http://elec3004.com

$\mathcal{Z}$-Transforms
(and Convolution Review)

ELEC 3004: **Systems**: Signals & Controls

Dr. Surya Singh

Lecture 8

(with material from Lathi and Cannon (Discrete Systems))

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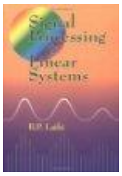
March 24, 2016

Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	24-Mar	z-Transform
	28-Mar	Holiday
	31-Mar	
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Feedback
9	3-May	Introduction to Feedback Control
	5-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	12-May	Digitial Control
11	16-May	Digital Control Design
	19-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

Today

- Chapter 11 (**Discrete-Time System Analysis Using the *z*-Transform**)
  - § 11.1 The $\mathcal{Z}$-Transform
  - § 11.2 Some Properties of the $\mathcal{Z}$-Transform

- Chapter 10 (**Discrete-Time System Analysis Using the *z*-Transform**)
  - § 10.3 Properties of DTFT
  - § 10.5 Discrete-Time Linear System analysis by DTFT
  - § 10.7 Generalization of DTFT to the $\mathcal{Z}$-Transform

Next Time

Announcements

Announcements Add

Equation Editors & Tips! Edit Delete

3/23/16 12:55 PM

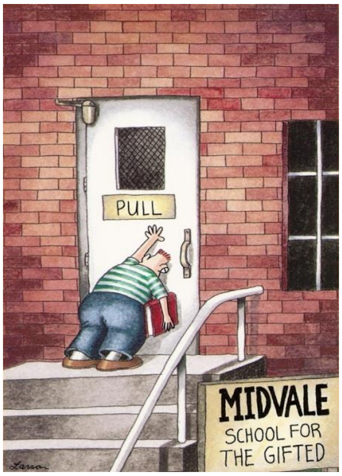
To post a response to a question asked by email...

A friendly reminder that, as noted in class, there are many equation editing interfaces that may make LaTeX entry easier to learn and/or may help with entering equations (such as 4×4 matrices).

Some links/tips that may (or may not) help:
List of Formula Editors: Available for many platforms and in many styles (e.g., LaTeX4technics, MathMagic, EqualX, EQ Editor, etc.)
Matlab will export symbolic equations as LaTeX via the `latex` command
There are many introductions and online generator tools e.g., LaTeX-Tutorial and Table Generator
For inserting some quick symbols -- try Unicode.
I find Unicode Lookup and
Unicode characters and corresponding LaTeX math mode page helpful.

Thanks!
View on Piazza

Announcements



Announcements

+ Add

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Thanks!
View on Piazza

$\mathcal{Z}$ Transform

Coping with Complexity

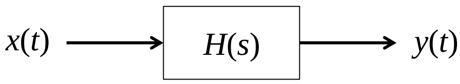
Transfer functions help control complexity

– Recall the Laplace transform:

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

where

$$\mathcal{L}\{\dot{f}(t)\} = sF(s)$$



- Is there a something similar for sampled systems?

Flashback: Euler's approximation (L7, p.26)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} \quad \Longrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} \approx \frac{x_{k+1} - x_k}{T}$$

For small enough T, this can be used to approximate a continuous controller by a discrete controller:

1. Laplace transform $\longrightarrow$ differential equation

   e.g.

$$D(s) = \frac{U(s)}{E(s)} = \frac{K(s+a)}{(s+b)} \quad \Longrightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}t} + b\,u = K\left(\frac{de}{dt} + a\,e\right)$$

2. Differential equation $\longrightarrow$ difference equation

   e.g.

$$\frac{u_{k+1} - u_k}{T} + b\,u_k = K\left(\frac{e_{k+1} - e_k}{T} + a\,e_k\right)$$

$$\Longrightarrow u_{k+1} = (1 - bT)u_k + K\,e_{k+1} + K(aT - 1)e_k$$
$$= -a_1 u_k + b_0 e_{k+1} + b_1 e_k$$

Discrete transfer function

Compare the discrete system time domain model:

$$\begin{aligned} u_k &= -a_1 u_{k-1} - \cdots - a_n u_{k-n} + b_0 e_k + \cdots + b_m e_{k-m} \\ &= -\sum_{i=1}^{n} a_i u_{k-i} + \sum_{j=0}^{m} b_j e_{k-j} \end{aligned}$$

recurrence equation

with the continuous system model:

$$u(t) = -a_1 \frac{\mathrm{d}u}{\mathrm{d}t} - \cdots a_n \frac{\mathrm{d}^n u}{\mathrm{d}t^n} + b_0 e + b_1 \frac{\mathrm{d}e}{\mathrm{d}t} + \cdots + b_m \frac{\mathrm{d}^m e}{\mathrm{d}t^m}$$

differential equation

↓ Laplace transform ↓

$$U(s) = -a_1 s U(s) - \cdots - a_n s^n U(s) + b_0 E(s) + b_1 s E(s) + \cdots + b_m s^m E(s)$$

$$\therefore \frac{U(s)}{E(s)} = D(s) = \frac{b_0 + b_1 s + b_2 s^2 + \cdots + b_m s^m}{1 + a_1 s + a_2 s^2 + \cdots + a_n s^n}$$

transfer function

Can we define a transfer function for the discrete system?

Discrete transfer function [2]

Suppose $u_k = u(kT)$ has transform $U'(s)$ ...

... then how can we represent u_{k-1}, u_{k-2}, etc.?

- If $x(t) \xrightarrow{L.T.} X(s)$, then $x(t-T) \xrightarrow{L.T.} e^{-sT} X(s)$, so

$$\begin{aligned} u_k &\longrightarrow U'(s) \\ u_{k-1} &\longrightarrow e^{-sT} U'(s) \\ u_{k-2} &\longrightarrow e^{-2sT} U'(s) \end{aligned}$$

etc

- Define the discrete frequency domain operator

$$\boxed{z = e^{sT}}$$

then

$$\begin{aligned} u_k &\longrightarrow U(z) \\ u_{k-1} &\longrightarrow z^{-1} U(z) \\ u_{k-2} &\longrightarrow z^{-2} U(z) \end{aligned}$$

etc

Discrete transfer function [3]

Comparison

- system representations:

Continuous

$$u(t) = -a_1\frac{\mathrm{d}u}{\mathrm{d}t} - a_2\frac{\mathrm{d}^2u}{\mathrm{d}t^2} - \cdots$$
$$+ b_0 e + b_1\frac{\mathrm{d}e}{\mathrm{d}t} + \cdots$$

Discrete

$$u_k = -a_1u_{k-1} - a_2u_{k-2} - \cdots$$
$$+ b_0e_k + b_1e_{k-1} + \cdots$$

- operators:

Continuous

$$\frac{\mathrm{d}u}{\mathrm{d}t} \longrightarrow s\,U(s)$$

differential

Discrete

$$u_{k-1} \longrightarrow z^{-1}\,U(z)$$

delay

Discrete transfer function [4]

Apply the transformation to the linear recurrence equation:

$$u_k = -a_1u_{k-1} - a_2u_{k-2} - \cdots - a_nu_{k-n}$$
$$+ b_0e_k + b_1e_{k-1} + \cdots b_ne_{k-n}$$

↓ transform ↓

$$U(z) = -a_1z^{-1}U(z) - a_2z^{-2}U(z) - \cdots a_nz^{-n}U(z)$$
$$+ b_1E(z) + b_2z^{-1}E(z) + \cdots + b_nz^{-n}E(z)$$

This gives the z domain transfer function:

$$\frac{U(z)}{E(z)} = D(z) = \frac{b_0 + b_1z^{-1} + b_2z^{-2} + \cdots b_mz^{-m}}{1 + a_1z^{-1} + a_2z^{-2} + \cdots + a_nz^{-n}}$$

Rationalize by multiplying top and bottom by z^n

$$\boxed{D(z) = \frac{b_0z^n + b_1z^{n-1} + b_2z^{n-2} + \cdots b_mz^{n-m}}{z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_n}}$$

Discrete transfer function [5]

Analysis tools based on s domain transfer functions:

e.g.

Pole & zero locations	$\longrightarrow$	damping, natural frequency, settling time & overshoot
Frequency response	$\longrightarrow$	gain & phase margins

. . . also apply to z domain transfer functions

Poles and zeros of $D(z)$:

$$D(z) = b_0 \frac{\Pi_{i=1}^{m}(z - z_j)}{\Pi_{i=1}^{n}(z - p_i)} z^{n-m}$$

zeros: z_i
poles: p_j

- z_i & p_i are real or in complex conjugate pairs
- n poles, n zeros, with $n - m$ zeros at $z = 0$
- at least as many poles as zeros

The z-Transform

- It is defined by:

$$z = re^{j\omega}$$

- Or in the Laplace domain (often $r = 1$):

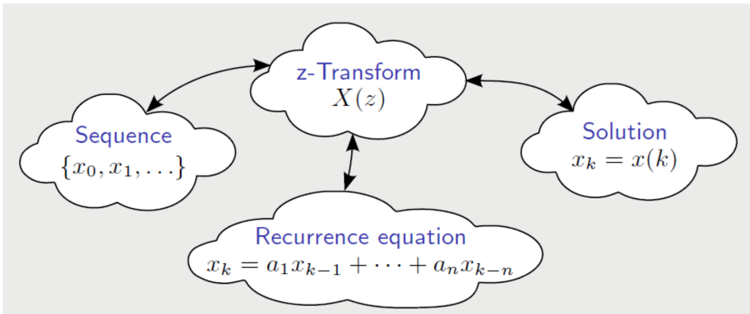
$$z = re^{sT}$$

- That is → it is a discrete version of the Laplace:

$$f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{z}{z - e^{-aT}}$$

The z-Transform [2]

To summarise:



$X(z)$ provides an easy way to convert between sequences, recurrence equations and their closed-form solutions.

The z-Transform [3]

So far we have considered z^{-1} as a delay operator acting on sequences

But to find $E(z)$ from $e(kT)$ we need to define the z-transform:

$$\boxed{\begin{aligned} E(z) &= \mathcal{Z}\{e(kT)\} = \mathcal{Z}\{e_k\} \\ &= \sum_{k=0}^{\infty} e(kT) z^{-k} = \sum_{k=0}^{\infty} e_k z^{-k} \end{aligned}}$$

Note:

- ⋆ Single-sided z-transform — all variables are assumed to be zero for $k < 0$ [Franklin uses a different definition]
- ⋆ Strictly speaking, we should give bounds on $|z|$ for convergence, e.g.

  $$r_0 < |z| < R_0$$

  where r_0, R_0 depend on $e(kT)$
  (these bounds are only needed in order to invert $E(z)$ by integration)

The z-Transform [4]

- Thus:

$$Y(z) = \sum_{k=-\infty}^{\infty} h[k] z^{-k} \qquad\qquad y[n] \xleftrightarrow{\mathcal{Z}} Y(z)$$

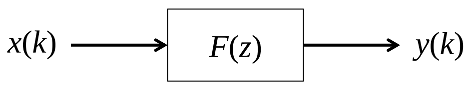
- z-Transform is analogous to other transforms:

$$\mathcal{Z}\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$$

and

$$\mathcal{Z}\{f(k-1)\} = z^{-1} F(z)$$

∴ Giving:



The z-Transform [5]

- The z-Transform may also be considered from the Laplace transform of the impulse train representation of sampled signal

$$u^*(t) = u_0 \delta(t) + u_1 \delta(t-T) + \ \ldots + u_{k(t-kT)} + \ \ldots$$

$$= \sum_{k=0}^{\infty} u_k \delta(t-kT)$$

$$U^*(s) = u_0 + u_1 e^{-sT} + \cdots + u_k e^{-skT} + \cdots$$

$$= \sum_{k=0}^{\infty} u_k \, e^{-ksT}$$

$$U(z) = \sum_{k=0}^{\infty} u_k \, z^{-k}, \qquad z = e^{sT}$$

The z-Transform [6]

Example – z-transform of a decaying exponential

Sample $x(t) = Ce^{-at}\mathcal{U}(t)$: ($\mathcal{U}(t)$ = unit step at $t = 0$)

$$x_k = Ce^{-akT}, \quad k \geq 0$$

and take the z-transform:

$$X(z) = \sum_{k=0}^{\infty} x_k z^{-k} = C\sum_{k=0}^{\infty} e^{-akT} z^{-k} = C\sum_{k=0}^{\infty} (e^{-aT} z^{-1})^k$$

this is a geometric series which converges if $|z| > e^{-aT}$:

$$X(z) = \frac{C}{1 - e^{-aT} z^{-1}} = \frac{Cz}{z - e^{-aT}}$$

$$\Downarrow$$

z-transform of exponental = rational polynomial (like Laplace)

The z-Transform [7]

Effect of delay:

$$\mathcal{Z}\{e(kT - T)\} = z^{-1}E(z) \quad \text{where} \quad E(z) = \mathcal{Z}\{e(kT)\}$$

Example – z-transform of a delayed sequence

Take a finite length sequence

$$e_0, e_1, e_2, e_3, e_4, \ldots = 1.5, 1.6, 1.7, 0, 0, \ldots$$

introduce a delay of one sampling interval:

$$f_0, f_1, f_2, f_3, f_4, \ldots = 0, 1.5, 1.6, 1.7, 0, \ldots$$

take z-transforms:

$$E(z) = \sum_{k=0}^{\infty} e_k z^{-k} = 1.5 + 1.6z^{-1} + 1.7z^{-2}$$

$$F(z) = \sum_{k=0}^{\infty} f_k z^{-k} = 1.5z^{-1} + 1.6z^{-2} + 1.7z^{-3}$$
$$= z^{-1}E(z)$$

The z-Transform [8]

Example – z-transform of a delayed exponential

Delay $x(t) = Ce^{-at}\mathcal{U}(t)$ by a time T:

$$y(t) = x(t-T) \quad \Longrightarrow \quad y(t) = Ce^{-a(t-T)}\mathcal{U}(t-T)$$

sample $y(t)$ with sample interval T:

$$y_k = \begin{cases} 0 & k = 0 \\ Ce^{-a(k-1)T} & k = 1, 2, \ldots \end{cases}$$

z-transform:

$$Y(z) = \sum_{k=0}^{\infty} y_k\, z^{-k} = \sum_{k=1}^{\infty} Ce^{-a(k-1)T} z^{-k}$$

$$= Cz^{-1} \sum_{j=0}^{\infty} (e^{-aT} z^{-1})^j = \frac{C}{z - e^{-aT}}$$

Comparing $X(z)$ and $Y(z)$:

$$X(z) = \frac{Cz}{z - e^{-aT}} \quad \Longrightarrow \quad Y(z) = z^{-1}X(z)$$

The z-Transform [9]

- In practice, you'll use look-up tables or computer tools (ie. Matlab) to find the z-transform of your functions

$F(s)$	$F(kt)$	$F(z)$
$\frac{1}{s}$	1	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\frac{1}{s^2+a^2}$	$\sin(akT)$	$\frac{z \sin aT}{z^2 - (2\cos aT)z + 1}$

z-Transform Example [10]

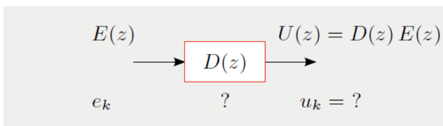
- Obtain the z-Transform of the sequence:

$$x[k] = \{3, 0, 1, 4, 1, 5, \dots\}$$

- Solution:

$$X(z) = 3 + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5}$$

Pulse Response



For continuous systems: $D(s) = \mathcal{L}\{d(t)\}$ $d(t)$ = plant impulse response

What is the equivalent property for the discrete transfer function $D(z)$?

- ⋆ Let $e(kT)$ = discrete unit pulse:

$$e_k = \delta_k = \begin{cases} 1 & k = 0 \\ 0 & k = 1, 2, \dots \end{cases} \iff E(z) = \sum_{k=0}^{\infty} e_k z^{-k} = 1$$

- ⋆ then

$$\boxed{U(z) = D(z)E(z) = D(z)}$$

i.e. $D(z) = \mathcal{Z}\{d(kT)\}$ = z-transform of the plant pulse response

Pulse Response [2]

Example – The recurrence equation $u_k = u_{k-1} + \frac{T}{2}(e_k + e_{k-1})$

has transfer function $D(z) = \frac{U(z)}{E(z)} = \frac{T}{2}\frac{(z+1)}{(z-1)}$

Check this by finding the pulse response and taking its z-transform

$e_k = \delta_k$ gives

k	u_{k-1}	e_k	e_{k-1}	u_k
0	0	1	0	$T/2$
1	$T/2$	0	1	T
2	T	0	0	T
3	T	0	0	T
$\vdots$				$\vdots$

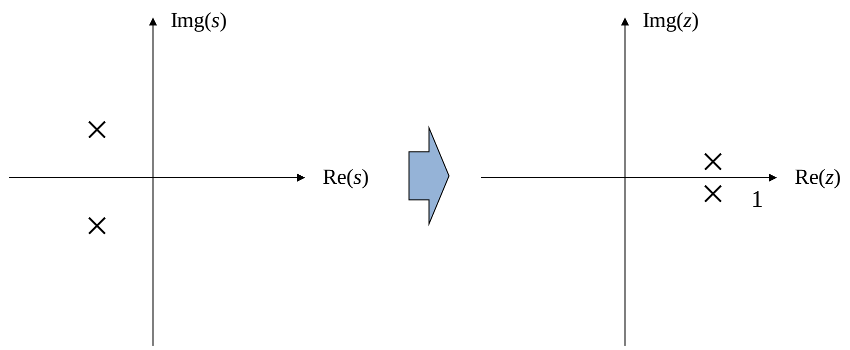
i.e. $u_k = T/2, T, T, \ldots$

so $$U(z) = \sum_{k=0}^{\infty} Tz^{-k} - T/2 = \frac{T}{1-z^{-1}} - \frac{T}{2} = \frac{T}{2}\frac{(z+1)}{(z-1)}$$

The z-Plane

z-domain poles and zeros can be plotted just like s-domain poles and zeros (of the $\mathcal{L}$):

- S-plane:
- $z = e^{sT}$ Plane

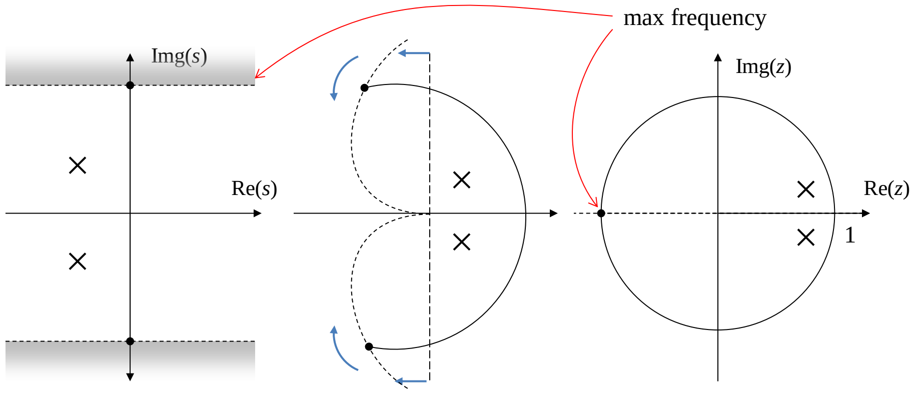


– λ – Plane

– γ – Plane

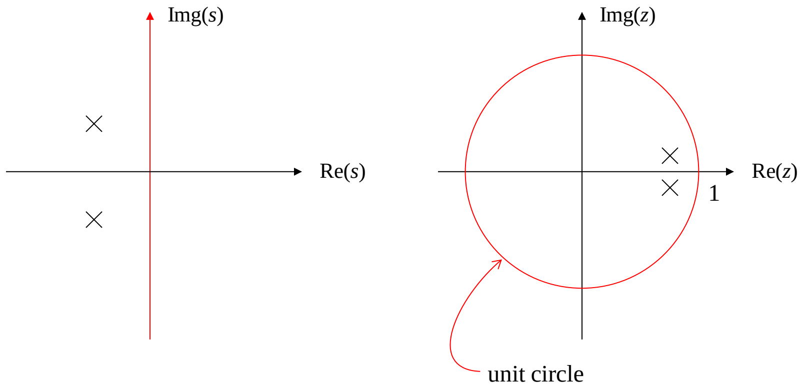
Deep insight #1

The mapping between continuous and discrete poles and zeros acts like a distortion of the plane



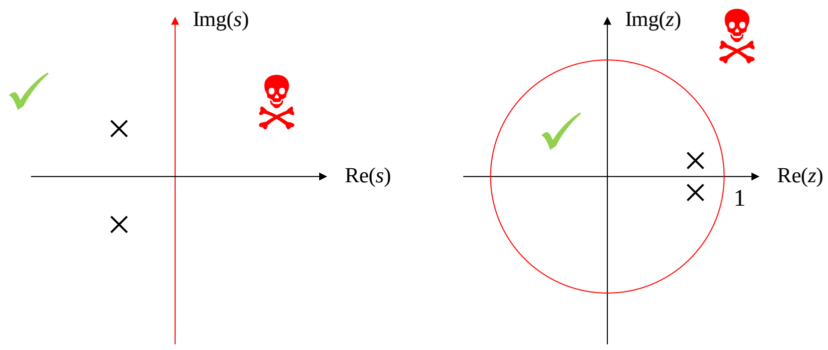
γ-plane Stability

- For a γ-Plane (e.g. the one the *z*-domain is embedded in) the unit circle is the system stability bound



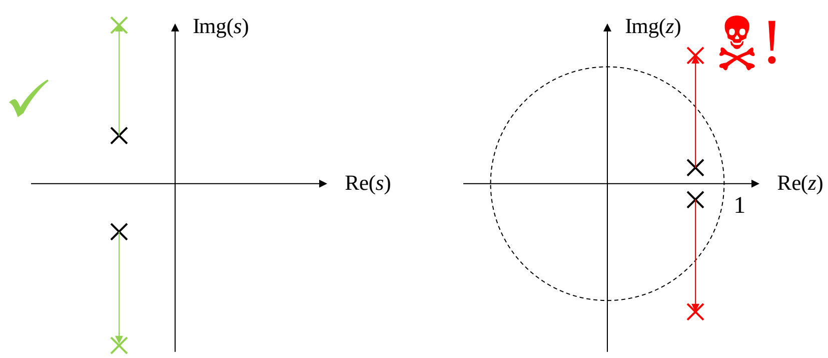
γ-plane Stability

- That is, in the *z*-domain,
  the unit circle is the system stability bound



z-plane stability

- The z-plane root-locus in closed loop feedback behaves just like the s-plane:



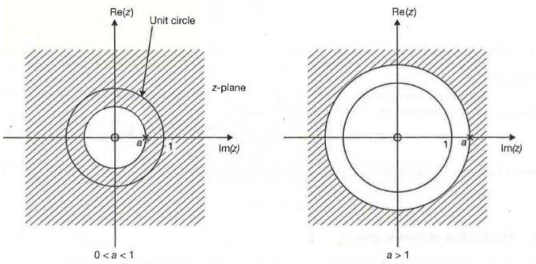
Region of Convergence

- For the convergence of X(z) we require that

$$\sum_{n=0}^{\infty}\left|az^{-1}\right|^{n}<\infty$$

- Thus, the ROC is the range of values of z for which $|az^{-1}|< 1$ or, equivalently, $|z| > |a|$. Then

$$X(z)=\frac{z}{z-a} \qquad |z|>|a|$$



An example!

- Back to our difference equation:

$$y(k) = x(k) + Ax(k-1) - By(k-1)$$

becomes

$$Y(z) = X(z) + Az^{-1}X(z) - Bz^{-1}Y(z)$$
$$(z+B)Y(z) = (z+A)X(z)$$

which yields the transfer function:

$$\frac{Y(z)}{X(z)} = \frac{z+A}{z+B}$$

Note: It is also not uncommon to see systems expressed as polynomials in z^{-n}

This looks familiar…

- Compare:

$$\frac{Y(s)}{X(s)} = \frac{s+2}{s+1} \text{ vs } \frac{Y(z)}{X(z)} = \frac{z+A}{z+B}$$

How are the Laplace and *z* domain representations related?

Linearity:

$$a_1 y_1[n] + a_2 y_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} a_1 Y_1(z) + a_2 Y_2(z)$$

Z-Transform Properties: Time Shifting

$$y[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} Y(z)$$

$$\begin{aligned} y_2[n] &= y[n-n_0] \\ Y_2(e^{jw}) &= \sum_{k=-\infty}^{\infty} y[k-n_0] z^{-k} \\ &= \sum_{l=-\infty}^{\infty} y[l] z^{-(l+n_0)} \\ &= z^{-n_0} Y(z) \end{aligned}$$

- Two Special Cases:
- z^{-1}: the *unit-delay operator:*

$$x[n-1] \leftrightarrow z^{-1} X(z) \qquad R' = R \cap \{0 < |z|\}$$

- z: *unit-advance operator:*

$$x[n+1] \leftrightarrow zX(z) \qquad R' = R \cap \{|z| < \infty\}$$

More Z-Transform Properties

- Time Reversal

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right) \qquad R' = \frac{1}{R}$$

- Multiplication by z^n

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$z_0^n x[n] \leftrightarrow X\left(\frac{z}{z_0}\right) \qquad R' = |z_0| R$$

- Multiplication by n (or Differentiation in z):

$$x[n] \leftrightarrow X(z) \qquad \text{ROC} = R$$

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \qquad R' = R$$

- Convolution

$$x_1[n] \leftrightarrow X_1(z) \qquad \text{ROC} = R_1$$
$$x_2[n] \leftrightarrow X_2(z) \qquad \text{ROC} = R_2$$

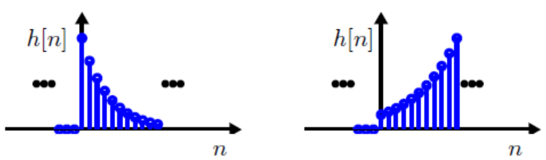
$$x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \qquad R' \supset R_1 \cap R_2$$

z-Transforms for Difference Equations

- First-order linear constant coefficient difference equation:

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$



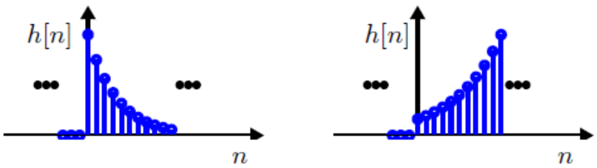
$$h[n] = \begin{cases} ba^n & n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H(z) = \sum_{k=0}^{\infty} ba^k z^{-k} = b\sum_{k=0}^{\infty}\left(\frac{a}{z}\right)^k = \frac{b}{1-az^{-1}}, \quad \text{when } |z| > |a|.$$

z-Transforms for Difference Equations

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$



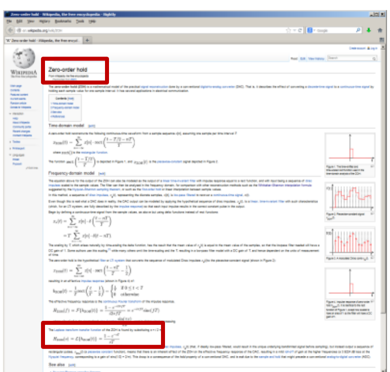
$$y[n] - ay[n-1] = bu[n]$$
$$\updownarrow$$
$$Y(z) - az^{-1}Y(z) = bU(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1-az^{-1}}, \text{ when does it converge?}$$

$\mathcal{L}$(ZOH)=??? : What is it?

$$\frac{1-e^{-Ts}}{Ts}$$

- Wikipedia

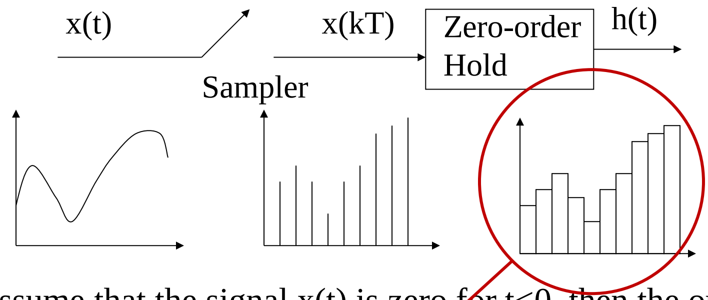


$$\frac{1-e^{-Ts}}{s}$$

- Lathi
- Franklin, Powell, Workman
- Franklin, Powell, Emani-Naeini
- Dorf & Bishop
- Oxford Discrete Systems: (Mark Cannon)
- MIT 6.002 (Russ Tedrake)
- Matlab

Proof!

Zero-order-hold (ZOH)



- Assume that the signal x(t) is zero for t<0, then the output h(t) is related to x(t) as follows:

$$h(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] + \cdots$$
$$= \sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]$$

Transfer function of Zero-order-hold (ZOH)

- Recall the Laplace Transforms ($\mathcal{L}$) of:

$$\mathcal{L}[\delta(t)] = 1 \qquad \mathcal{L}[f(t-kT)] = F(s)e^{-kTs}$$
$$\mathcal{L}\left[\delta(t-kT)\right] = e^{-kTs} \qquad \mathcal{L}[1(t-kT)] = \frac{e^{-kTs}}{s}$$

- Thus the $\mathcal{L}$ of h(t) becomes:

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$
$$= \sum_{k=0}^{\infty} x(kT)\mathcal{L}[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$
$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Transfer function of Zero-order-hold (ZOH)

… Continuing the $\mathcal{L}$ of h(t) …

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\rightarrow X(s) = \mathcal{L}\left[\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right] = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\therefore H(s) = \mathcal{L}[h(t)] = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}X(s)$$

➔ Thus, giving the transfer function as:

$$\boxed{G_{ZOH}(s) = \frac{H(s)}{X(s)} = \frac{1-e^{-Ts}}{s}} \xrightarrow{\mathcal{Z}} \boxed{G_{ZOH}(z) = \frac{\left(1-e^{-aT}\right)}{z-e^{-aT}}}$$

Example:

- Is this system stable?

$$u(k) = 0.9u(k-1) - 0.2u(k-2)$$

- Time-shift it:

$$u(k+2) = 0.9u(k+1) - 0.2u(k)$$

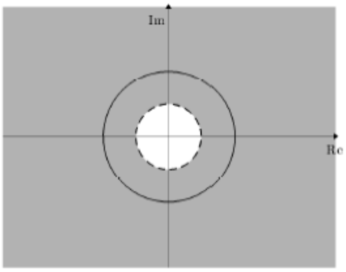
- z-Transform:

$$(1)z^2 - 0.9z + 0.2 = 0$$

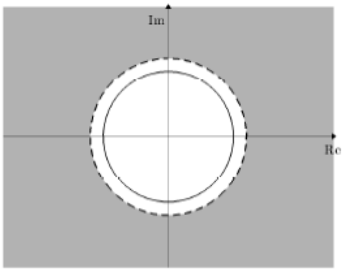
- Characteristic Roots:
  z=0.5, z=0.4 ➔ STABLE!

Region of Convergence (ROC) Plots

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1 - az^{-1}}, \quad |z| > |a|$$



$a = .5$



$a = 1.2$

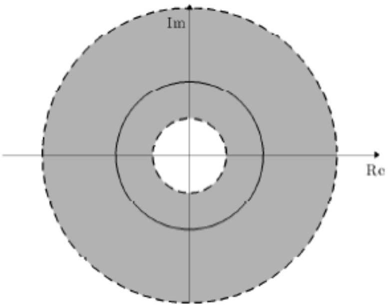
Combinations of Signals

$$y_1[n] = \begin{cases} ba^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$a = .5$

$$y_2[n] = \begin{cases} 0 & n \geq 0 \\ -ba^n & n < 0 \end{cases}$$

$a = 2$

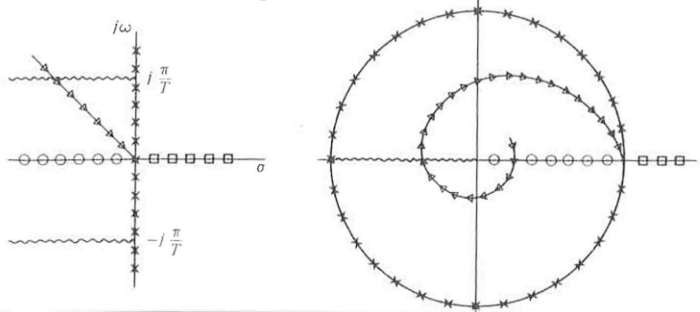


ROC for $\alpha_1 y_1[n] + \alpha_2 y_2[n]$

$s \leftrightarrow \mathcal{Z}$

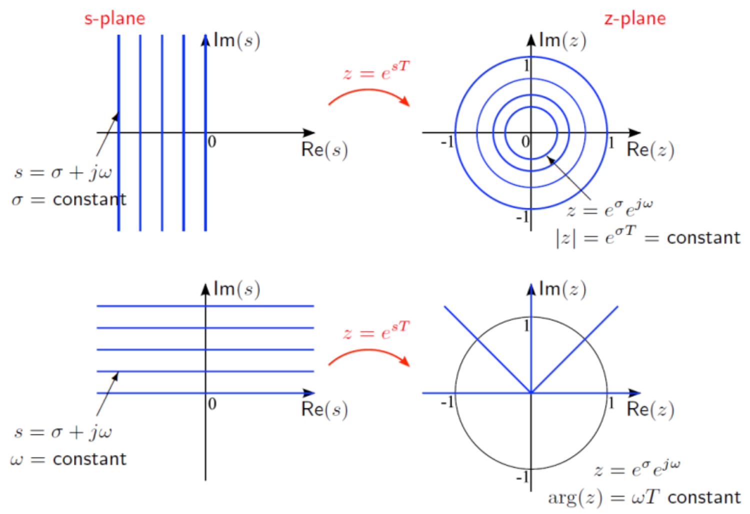
(More than "*American English*")

Hint: Use γ to Transform $s \leftrightarrow z$: $z = e^{sT}$



s-plane	Symbol	z-plane
$s = j\omega$ Real frequency axis	× × ×	$\lvert z \rvert = 1$ Unit circle
$s = \sigma \geq 0$	□□□	$z = r \geq 1$
$s = \sigma \leq 0$	○○○	$z = r,\ 0 \leq r \leq 1$
$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -a + jb$ Constant damping ratio if ζ is fixed and ω_n varies	△△△	$z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T) = e^{-aT}$, $\theta = \omega_n T\sqrt{1-\zeta^2} = bT$ Logarithmic spiral
$s = \pm j(\pi/T) + \sigma, \sigma \leq 0$	〰〰	$z = -r$

S-Plane to z-Plane [1/2]



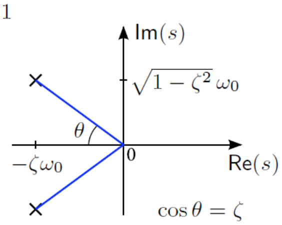
S-Plane to z-Plane [2/2]

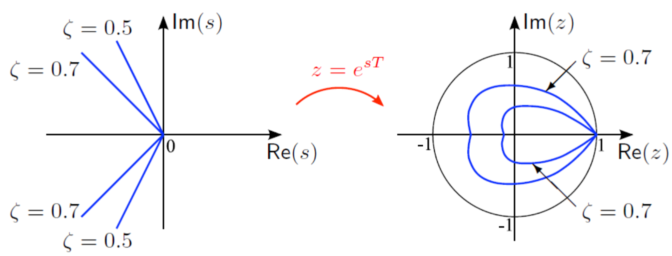
Pole locations for constant damping ratio $\zeta < 1$

$$s^2 + \zeta\omega_0 s + \omega_0^2 = 0$$

$$\Downarrow$$

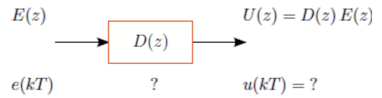
$$s = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\,\omega_0$$





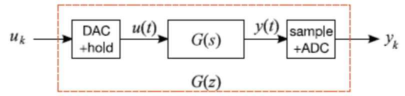
$s = -\zeta\omega_0 + j\sqrt{1-\zeta^2}\,\omega_0$: ζ = constant $\qquad z = e^{-\zeta\omega_0 T} e^{-j\sqrt{1-\zeta^2}\,\omega_0 T}$

$s \leftrightarrow z$: Pulse Transfer Function Models



- Pulse in Discrete is equivalent to Dirac-δ

$$e_k = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$



➔

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}_{t=kT}\right\} = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

Source: Oxford 2A2 Discrete Systems, Tutorial Notes p. 26

Relationship with s-plane poles and z-plane transforms

If $F(s)$ has a pole at $s = a$
then $F(z)$ has a pole at $z = e^{aT}$

↑

consistent with $z = e^{sT}$

What about transfer functions?

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

↓

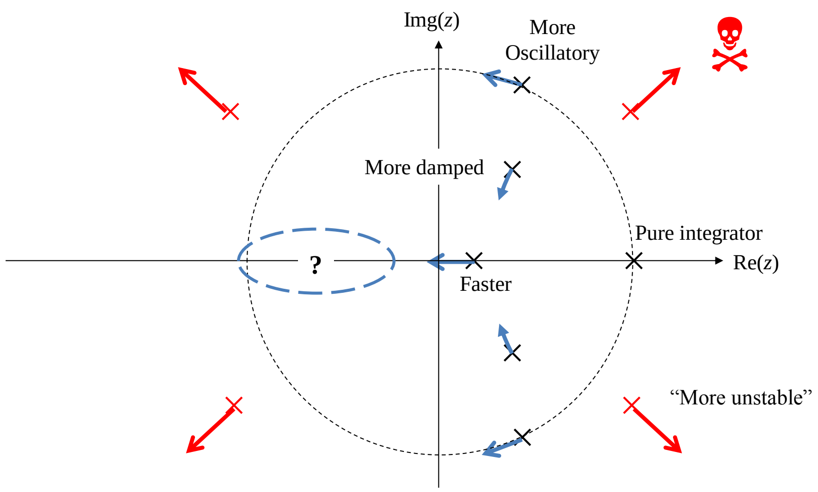
If $G(s)$ has poles $s = a_i$
then $G(z)$ has poles $z = e^{a_i T}$

but the zeros are unrelated

$\mathcal{F}(s)$	$f(kT)$	$F(z)$
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-1}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$
$\frac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$

Recall dynamic responses

- Ditto the z-plane:

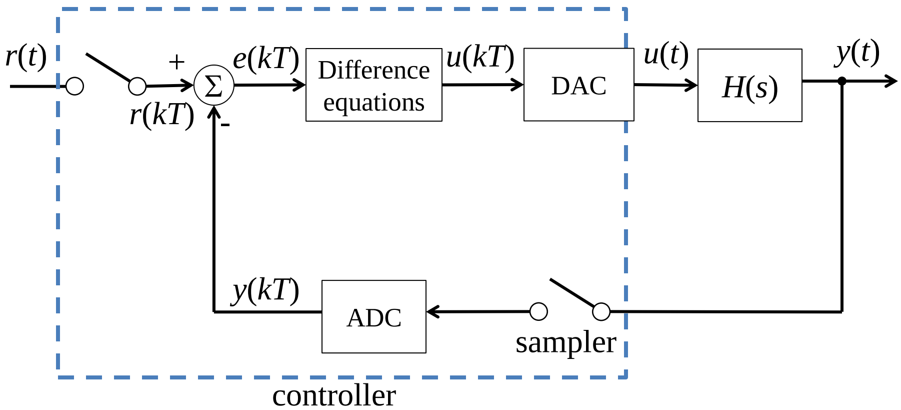


Discrete-Time

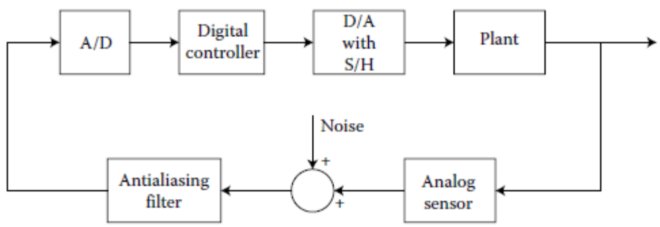
System Analysis

Digitisation

- Continuous signals sampled with period T
- kth control value computed at $t_k = kT$



Remember: Selection of Sampling Frequency!

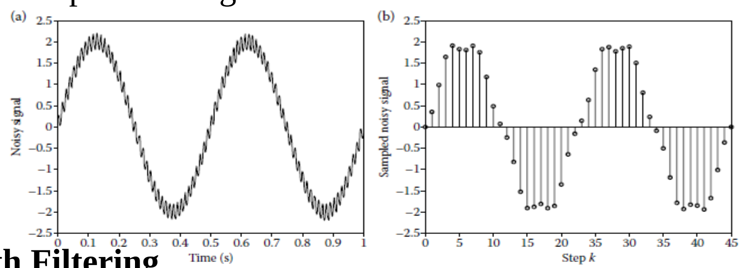


To Prevent Aliasing:

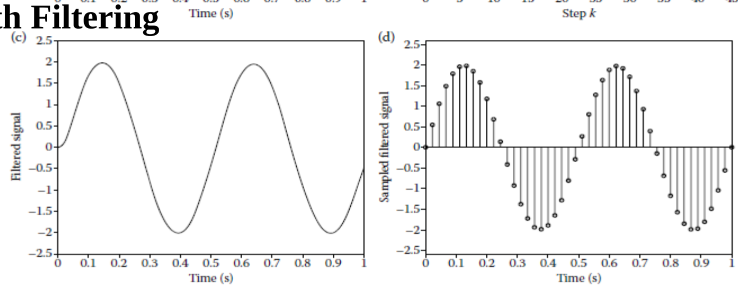
- Make sure your sampling frequency is greater than twice of the highest frequency component of the signal. In practice, take it ten times the highest frequency component.
- Pre-filtering of the analog signal
- Set your sampling frequency to the maximum if possible

Remember: Effect of Noise…

- Without pre-filtering:

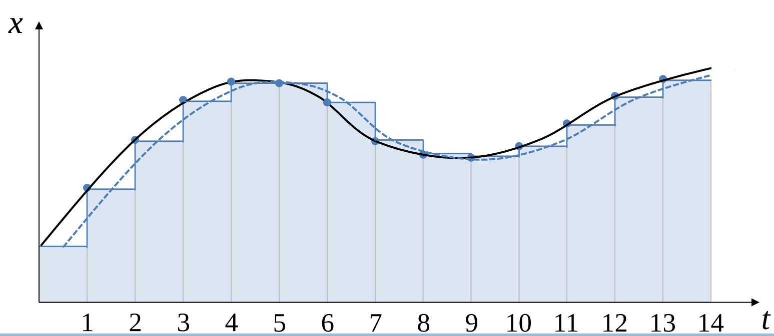


- **With Filtering**



Zero Order Hold

- An output value of a synthesised signal is held constant until the next value is ready
  - This introduces an effective delay of $T/2$



Effect of ZOH Sampling

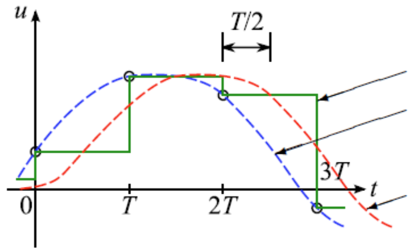
Lower sample rate $\Longrightarrow$ more oscillatory response

— Why?

Sampling and reconstruction introduces:

delay in time domain
& phase lag in freq. domain $\leftarrow$ can destabilize the closed loop system

On average $u(kT)$ is delayed by $T/2$ relative to $u(t)$ due to the ZOH:



Effect of ZOH Sampling

The ZOH delay of $T/2$ (sec) causes

phase lag $= \omega T/2$ (rad) at ω rad s^{-1}
phase lag $= \pi/2 = 90^\circ$ at $\omega = \pi/T$ [= Nyquist rate]
phase lag $= \pi/30 = 6^\circ$ at $\omega = \pi/(15T)$

⋆ 90° phase lag could be catastrophic

⋆ If $\omega_{\text{samp}} > 30 \times \omega_{\max}$,

then system bandwidth: $\omega_{\max} < \pi/(15T)$,
so the maximum phase lag is less than 6°

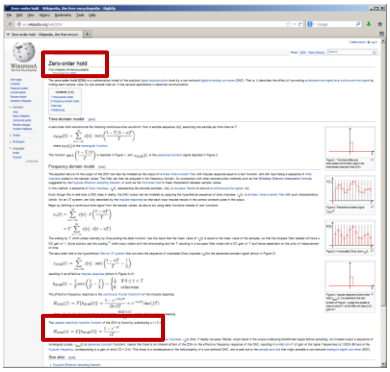
$\nwarrow$ usually safe to ignore

⋆ Any time needed to compute u_k causes additional delay (!)

$\mathcal{L}$(ZOH)=??? : What is it?

$$\frac{1-e^{-Ts}}{Ts}$$

- Wikipedia

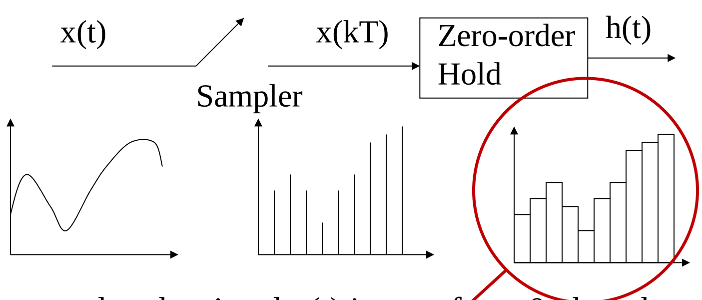


$$\frac{1-e^{-Ts}}{s}$$

- Lathi
- Franklin, Powell, Workman
- Franklin, Powell, Emani-Naeini
- Dorf & Bishop
- Oxford Discrete Systems: (Mark Cannon)
- MIT 6.002 (Russ Tedrake)
- Matlab

Proof!

Zero-order-hold (ZOH)



- Assume that the signal x(t) is zero for t<0, then the output h(t) is related to x(t) as follows:

$$h(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] + \cdots$$

$$= \sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]$$

Transfer function of Zero-order-hold (ZOH)

- Recall the Laplace Transforms ($\mathcal{L}$) of:

$$\mathcal{L}[\delta(t)] = 1 \qquad \mathcal{L}[f(t-kT)] = F(s)e^{-kTs}$$

$$\mathcal{L}\left[\delta(t-kT)\right] = e^{-kTs} \qquad \mathcal{L}[1(t-kT)] = \frac{e^{-kTs}}{s}$$

- Thus the $\mathcal{L}$ of h(t) becomes:

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Transfer function of Zero-order-hold (ZOH)

… Continuing the $\mathcal{L}$ of h(t) …

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\rightarrow X(s) = \mathcal{L}\left[\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right] = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\therefore H(s) = \mathcal{L}[h(t)] = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}X(s)$$

➔ Thus, giving the transfer function as:

$$\boxed{G_{\text{ZOH}}(s) = \frac{H(s)}{X(s)} = \frac{1-e^{-Ts}}{s}} \xrightarrow{\mathcal{Z}} \boxed{G_{ZOH}(z) = \frac{\left(1-e^{-aT}\right)}{z-e^{-aT}}}$$

Direct Design: Second Order Digital Systems

Consider the z-transform of a decaying exponential signal:

$$y(t) = e^{-at}\cos(bt)\,\mathcal{U}(t) \qquad\qquad (\mathcal{U}(t) = \text{unit step})$$

⋆ sample: $y(kT) = r^k \cos(k\theta)\,\mathcal{U}(kT)$ with $r = e^{-aT}$ & $\theta = bT$

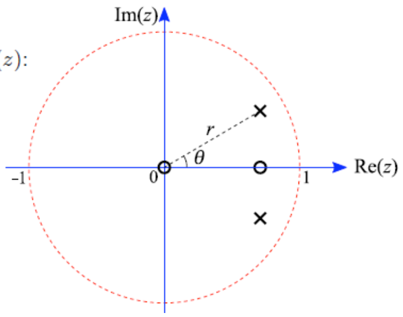
⋆ transform: $$Y(z) = \frac{1}{2}\frac{z}{(z - re^{j\theta})} + \frac{1}{2}\frac{z}{(z - re^{-j\theta})} = \frac{z(z - r\cos\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

⋆ e.g. y_k is the pulse response of $G(z)$:

$$G(z) = \frac{z(z - r\cos\theta)}{(z - re^{j\theta})(z - re^{-j\theta})}$$

poles: $\begin{cases} z = re^{j\theta} \\ z = re^{-j\theta} \end{cases}$

zeros: $\begin{cases} z = 0 \\ z = r\cos\theta \end{cases}$



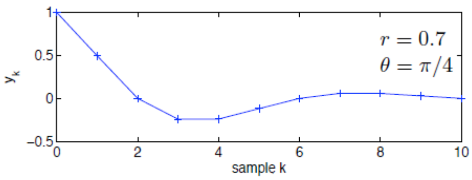
Response of 2nd order system [1/3]

Responses for varying r:

▷ $r < 1$

⇓

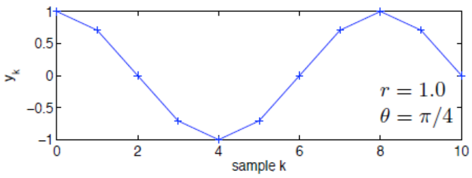
exponentially decaying envelope



▷ $r = 1$

⇓

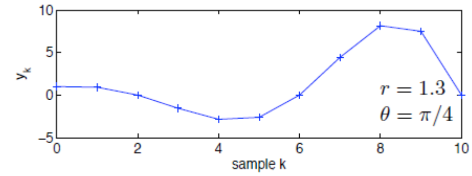
sinusoidal response with $2\pi/\theta$ samples per period



▷ $r > 1$

⇓

exponentially increasing envelope



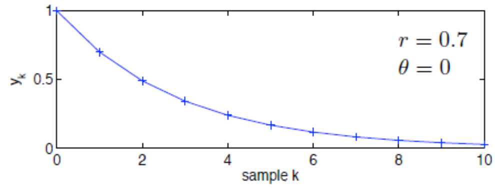
Response of 2nd order system [2/3]

Responses for varying θ:

- $\theta = 0$

  $\Downarrow$

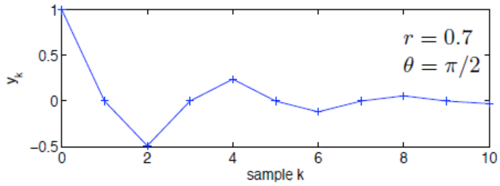
  decaying exponential



- $\theta = \pi/2$

  $\Downarrow$

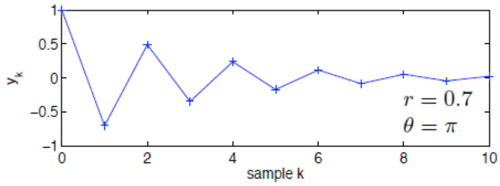
  $2\pi/\theta = 4$ samples per period



- $\theta = \pi$

  $\Downarrow$

  2 samples per period



Response of 2nd order system [3/3]

Some special cases:

- for $\theta = 0$, $Y(z)$ simplifies to:

$$Y(z) = \frac{z}{z - r}$$

$\Longrightarrow$ exponentially decaying response

- when $\theta = 0$ and $r = 1$:

$$Y(z) = \frac{z}{z - 1}$$

$\Longrightarrow$ unit step

- when $r = 0$:

$$Y(z) = 1$$

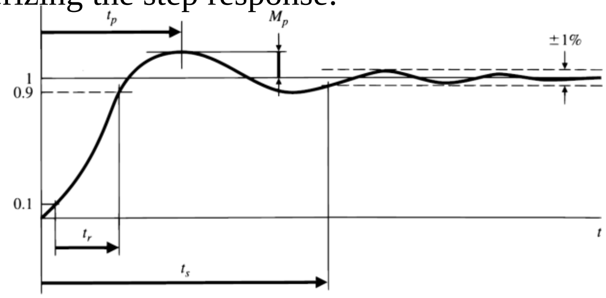
$\Longrightarrow$ unit pulse

- when $\theta = 0$ and $-1 < r < 0$:

samples of alternating signs

2nd Order System Specifications

Characterizing the step response:

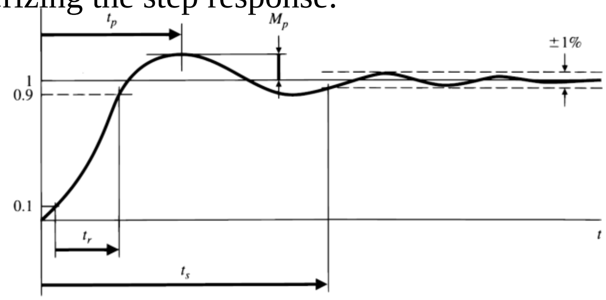


- Rise time (10% → 90%): $t_r \approx \frac{1.8}{\omega_0}$
- Overshoot: $M_p \approx \frac{e^{-\pi\zeta}}{\sqrt{1-\zeta^2}}$
- Settling time (to 1%): $t_s = \frac{4.6}{\zeta\omega_0}$
- Steady state error to unit step: e_{ss}
- Phase margin:

$$\phi_{PM} \approx 100\zeta$$

2nd Order System Specifications

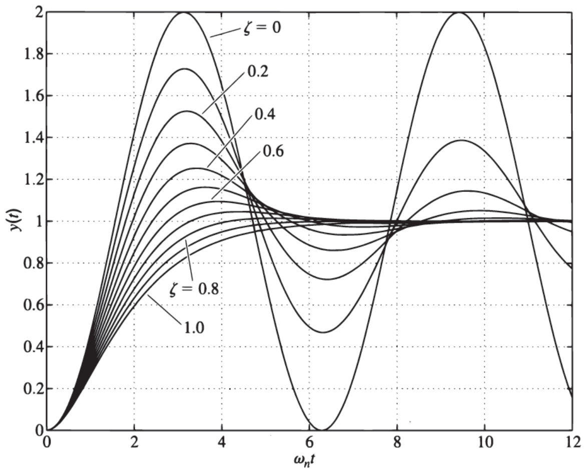
Characterizing the step response:



- Rise time (10% → 90%) & Overshoot:
  t_r, M_p → ζ, ω_0 : Locations of dominant poles
- Settling time (to 1%):
  t_s → radius of poles: $|z| < 0.01^{\frac{T}{t_s}}$
- Steady state error to unit step:
  e_{ss} → final value theorem $e_{ss} = \lim_{z \to 1} \{(z-1)\, F(z)\}$

2nd Order System Response

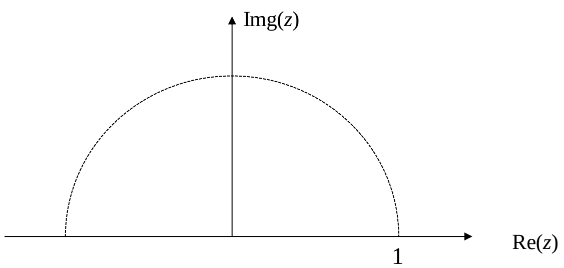
- Response of a 2nd order system to increasing levels of damping:



The z-plane [for all pole systems]

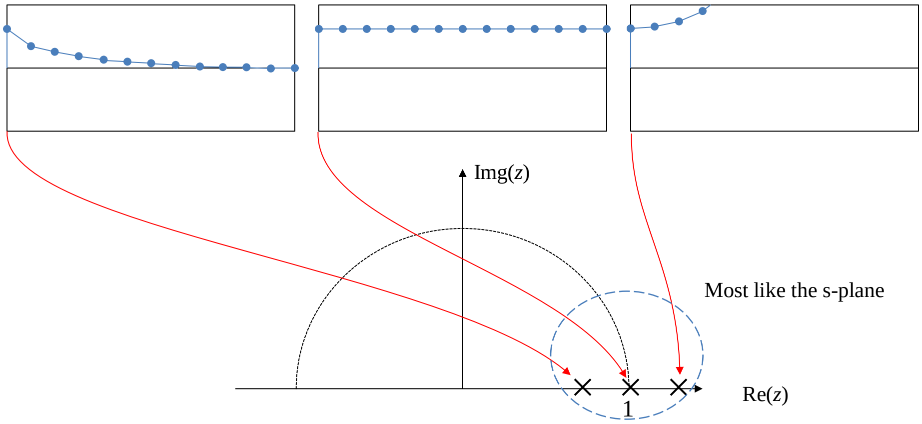
- We can understand system response by pole location in the *z*-plane

[Adapted from Franklin, Powell and Emami-Naeini]



Effect of pole positions

- We can understand system response by pole location in the *z*-plane



Effect of pole positions

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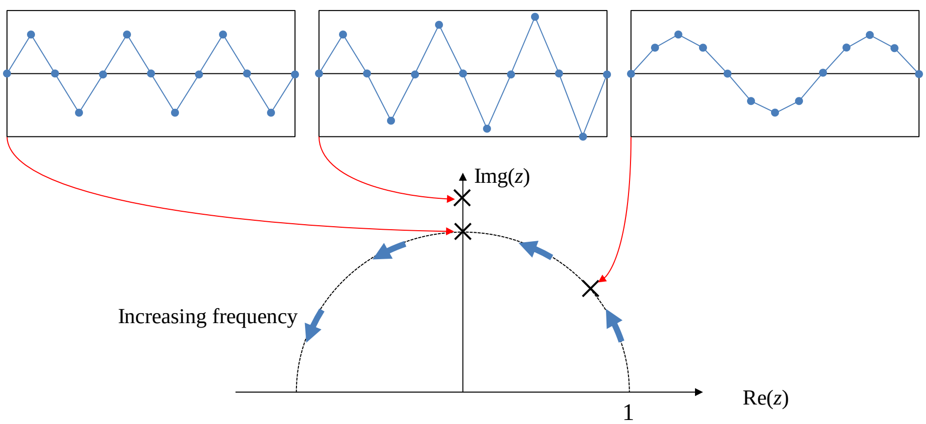
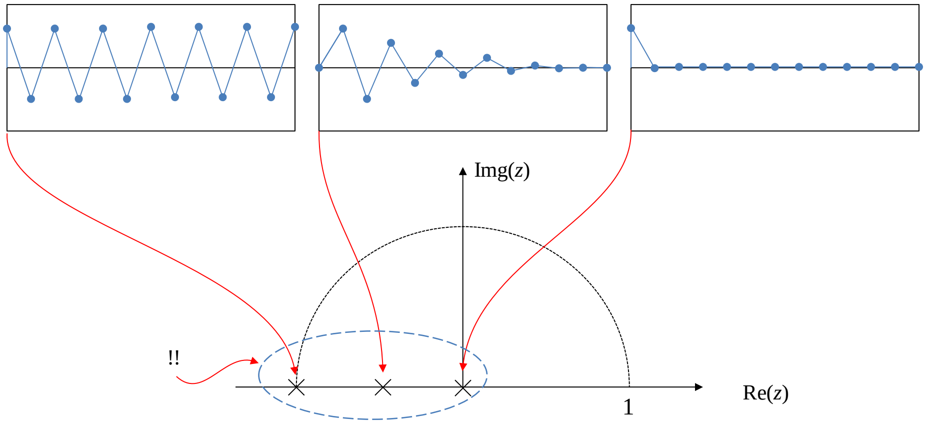


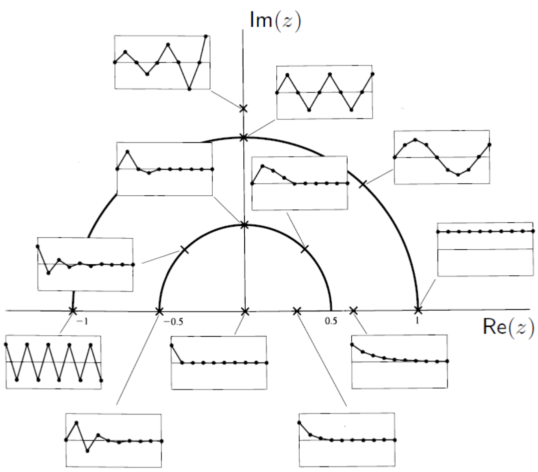
Effect of pole positions

- We can understand system response by pole location in the z-plane



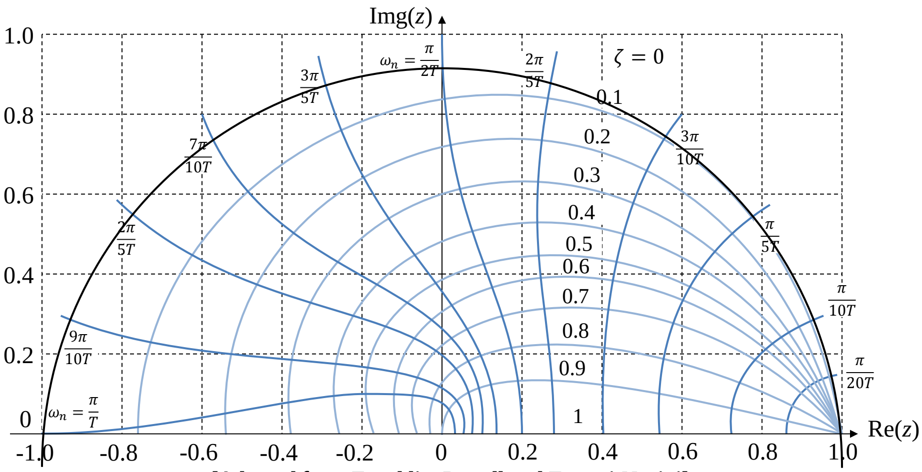
Pole positions in the z-plane

- Poles inside the unit circle are **stable**
- Poles outside the unit circle **unstable**
- Poles on the unit circle are oscillatory
- Real poles at 0 < z < 1 give exponential response
- Higher frequency of oscillation for larger
- Lower apparent damping for larer and r



Damping and natural frequency

$$z = e^{sT} \text{ where } s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$



[Adapted from Franklin, Powell and Emami-Naeini]

Ex: System Specifications → Control Design [1/4]

Design a controller for a system with:

- A continuous transfer function: $G(s) = \dfrac{0.1}{s(s+0.1)}$
- A discrete ZOH sampler
- Sampling time (T_s): T_s= 1s
- Controller:

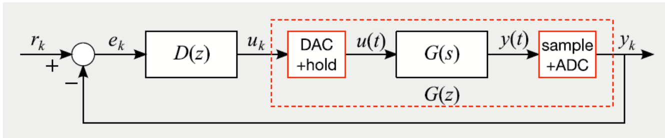
$$u_k = -0.5u_{k-1} + 13\left(e_k - 0.88e_{k-1}\right)$$

The closed loop system is required to have:

- M_p < 16%
- t_s < 10 s
- e_{ss} < 1

Ex: System Specifications → Control Design [2/4]

1. (a) Find the pulse transfer function of $G(s)$ plus the ZOH



$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{(z-1)}{z}\mathcal{Z}\left\{\frac{0.1}{s^2(s+0.1)}\right\}$$

e.g. look up $\mathcal{Z}\{a/s^2(s+a)\}$ in tables:

$$G(z) = \frac{(z-1)}{z}\frac{z\Big((0.1 - 1 + e^{-0.1})z + (1 - e^{-0.1} - 0.1e^{-0.1})\Big)}{0.1(z-1)^2(z - e^{-0.1})}$$
$$= \frac{0.0484(z + 0.9672)}{(z-1)(z-0.9048)}$$

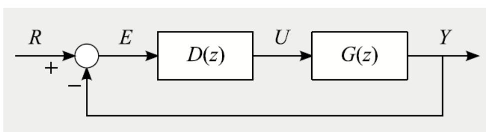
(b) Find the controller transfer function (using z = shift operator):

$$\frac{U(z)}{E(z)} = D(z) = 13\frac{(1 - 0.88z^{-1})}{(1 + 0.5z^{-1})} = 13\frac{(z - 0.88)}{(z + 0.5)}$$

Ex: System Specifications → Control Design [3/4]

2. Check the steady state error e_{ss} when r_k = unit ramp

$$e_{ss} = \lim_{k\to\infty} e_k = \lim_{z\to 1}(z-1)E(z)$$



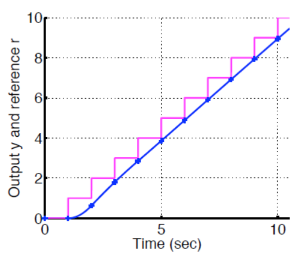
$$\frac{E(z)}{R(z)} = \frac{1}{1 + D(z)G(z)}$$
$$R(z) = \frac{Tz}{(z-1)^2}$$

so $$e_{ss} = \lim_{z\to 1}\left\{(z-1)\frac{Tz}{(z-1)^2}\frac{1}{1 + D(z)G(z)}\right\} = \lim_{z\to 1}\frac{T}{(z-1)D(z)G(z)}$$

$$= \lim_{z\to 1}\frac{T}{(z-1)\frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)}D(1)}$$

$$= \frac{1 - 0.9048}{0.0484(1 + 0.9672)D(1)} = 0.96$$

$\implies e_{ss} < 1$ (as required)



Ex: System Specifications → Control Design [4/4]

3. Step response: overshoot $M_p < 16\% \implies \zeta > 0.5$

   settling time $t_s < 10 \implies |z| < 0.01^{1/10} = 0.63$

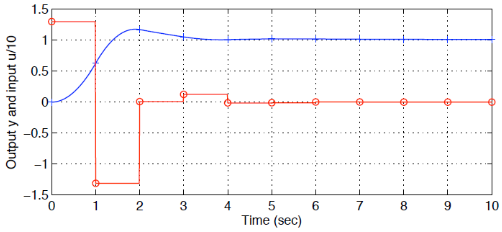
The closed loop poles are the roots of $1 + D(z)G(z) = 0$, i.e.

$$1 + 13\frac{(z-0.88)}{(z+0.5)}\frac{0.0484(z+0.9672)}{(z-1)(z-0.9048)} = 0$$

$$\implies z = 0.88,\ -0.050 \pm j0.304$$

But the pole at $z = 0.88$ is cancelled by controller zero at $z = 0.88$, and

$$z = -0.050 \pm j0.304 = re^{\pm j\theta} \implies \begin{cases} r = 0.31,\ \theta = 1.73 \\ \zeta = 0.56 \end{cases}$$



all specs satisfied!

Convolution

Convolution Definition

The convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as:

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$= f_1(t) * f_2(t)$$

Source: URI ELE436

Properties of Convolution

$$\boxed{f_1(t) * f_2(t) = f_2(t) * f_1(t)}$$

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=-\infty}^{\tau=\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$= \int_{t-\tau=-\infty}^{t-\tau=\infty} f_1(t-\tau) f_2[t-(t-\tau)] d(t-\tau)$$

$$= -\int_{\tau=\infty}^{\tau=-\infty} f_1(t-\tau) f_2(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau \;\; = f_2(t) * f_1(t)$$

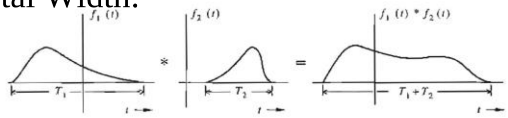
Source: URI ELE436

Convolution & Properties

$$f_1(t) * f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

Properties:

- Commutative: $f_1(t) * f_2(t) = f_2(t) * f_1(t)$
- Distributive: $f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$
- Associative: $f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t)$
- Shift:
  if $f_1(t)*f_2(t)=c(t)$, then $f_1(t-\mathbf{T})*f_2(t)= f_1(t)*f_2(t-\mathbf{T})=c(t-\mathbf{T})$
- Identity (Convolution with an Impulse):
  $f(t) * \delta(t) = f(t)$
- Total Width:



Based on Lathi, *SPLS*, Sec 2.4-1

Convolution & Properties [II]

- Convolution systems are **linear**:

$$h * (\alpha u_1 + \beta u_2) = \alpha(h * u_1) + \beta(h * u_2)$$

- Convolution systems are **causal**: the output y(t) at time t depends only on past inputs

- Convolution systems are **time-invariant**
  (if we shift the signal, the output similarly shifts)

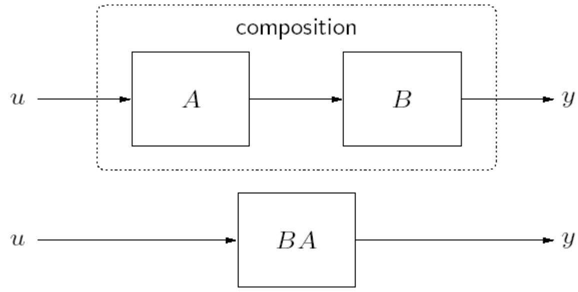
➔

$$\tilde{u}(t) = \begin{cases} 0 & t < T \\ u(t-T) & t \geq 0 \end{cases}$$

$$\tilde{y}(t) = \begin{cases} 0 & t < T \\ y(t-T) & t \geq 0 \end{cases}$$

Convolution & Properties [III]

- Composition of convolution systems corresponds to:
  - multiplication of transfer functions
  - convolution of impulse responses



- Thus:
  - We can manipulate block diagrams with transfer functions as if they were simple gains
  - convolution systems commute with each other

Convolution & Systems

- Convolution system with input u (u(t) = 0, t < 0) and output y:

$$y(t) = \int_0^t h(\tau)u(t-\tau)\,d\tau = \int_0^t h(t-\tau)u(\tau)\,d\tau$$

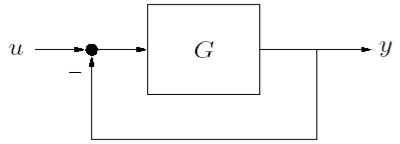
- abbreviated:

$$y = h * u$$

- in the frequency domain:

$$Y(s) = H(s)U(s)$$

Convolution & Feedback



- In the time domain:

$$y(t) = \int_0^t g(t-\tau)(u(\tau) - y(\tau))\, d\tau$$

- In the frequency domain:
  – Y=G(U-Y)
  ➔Y(s) = H(s)U(s)

$$H(s) = \frac{G(s)}{1+G(s)}$$

Graphical Understanding of Convolution

➔ For c(***τ***)= $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$:

1. Keep the function ***f (τ)*** fixed
2. **Flip** (invert) the function ***g(τ)*** about the vertical axis (τ=0)
   = this is ***g(-τ)***
3. **Shift** this frame (***g(-τ)***) along τ (horizontal axis) by $\mathbf{t_0}$.
   = this is ***g***$\mathbf{(t_0 - \tau)}$

➔ For c($\mathbf{t_0}$):

4. **c**$\mathbf{(t_0)}$ = the area under the product of ***f (τ)* and *g***$\mathbf{(t_0 - \tau)}$

5. Repeat this procedure, shifting the frame by different values (positive and negative) to obtain **c(t) for all values of t**.

Another (Better) View

e.g. convolution

x(n) = 1 2 3 4 5

h(n) = 3 2 1

x(k)	0 0 1 | 2 3 4 5	0 0 1 2 | 3 4 5	0 0 1 2 3 | 4 5	
h(n,k)	1 2 3 | 0 0 0 0	0 1 2 3 | 0 0 0	0 0 1 2 3 | 0 0	h(n-k)
y(n,k)	3	2 6	1 4 9	
y(n)	3	8	14	

Sum over all k

Notice the gain

Matrix Formulation of Convolution

$$\mathbf{y} = \mathbf{Hx}$$

$$\begin{bmatrix} 3 \\ 8 \\ 14 \\ 20 \\ 26 \\ 14 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

Toeplitz Matrix

Next Time…

- **DTFT**
  - **Then DT-*FFT!***
    - Then *DT-FFTW!*
- Review:
  - Chapter 10 of Lathi
- FFT-W: Is it really the Fastest FFT in the West?