scaled basis vectors.

This is not the case for a basis that is not orthogonal. Because the basis vectors are non-orthogonal, the solutions for k_i are dependent on all the vectors in the set.

Clearly, there are an infinite number of choices for an orthonormal basis, and by extension for an arbitrary basis too. However there are choices of basis which help solve some important problems. Today, we will discuss the Fourier basis, which is useful for the case where our signals are periodic.

The Discrete Time Fourier Transform:

Fourier provided the underlying math for decomposing a function into a set of sine waves, or more commonly a set of complex exponential functions.

In the discrete fourier transform we decompose a vector signal into a set of sampled complex exponential functions.

What do these basis vectors look like?

In the continuous case we have $f(t) = \cos(\omega t) + j\,\sin(\omega t)$ which alternatively is equal to the complex exponential $f(t) = e^{-2\pi j\omega t}$. To understand the FFT, we have to look at the problem in terms of the complex exponential, so we will continue with this from here on in.

When we sample f(t), we form a the following vector.

$$\bar{f}[n] = \{\, f(0) \quad f(T) \quad f(2T) \quad \cdots \quad f\big((n-1)\,T\,\big) \}$$

➥ *Assume that one of our fourier basis functions is* $\mathrm{f(t)} = \mathrm{e}^{\mathrm{j}0.5\pi\mathrm{t}}$ *(IE* $\omega = \frac{1}{4}$*). Find the first 4 terms of the corresponding fourier basis vector* $\bar{\mathrm{f}}[\mathrm{n}]$.

➥ *Find the first 4 terms of the fourier basis vector* $\mathrm{f(t)} = \mathrm{e}^{\mathrm{j}0\mathrm{t}}$*, IE for* $\omega = 0$ *. What would we call this basis function?*

➥ *Are these vectors unit length (ie would a set of these vectors form an orthonormal basis)?*

Fourier Functions are orthogonal, but what about when they are sampled?

While we won't prove this today, Fourier functions of different frequencies are orthogonal functions, which as we showed earlier is very useful. However, when we sample them, we find that they aren't always orthogonal, we have to make sure that the vector covers a whole unit period.

➥ *Are the fourier basis vectors given above are orthogonal? (use the inner product)*

This is because for a bounded DFT, we have to assume a periodic extension. For that extension to not have a discontinuity, it has to have a whole number of periods captured.

Also we don't need to define our basis vectors in terms of ω, as the basis vector values will be invariant with changes in sampling period. Consider a sampling period of 0.01s, if we halve the samping period (doubling the frequency), we also halve the time required to capture N samples, and the basis vectors will remain the same (however the frequency they represent doubles)

so we find that we can describe the DFT basis functions for a vector of dimension N as follows

$$f_n(t) = e^{-2\pi jtn/N}\ for\ n = 0{:}\,N-1$$

A Discrete Basis

By now you have calculated the first two basis vectors for the 4 point DFT. The other two vectors, correspond to n = 2, 3. The following are the 4 basis functions:

$$\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}, \begin{Bmatrix}-i\\-1\\i\\1\end{Bmatrix}, \begin{Bmatrix}-1\\1\\-1\\1\end{Bmatrix}, \begin{Bmatrix}i\\-1\\-i\\1\end{Bmatrix}$$

$$\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}, \begin{Bmatrix}1\\-i\\-1\\i\end{Bmatrix}, \begin{Bmatrix}1\\-1\\1\\-1\end{Bmatrix}, \begin{Bmatrix}1\\i\\-1\\-i\end{Bmatrix}$$

➥ *Find the DFT of* $\bar{f}[n] = \{\,0 \quad 1 \quad 0 \quad -1\,\}$
That is, a sampled vector of $\sin(0.5\pi t)$ *by whatever means you feel like.*

➥ *You should find only two terms are significant, why is this?*

➥ *You should also find that the terms are complex, why is this? If we consider this complex number in polar form, what does it correspond to?*