Discrete System Analysis

ELEC 3004: **Systems**: Signals & Controls
Dr. Surya Singh

Lecture 7
(with material from Lathi and Cannon (Discrete Systems))

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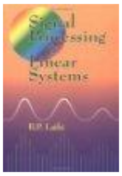
Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	**21-Mar**	**Discrete System Analysis**
	24-Mar	Convolution Review
	28-Mar	Holiday
	31-Mar	
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Feedback
9	3-May	Introduction to Feedback Control
	5-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	12-May	Digitial Control
11	16-May	Digital Control Design
	19-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
[TK5102.9.L38 1998]

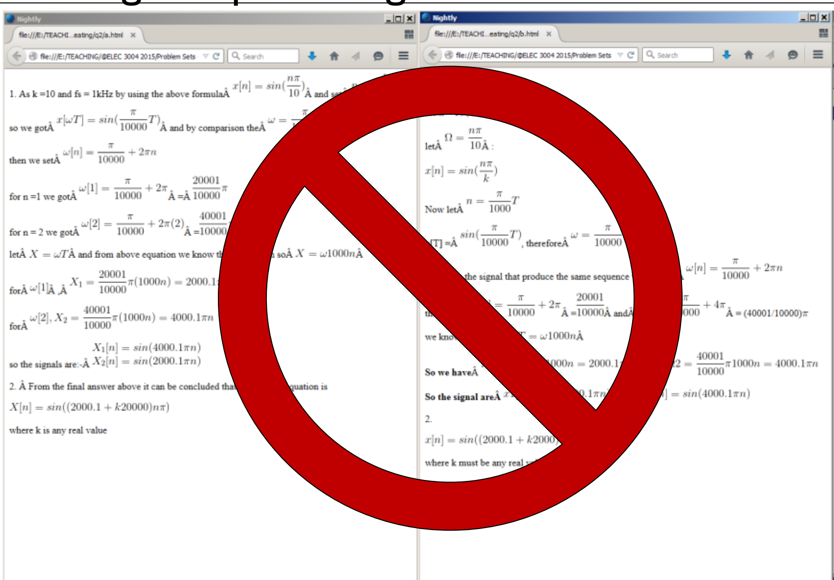
Today

- Chapter 8 (**Discrete-Time Signals and Systems**)
  - § 8.1 Introduction
  - § 8.2 **Some Useful Discrete-Time Signal Models**
  - § 8.3 **Sampling Continuous-Time Sinusoids & Aliasing**
  - § 8.4 Useful Signal Operations
  - § 8.5 Examples of Discrete-Time Systems

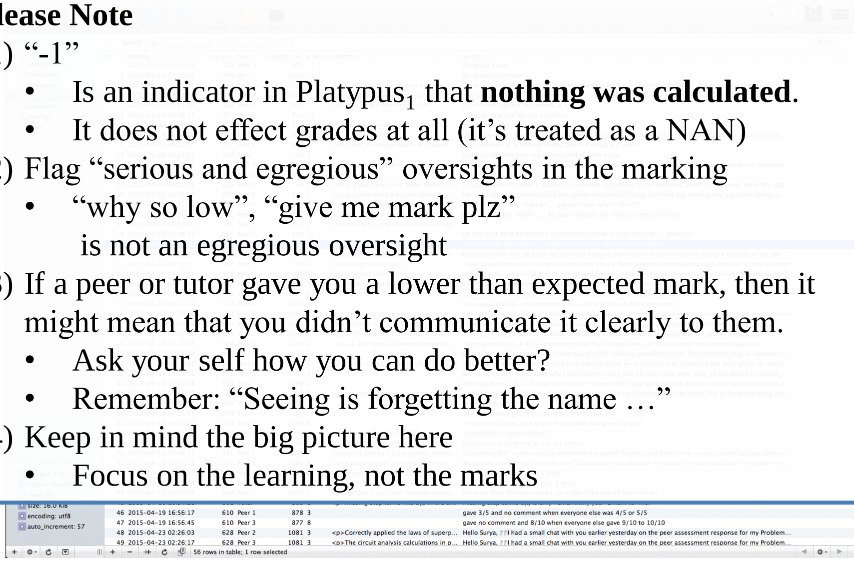
- Chapter 11 (**Discrete-Time System Analysis Using the *z*-Transform**)
  - § 11.1 The $\mathcal{Z}$-Transform
  - § 11.2 Some Properties of the *Z*-Transform

Next Time

Cheating: Despiration/Ignorance is not an excuse…



Feedback on the Peer Review/Flagged Answers



Please Note

(1) "-1"

- Is an indicator in Platypus$_1$ that **nothing was calculated**.
- It does not effect grades at all (it's treated as a NAN)

(2) Flag "serious and egregious" oversights in the marking

- "why so low", "give me mark plz" is not an egregious oversight

(3) If a peer or tutor gave you a lower than expected mark, then it might mean that you didn't communicate it clearly to them.

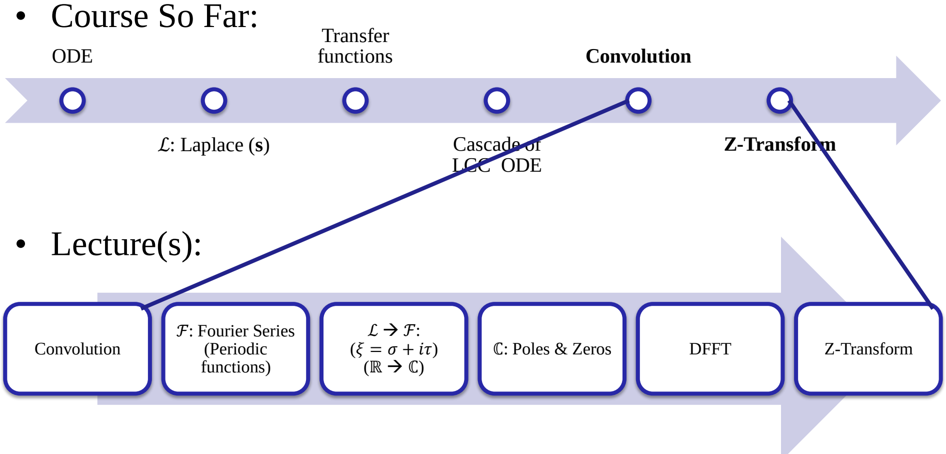
- Ask your self how you can do better?
- Remember: "Seeing is forgetting the name …"

(4) Keep in mind the big picture here

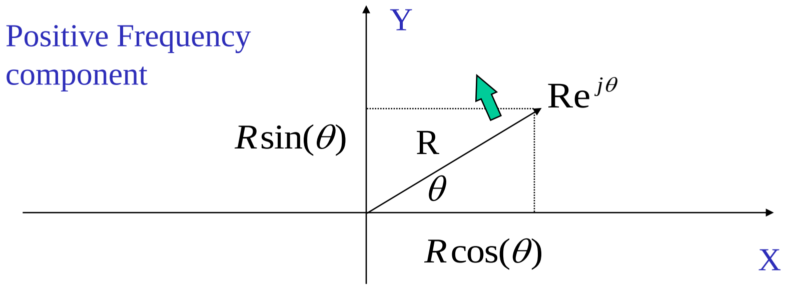
- Focus on the learning, not the marks

Lecture Overview

- Course So Far:
- Lecture(s):



Complex Numbers and Phasors

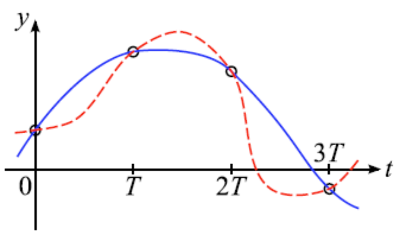


$$
\begin{aligned}
Re^{j\theta} &= (R\cos\theta, R\sin\theta) \\
&= R\cos\theta + jR\sin\theta \\
&= R(\cos\theta + j\sin\theta)
\end{aligned}
$$

Nyquist sampling theorem

What continuous signal is represented by a given set of samples?

Infinitely many continuous signals have the same discrete samples:



An answer is provided by Nyquist's sampling theorem:

> A signal $y(t)$ is uniquely defined by its samples $y(kT)$ if the sampling frequency is more than twice the bandwidth of $y(t)$.

Nyquist sampling theorem [2]

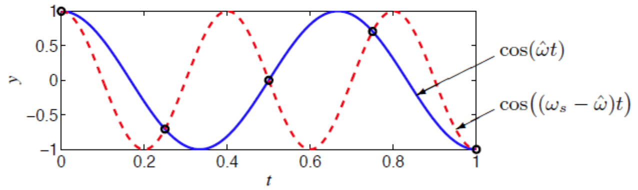
Example – Sampled sinusoidal signal

Sample $\cos(\hat\omega t)$ at frequency $\omega_s = 2\pi/T$:

$$y(t) = \cos(\hat\omega t) \xrightarrow{\text{sample}} y(kT) = \cos(k\,\hat\omega T) = \cos(2\pi k\,\hat\omega/\omega_s)$$

Identical samples are obtained from a sinusoid with frequency $\omega_s - \hat\omega$:

$$\cos\big((\omega_s - \hat\omega)t\big) \xrightarrow{\text{sample}} \cos\big(k(\omega_s - \hat\omega)T\big) = \cos(2\pi k - 2\pi k\,\hat\omega/\omega_s)$$
$$= \cos(2\pi k\,\hat\omega/\omega_s)$$



The spectrum of $y(kT)$ contains an alias at frequency $\omega_s - \hat\omega$!!

↑

(a copy of the original signal $y(t)$ shifted to a different frequency)

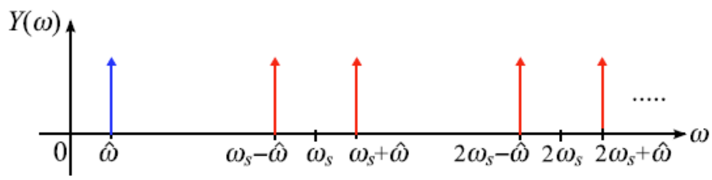
Nyquist sampling theorem

Example – Sampled sinusoidal signal

By the same argument, $y(kT)$ contains an infinite number of aliases at

$$\omega_s \pm \hat\omega,\ 2\omega_s \pm \hat\omega,\ 3\omega_s \pm \hat\omega, \ldots$$

spectrum of $y(kT)$:



The Nyquist sampling theorem requires $\omega_s > 2\hat\omega$

$\Downarrow$

$y(t)$ and alias spectra do not overlap

$y(t)$ can be recovered without distortion from $y(kT)$ (via low-pass filter)

Nonuniqueness of Discrete-Time Sinusoids [p. 553]

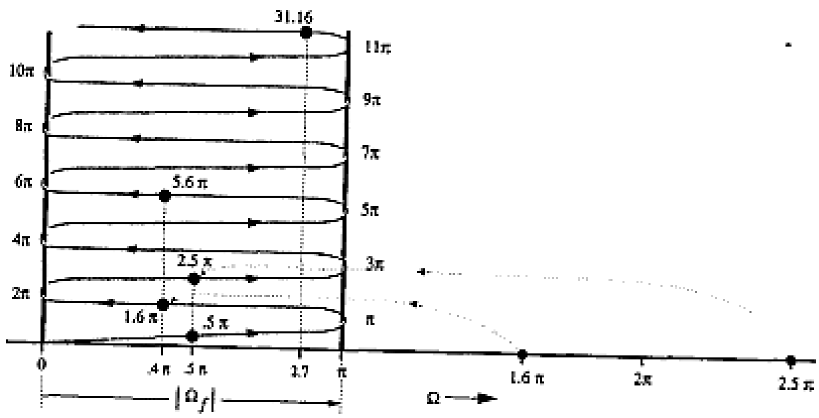
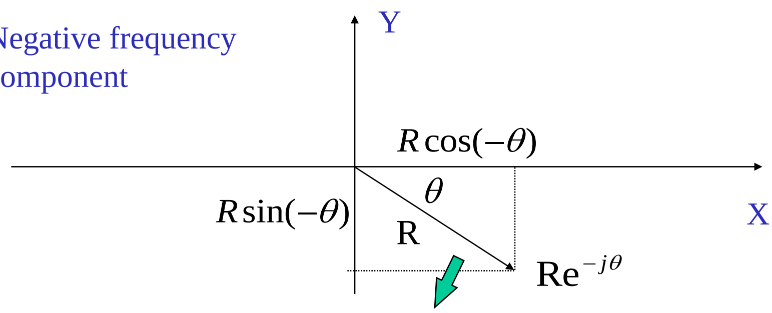


Fig. 8.11 A graphical artifice to determine the reduced frequency of a discrete-time sinusoid.

Complex Numbers and Phasors

Negative frequency component



$$\begin{aligned} Re^{-j\theta} &= (R\cos(-\theta), R\sin(-\theta)) \\ &= R\cos(-\theta) + jR\sin(-\theta) \\ &= R(\cos\theta - j\sin\theta) \end{aligned}$$

Positive and Negative Frequencies

- Frequency is the derivative of phase
  more nuanced than :

$$\frac{1}{\tau} = repetition\ rate$$

- Hence both positive and negative frequencies are possible.

- Compare
  - velocity vs speed
  - frequency vs repetition rate

Negative Frequency

- Q: What is negative frequency?
- A: A mathematical convenience

- Trigonometrical FS
  - periodic signal is made up from
  - sum 0 to ∞ of sine and cosines ‘harmonics’

- Complex Fourier Series & the Fourier Transform
  - use $\exp(\pm j\omega t)$ instead of $\cos(\omega t)$ and $\sin(\omega t)$
  - signal is sum from 0 to ∞ of $\exp(\pm j\omega t)$
  - same as sum -∞ to ∞ of $\exp(-j\omega t)$
  - which is more compact (i.e., less LaTeX!)

Linear Differential System Order

$$Q(D)y(t) = P(D)f(t)$$

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$
$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0$$

→ *y(t)=P(D)/Q(D) f(t)*
P(D): M
Q(D): N
(yes, N is de**N**ominator)

- In practice: $m \le n$

∵ **if** m > *n:*
then the system is an
$(m - n)^{th}$ -order differentiator of high-frequency signals!

- Derivatives magnify noise!

Zero-Input | Zero-State

Total response = zero-input response + zero-state response

Zero Input

- = The system response when the input *f(t)* = 0 so that it is the result of internal system conditions (such as energy storages, initial conditions) alone.
- It is **independent of the external input**.

Zero-State

- = the system response to the external input *f (t)* when the system is in zero state, meaning the absence of all internal energy storages; that is, all initial conditions are zero.

System Stability

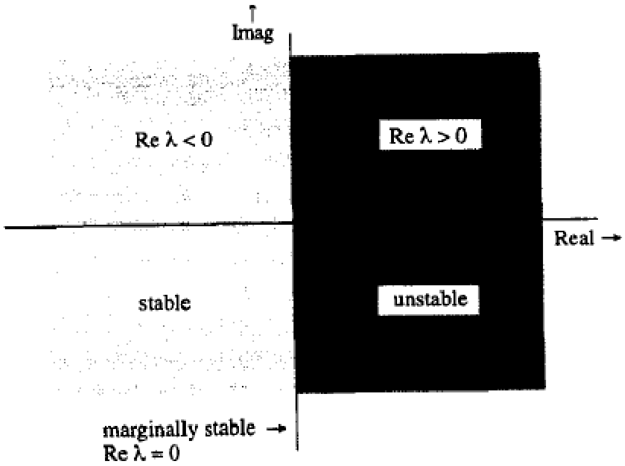
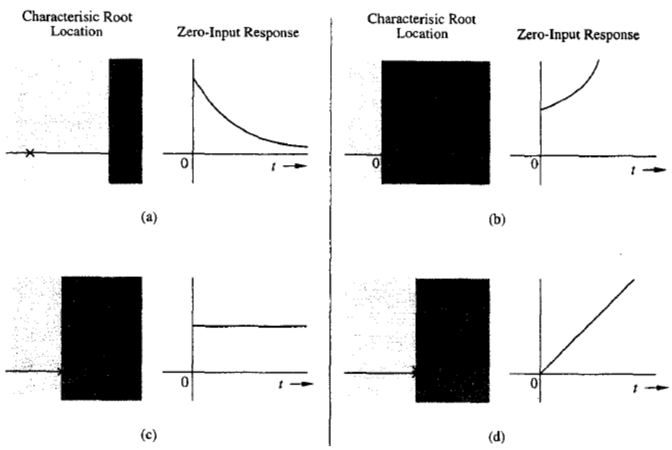


Fig. 2.15 Characteristic roots location and system stability.

Lathi, p. 149

System Stability [II]



Lathi, p. 150

System Stability [III]

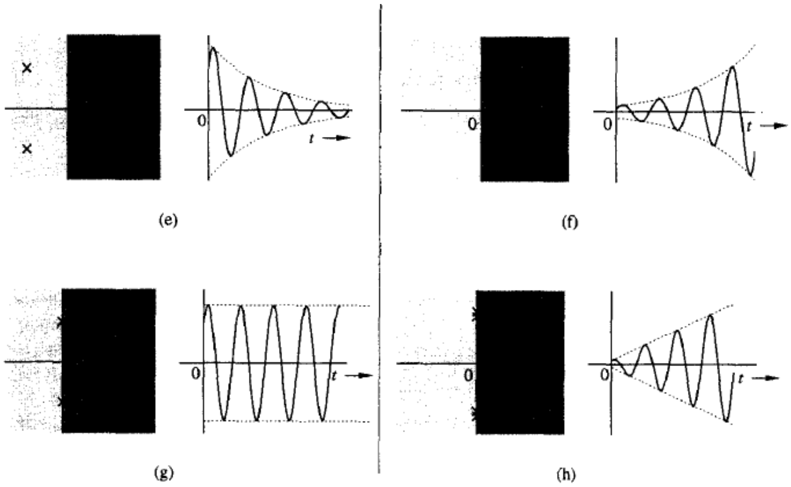
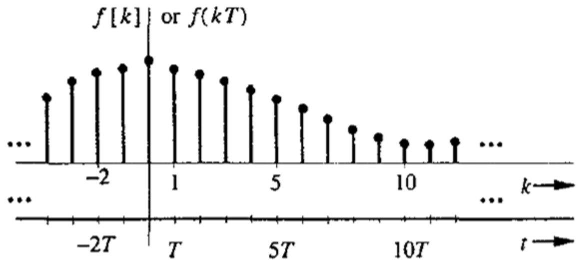


Fig. 2.16 Location of characteristic roots and the corresponding characteristic modes.

Discrete-Time
Signal Analysis

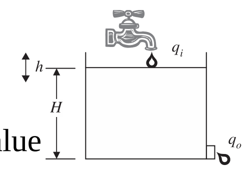
Discrete-Time Signal: *f[k]*



- Discrete-time signal:
  - May be denoted by *f(kT)*, where time *t* values are specified at *t* = *kT*
  - ***OR*** *f[k]* and viewed as a function of *k* (*k* ∈ integer)
- Continuous-time exponential:
  - $f(t) = e^{-t}$, sampled at *T* = 0.1 ➔ $f(kT) = e^{-kT} = e^{-0.1k}$

Why e^{-kT} ?

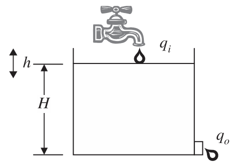
- Solution to First-Order ODE!
- Ex: "Tank" Fill
- Where:
  - H=steady-state fluid height in the tank
  - h=height perturbation from the nominal value
  - Q=steady-state flow rate through the tank
  - q_i=inflow perturbation from the nominal value
  - q_0=outflow perturbation from the nominal value



- Goal: Maintain H by adjusting Q.

Why e^{-kT} ? [2]

- $h = Rq_0$
- $\frac{dC(h+H)}{dt} = (q_i + Q) - (q_0 + Q)$
- $\frac{dh}{dt} + \frac{h}{\tau} = \frac{q_i}{C}$
- $\tau = RC$
- Solution:
  - $h(t) = e^{\frac{t-t_0}{\tau}} h(t_0) + \frac{1}{C}\int_{t_0}^{t} e^{\frac{t-\lambda}{\tau}} q_i(\lambda) d\lambda$
- For a fixed period of time (T) and steps k=0,1,2,…:
  - $h(k+1) = e^{\frac{-T}{\tau}} h(k) + R\left[1 - e^{-\frac{T}{\tau}}\right] q_i(k)$



So Why Is this a Concern? **Difference equations**

Difference equations arise in problems where the independent variable, usually time, is assumed to have a discrete set of possible values. The nonlinear difference equation

$$y(k+n) = f[y(k+n-1),\ y(k+n-2),\ \ldots,\ y(k+1),\ y(k),\ u(k+n),\ u(k+n-1),\ \ldots,\ u(k+1),\ u(k)] \quad (2.1)$$

with forcing function $u(k)$ is said to be of order n because the difference between the highest and lowest time arguments of $y(.)$ and $u(.)$ is n. The equations we deal with in this text are almost exclusively linear and are of the form

$$\begin{aligned} y(k+n) &+ a_{n-1}y(k+n-1) + \cdots + a_1 y(k+1) + a_0 y(k) \\ &= b_n u(k+n) + b_{n-1}u(k+n-1) + \cdots + b_1 u(k+1) + b_0 u(k) \end{aligned} \quad (2.2)$$

We further assume that the coefficients a_i, b_i, $i = 0, 1, 2, \ldots$, are constant. The difference equation is then referred to as linear time invariant, or LTI. If the forcing function $u(k)$ is equal to zero, the equation is said to be *homogeneous*.

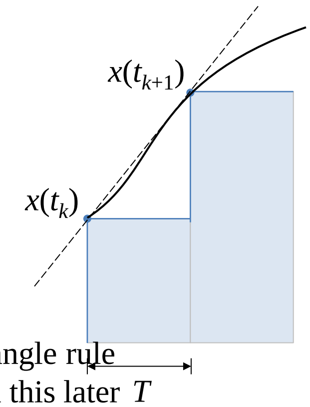
Difference equations can be solved using classical methods analogous to those available for differential equations. Alternatively, z-transforms provide a convenient approach for solving LTI equations, as discussed in the next section.

Euler's method*

- Dynamic systems can be approximated† by recognising that:

$$\dot{x} \cong \frac{x(k+1) - x(k)}{T}$$

- As $T \to 0$, approximation error approaches 0



*Also known as the forward rectangle rule
†Just an approximation – more on this later

Difference Equation: Euler's approximation

$$\frac{dx}{dt} = \lim_{\delta t \to 0} \frac{x(t+\delta t) - x(t)}{\delta t} \quad \Longrightarrow \quad \frac{dx}{dt} \approx \frac{x_{k+1} - x_k}{T}$$

For small enough T, this can be used to approximate a continuous controller by a discrete controller:

1. Laplace transform $\longrightarrow$ differential equation

   e.g.
   $$D(s) = \frac{U(s)}{E(s)} = \frac{K(s+a)}{(s+b)} \quad \Longrightarrow \quad \frac{du}{dt} + b\,u = K\left(\frac{de}{dt} + a\,e\right)$$

2. Differential equation $\longrightarrow$ difference equation

   e.g.
   $$\frac{u_{k+1} - u_k}{T} + b\,u_k = K\left(\frac{e_{k+1} - e_k}{T} + a\,e_k\right)$$
   $$\Longrightarrow u_{k+1} = (1 - bT)u_k + K e_{k+1} + K(aT - 1)e_k$$
   $$= -a_1 u_k + b_0 e_{k+1} + b_1 e_k$$

Difference Equation: Euler's approximation [2]

Discrete controller recurrence equation:

$$u_k = -a_1 u_{k-1} - a_2 u_{k-2} - \ldots + b_0 e_k + b_1 e_{k-1} + \ldots$$

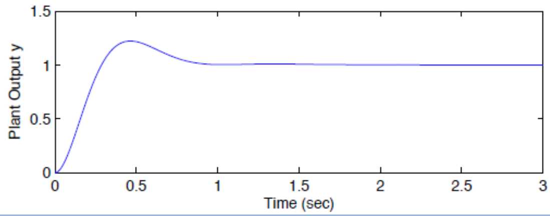
coefficients $a_1, a_2, \ldots,\ b_0, b_1, \ldots$ depend on T

Example

Controller: $D(s) = \frac{K(s+a)}{(s+b)}$, $K = 70,\ a = 2$ rad s^{-1}, $b = 10$ rad s^{-1}

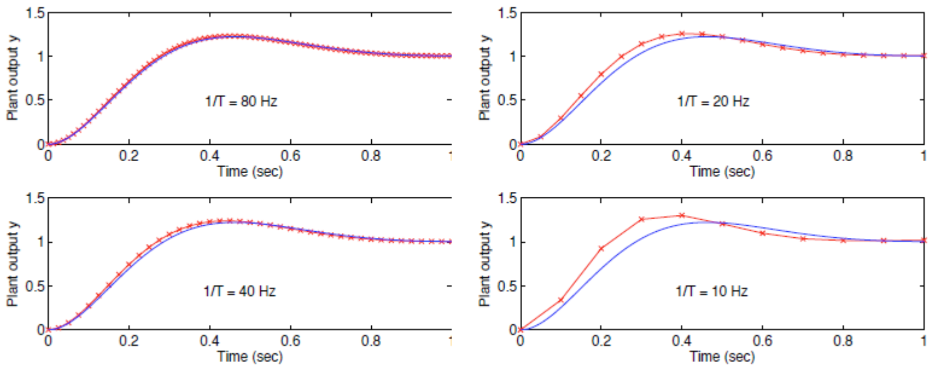
Plant: $G(s) = \frac{1}{s(s+1)}$

- Step response with continuous controller:



Difference Equation: Euler's approximation [3]

- Step responses with discrete controller:



Difference Equation: Euler's approximation [4]

- At high enough sample rates Euler's approximation works well:
  - discrete controller ≈ continuous controller
- **But** if sampling is not fast enough the approximation is poor:

$$\frac{1}{T} > 30 \times [System\ Bandwidth]$$

- Works, but Not Efficient (**η**)
- Later (May) We consider:
  - better ways of representing continuous systems in discrete-time
  - ways of analysing discrete controllers directly

Linear Differential Systems

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) = b_m\frac{d^m f}{dt^m} + b_{m-1}\frac{d^{m-1}f}{dt^{m-1}} + \cdots + b_1\frac{df}{dt} + b_0 f(t) \qquad (2.1a)$$

where all the coefficients a_i and b_i are constants. Using operational notation D to represent d/dt, we can express this equation as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)\,y(t) = (b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0)\,f(t) \qquad (2.1b)$$

or

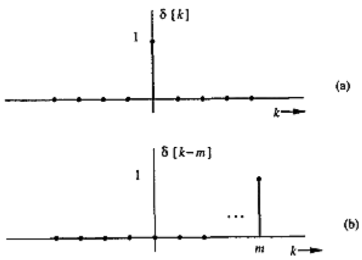
$$Q(D)y(t) = P(D)f(t) \qquad (2.1c)$$

where the polynomials $Q(D)$ and $P(D)$ are

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 \qquad (2.2a)$$

$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0 \qquad (2.2b)$$

Discrete-Time Impulse Function $\delta[k]$



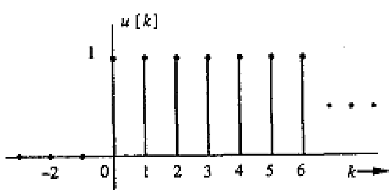
The discrete-time counterpart of the continuous-time impulse function $\delta(t)$ is $\delta[k]$, defined by

$$\delta[k] = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{8.1}$$

This function, also called the unit impulse sequence, is shown in Fig. 8.3a. The time-shifted impulse sequence $\delta[k-m]$ is depicted in Fig. 8.3b. Unlike its continuous-time counterpart $\delta(t)$, this is a very simple function without any mystery.

Later, we shall express an arbitrary input $f[k]$ in terms of impulse components. The (zero-state) system response to input $f[k]$ can then be obtained as the sum of system responses to impulse components of $f[k]$.

Discrete-Time Unit Step Function $u[k]$



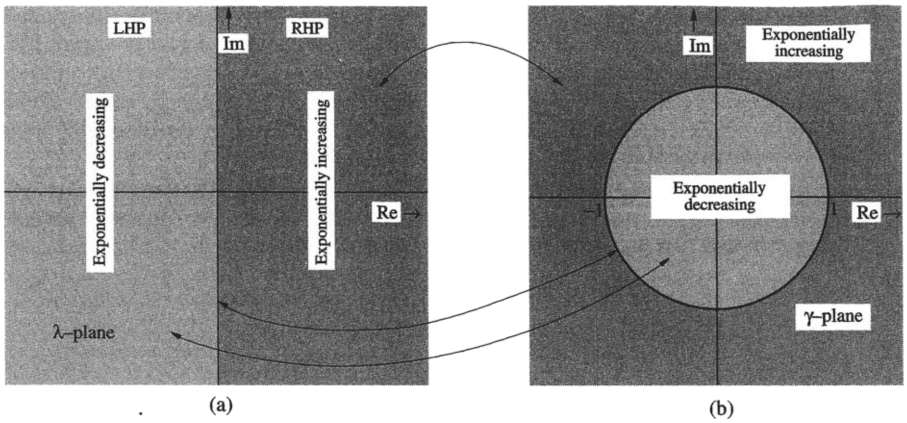
The discrete-time counterpart of the unit step function $u(t)$ is $u[k]$ (Fig. 8.4), defined by

$$u[k] = \begin{cases} 1 & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases} \tag{8.2}$$

If we want a signal to start at $k = 0$ (so that it has a zero value for all $k < 0$), we need only multiply the signal with $u[k]$.

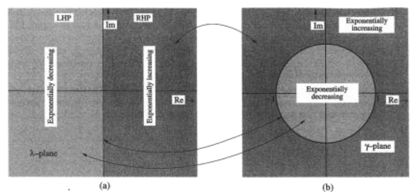
Discrete-Time Exponential γ^k

$$e^{\lambda k} = \gamma^k$$



Discrete-Time Exponential γ^k

- $e^{\lambda k} = \gamma^k$
- $\gamma = e^{\lambda}$ or $\lambda = \ln \gamma$

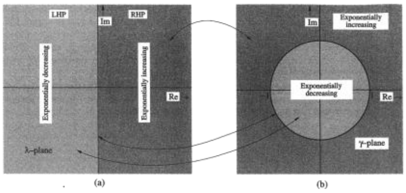


- In discrete-time systems, unlike the continuous-time case, the form γ^k proves more convenient than the form $e^{\lambda k}$

Why?

- Consider $e^{j\Omega k}$ ($\lambda = j\Omega$ ∴ constant amplitude oscillatory)
- $e^{j\Omega k}$ ➔ γ^k, for $\gamma \equiv e^{j\Omega}$
- $\left|e^{j\Omega}\right| = 1$, hence $|\gamma| = 1$

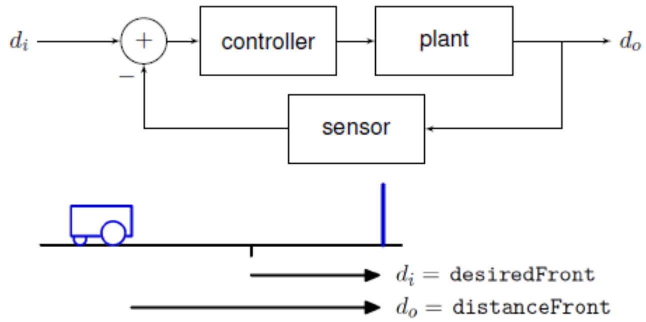
Discrete-Time Exponential γ^k



- Consider $e^{\lambda k}$
  When λ: LHP
- Then
- $\gamma = e^{\lambda}$
- $\gamma = e^{\lambda} = e^{a+jb} = e^{a}e^{jb}$
- $|\gamma| = \left|e^{a}e^{jb}\right| = |e^{a}| \because \left|e^{jb}\right| = 1$

Discrete-Time **System Analysis**

Simple Controller Goes Digital



plant: $y[n] = y[n-1] - Tu[n-1]$

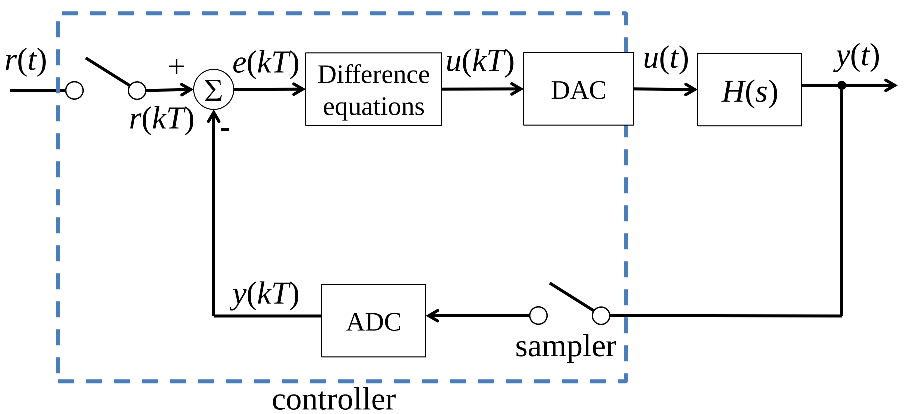
sensor: $y[n] = u[n-1]$

controller: $y[n] = Ku[n]$

Complex system behaviors, depending on K

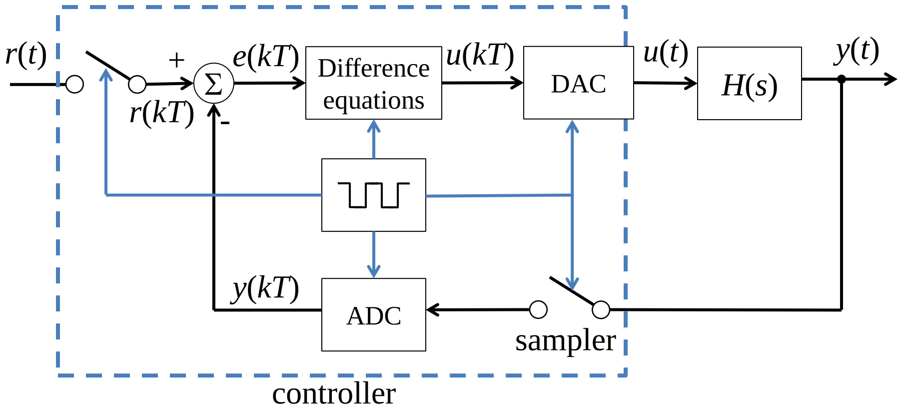
Digitisation

- Continuous signals sampled with period T
- kth control value computed at $t_k = kT$



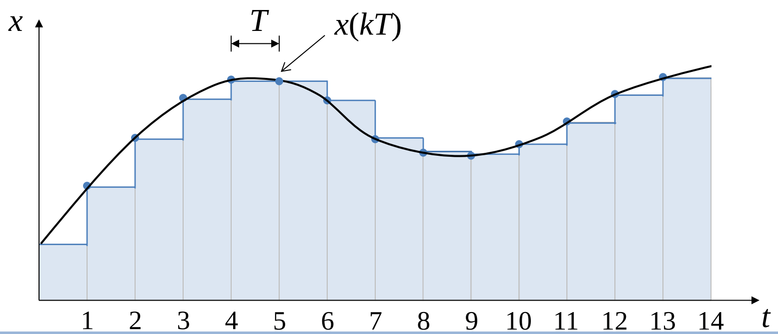
Digitisation

- Continuous signals sampled with period T
- kth control value computed at $t_k = kT$



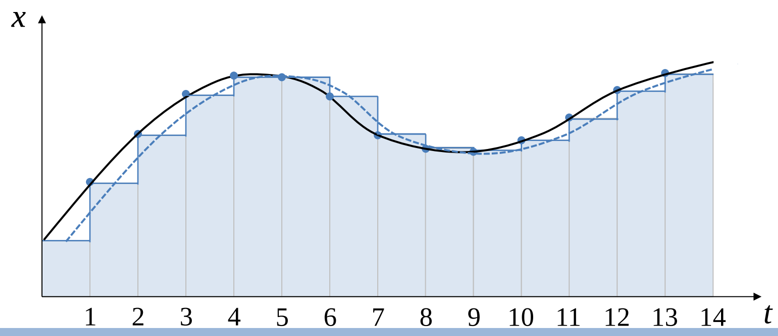
Return to the discrete domain

- Recall that continuous signals can be represented by a series of samples with period T



Zero Order Hold

- An output value of a synthesised signal is held constant until the next value is ready
  - This introduces an effective delay of $T/2$



Effect of ZOH Sampling

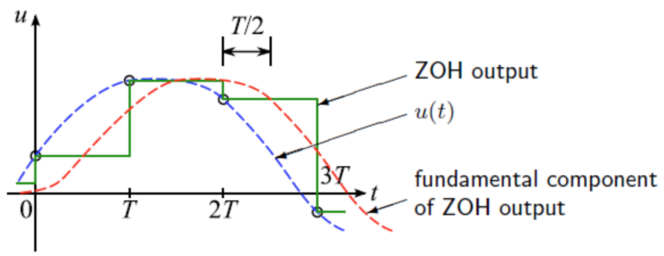
Lower sample rate $\Longrightarrow$ more oscillatory response

— Why?

Sampling and reconstruction introduces:

delay in time domain
& phase lag in freq. domain $\leftarrow$ can destabilize the closed loop system

On average $u(kT)$ is delayed by $T/2$ relative to $u(t)$ due to the ZOH:



Effect of ZOH Sampling

The ZOH delay of $T/2$ (sec) causes

phase lag $= \omega T/2$ (rad) at ω rad s^{-1}

phase lag $= \pi/2 = 90°$ at $\omega = \pi/T$ [= Nyquist rate]

phase lag $= \pi/30 = 6°$ at $\omega = \pi/(15T)$

⋆ $90°$ phase lag could be catastrophic

⋆ If $\omega_{\text{samp}} > 30 \times \omega_{\max}$,

then system bandwidth: $\omega_{\max} < \pi/(15T)$,
so the maximum phase lag is less than $6°$

↖ usually safe to ignore

⋆ Any time needed to compute u_k causes additional delay (!)

Properties of the ROC

➔The ROC is always defined by circles centered around the origin.

$h[k]r^{-k}$ is absolutely summable, where $r = |z|$.

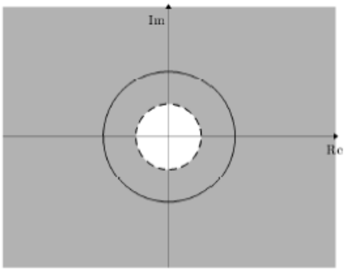
➔Right-sided signals have "outsided" ROCs.

if $\exists n_0$ such that $h[n] = 0 \; \forall n < n_0$, then if $r_0 \in$ ROC, then $\forall r$ with $r_0 < r < \infty$ are also in the ROC.

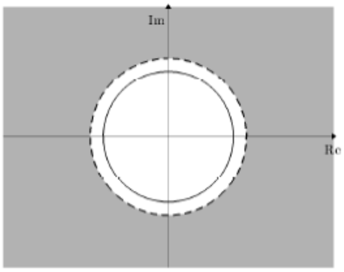
➔Left-sided signals have "insided" ROCs.
(with $\forall$r within $0<r<r_0$)

Region of Convergence (ROC) Plots

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1 - az^{-1}}, \quad |z| > |a|$$



$a = .5$



$a = 1.2$

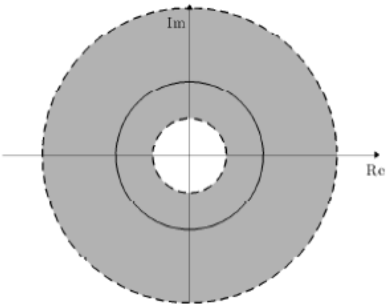
Combinations of Signals

$$y_1[n] = \begin{cases} ba^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$a = .5$

$$y_2[n] = \begin{cases} 0 & n \geq 0 \\ -ba^n & n < 0 \end{cases}$$

$a = 2$



ROC for $\alpha_1 y_1[n] + \alpha_2 y_2[n]$

Back to the future

A quick note on causality:

- Calculating the "(k+1)th" value of a signal using

$$y(k+1) = \underbrace{x(k+1)}_{\text{future value}} + \underbrace{Ax(k) - By(k)}_{\text{current values}}$$

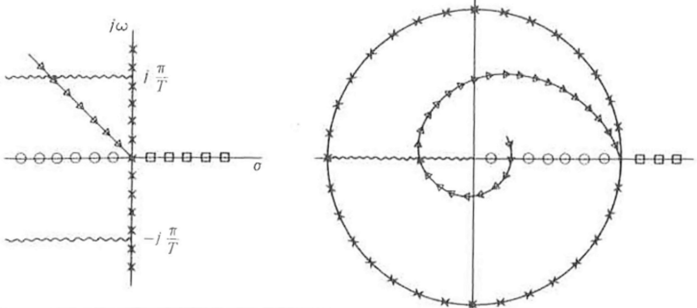
relies on also knowing the next (future) value of $x(t)$.

(this requires very advanced technology!)

- Real systems always run with a delay:

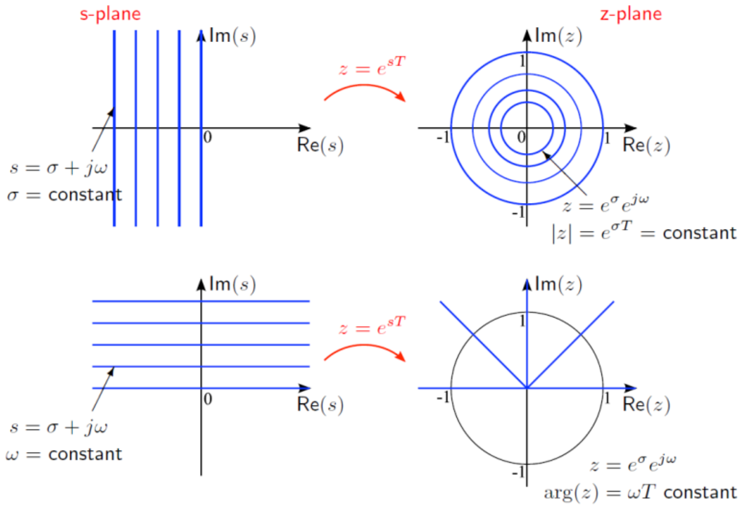
$$y(k) = x(k) + Ax(k-1) - By(k-1)$$

Hint: Use γ to Transform s ↔ z: z=e^{sT}



s-plane s-plane	**Symbol**	**z-plane** z-plane
$s = j\omega$ Real frequency axis (a)	× × ×	$\lvert z \rvert = 1$ Unit circle (b)
$s = \sigma \geq 0$	□□□	$z = r \geq 1$
$s = \sigma \leq 0$	○○○	$z = r,\ 0 \leq r \leq 1$
$s = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} = -a + jb$ Constant damping ratio if ζ is fixed and ω_n varies	△△△	$z = re^{j\theta}$ where $r = \exp(-\zeta\omega_n T) = e^{-aT}$, $\theta = \omega_n T\sqrt{1-\zeta^2} = bT$ Logarithmic spiral
$s = \pm j(\pi/T) + \sigma, \sigma \leq 0$	∼∼∼∼	$z = -r$

S-Plane to z-Plane [1/2]



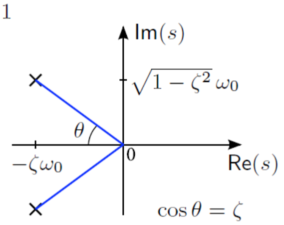
S-Plane to z-Plane [2/2]

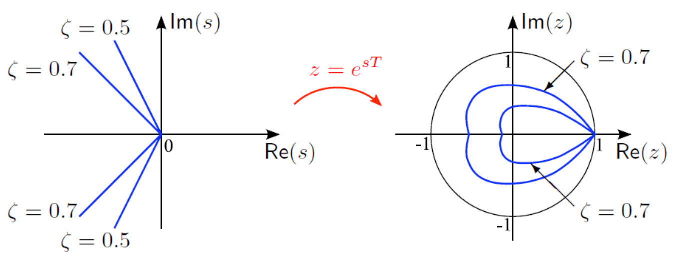
Pole locations for constant damping ratio $\zeta < 1$

$$s^2 + \zeta\omega_0 s + \omega_0^2 = 0$$

$$\Downarrow$$

$$s = -\zeta\omega_0 \pm j\sqrt{1-\zeta^2}\,\omega_0$$





$s = -\zeta\omega_0 + j\sqrt{1-\zeta^2}\,\omega_0$: $\zeta =$ constant $\quad z = e^{-\zeta\omega_0 T} e^{-j\sqrt{1-\zeta^2}\,\omega_0 T}$

Relationship with s-plane poles and z-plane transforms

If $F(s)$ has a pole at $s = a$
then $F(z)$ has a pole at $z = e^{aT}$

$\uparrow$

consistent with $z = e^{sT}$

What about transfer functions?

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

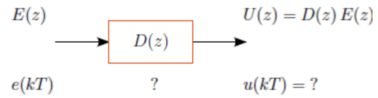
$\downarrow$

If $G(s)$ has poles $s = a_i$
then $G(z)$ has poles $z = e^{a_i T}$

but the zeros are unrelated

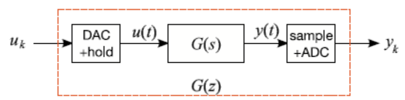
$\mathcal{F}(s)$	$f(kT)$	$F(z)$
$\frac{1}{s}$	$1(kT)$	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\frac{a}{s(s+a)}$	$1-e^{-akT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\frac{b-1}{(s+a)(s+b)}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
$\frac{a}{s^2+a^2}$	$\sin akT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$
$\frac{b}{(s+a)^2+b^2}$	$e^{-akT}\sin bkT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$

$s \leftrightarrow z$: Pulse Transfer Function Models



- Pulse in Discrete is equivalent to Dirac-δ

$$e_k = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k > 0 \end{cases}$$



➔

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}_{t=kT}\right\} = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

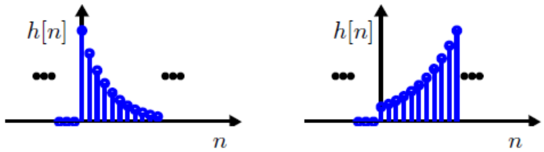
Source: Oxford 2A2 Discrete Systems, Tutorial Notes p. 26

z-Transforms for Difference Equations

- First-order linear constant coefficient difference equation:

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$



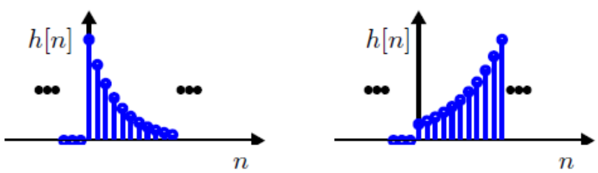
$$h[n] = \begin{cases} ba^n & n \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$H(z) = \sum_{k=0}^{\infty} ba^k z^{-k} = b \sum_{k=0}^{\infty} \left(\frac{a}{z}\right)^k = \frac{b}{1 - az^{-1}}, \quad \text{when } |z| > |a|.$$

z-Transforms for Difference Equations

First-order linear constant coefficient difference equation:

$$y[n] = ay[n-1] + bu[n]$$



$$y[n] - ay[n-1] = bu[n]$$

$$\updownarrow$$

$$Y(z) - az^{-1}Y(z) = bU(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1 - az^{-1}}, \text{ when does it converge?}$$

Properties of the the *z*-transform

- Some useful properties
  - **Delay by $\boldsymbol{n}$ samples**: $\mathcal{Z}\{f(k-n)\} = z^{-n}F(z)$
  - **Linear**: $\mathcal{Z}\{af(k)+bg(k)\} = \mathrm{a}F(z)+bG(z)$
  - **Convolution**: $\mathcal{Z}\{f(k)*g(k)\} = F(z)G(z)$

So, all those block diagram manipulation tools you know and love will work just the same!

Next Time…

- **z-Transforms!**

- Review:
  - Chapter 11 of Lathi

- Lower Sampling Rate means More Oscillation ☹

OTHERS