http://elec3004.com

Sampling Theory: Antialiasing Filters

ELEC 3004: **Systems**: Signals & Controls

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Lecture 6

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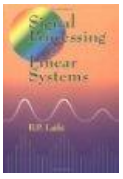
Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	24-Mar	Convolution Review
	28-Mar	Holiday
	31-Mar	
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Feedback
9	3-May	Introduction to Feedback Control
	5-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	12-May	Digitial Control
11	16-May	Digital Control Design
	19-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	31-May	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 5 (**Sampling**)
  - § 5.1: **The Sampling Theorem**

Sampling!

REVIEW

Sampling Theorem

- The Nyquist criterion states:

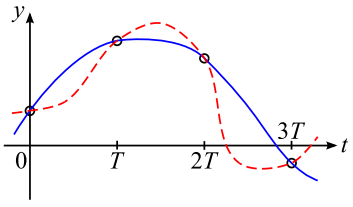
To prevent aliasing, a ***bandlimited*** *signal of bandwidth* w_B *rad/s must be sampled at a rate greater than* $2w_B$ *rad/s*

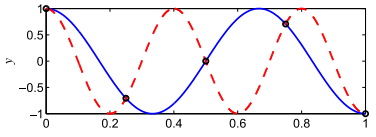
$$w_s > 2w_B$$

Note: this is a > sign not a ≥

Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

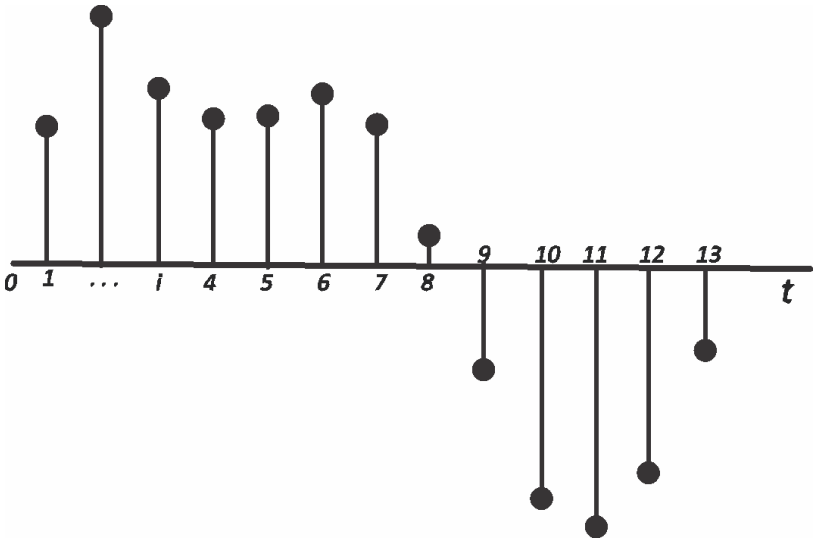
Nyquist Sampling Theorem and Aliasing





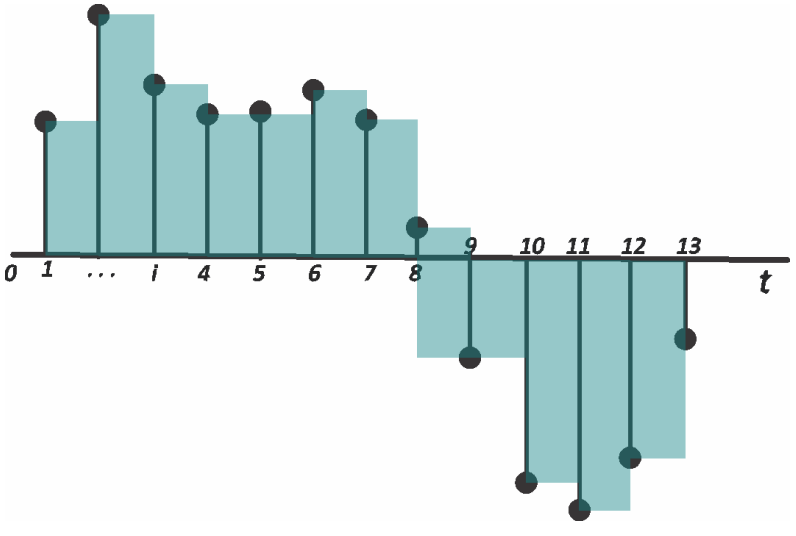
- A signal $y(t)$ is uniquely defined by its samples $y(kT)$ if the sampling frequency is more than twice the bandwidth of $y(t)$.

Reconstruction



Reconstruction

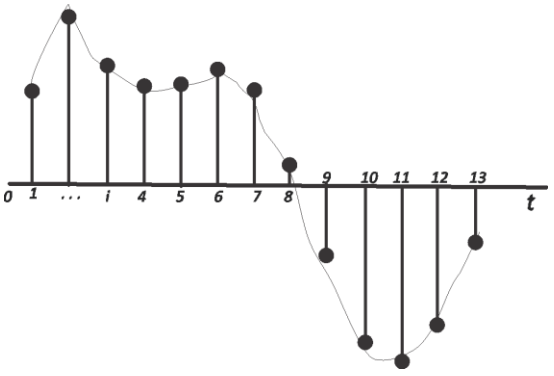
- Zero-Order Hold [ZOH]



Reconstruction

- Whittaker–Shannon interpolation formula

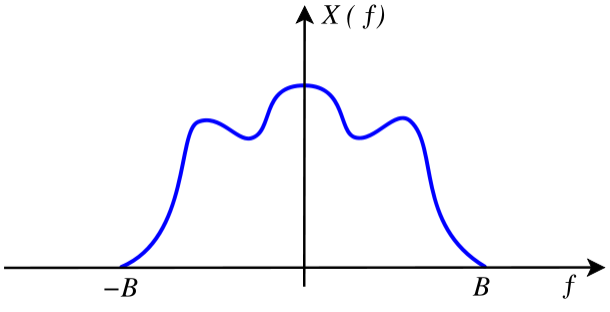
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \,\mathrm{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$



Reconstruction

- Whittaker–Shannon interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$



Aliasing

Alliasing

- Aliasing - through sampling, two entirely different analog sinusoids take on the same "discrete time" identity

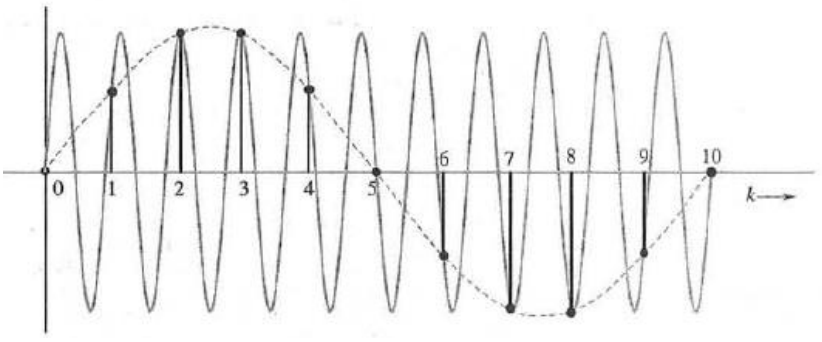
For f[k]=cosΩk, Ω=ωT:

The period has to be less than F_h (highest frequency): $T \leq \frac{1}{2\mathcal{F}_h}$

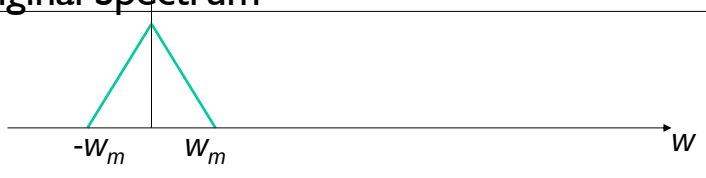
Thus: $0 \leq \mathcal{F} \leq \frac{\mathcal{F}_s}{2}$

ω_f: aliased frequency: $\omega T = \omega_f T + 2\pi m$

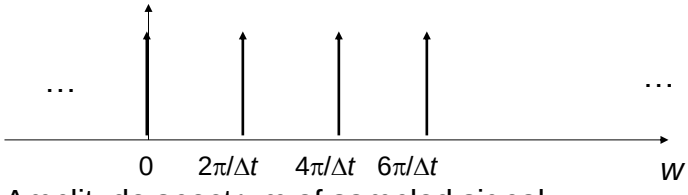
Aliasing: Another view of this



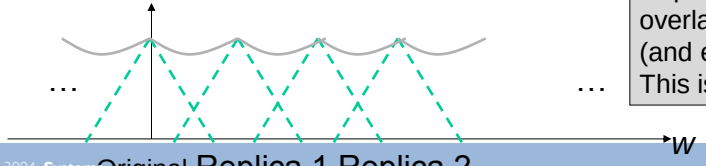
Original Spectrum



Fourier transform of impulse train (sampling signal)



Amplitude spectrum of sampled signal

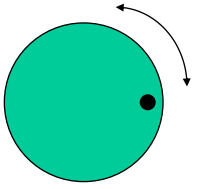


Original Replica 1 Replica 2 …

Replica spectrums overlap with original (and each other) This is **Aliasing**

Another way to see Aliasing Too!

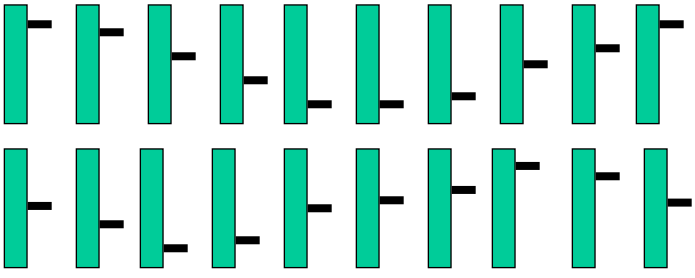
Rotating wheel and peg



Need both top and front view to determine rotation

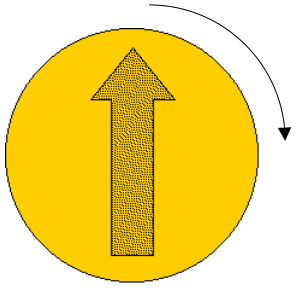
Top View

Front View

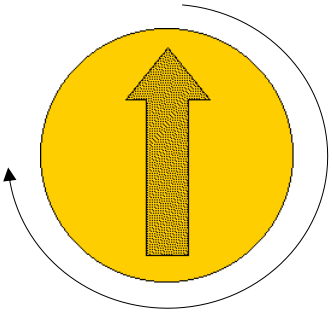


Temporal Aliasing

90^o clockwise rotation/frame clockwise rotation perceived



270^o clockwise rotation/frame (90^o) anticlockwise rotation perceived i.e., aliasing



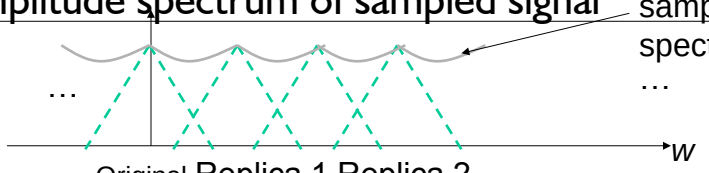
Require LPF to 'blur' motion

Sampling & Antialiasing

Practical Anti-aliasing Filter

- Non-ideal filter
  - $w_c = \frac{w_s}{2}$
- Filter usually 4th – 6th order (e.g., Butterworth)
  - so frequencies > w_c may still be present
  - not higher order as phase response gets worse
- Luckily, most real signals
  - are lowpass in nature
    - signal power reduces with increasing frequency
  - e.g., speech naturally bandlimited (say < 8KHz)
  - Natural signals have a $\sim\frac{1}{f}$ spectrum
  - so, in practice aliasing is not (usually) a problem

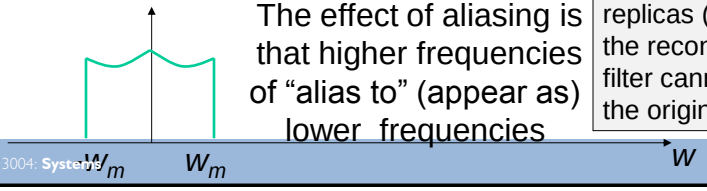
Amplitude spectrum of sampled signal



Original Replica 1 Replica 2 …

Reconstruction filter (ideal lowpass filter)

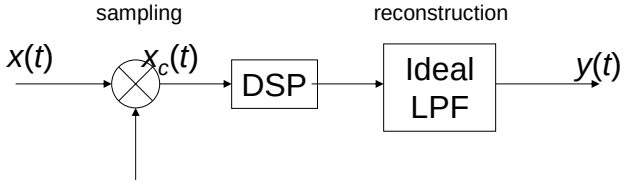
Spectrum of reconstructed signal



The effect of aliasing is that higher frequencies of "alias to" (appear as) lower frequencies

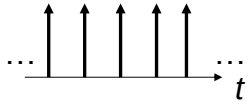
Due to overlapping replicas (aliasing) the reconstruction filter cannot recover the original spectrum

Mathematics of Sampling and Reconstruction

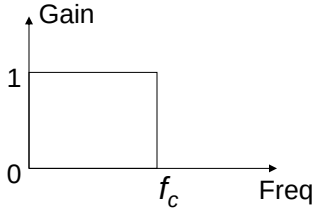


Impulse train

$\delta_T(t) = \sum\delta(t - n\Delta t)$



Sampling frequency $f_s = 1/\Delta t$



Cut-off frequency = f_c

Frequency Domain Analysis of Sampling

- Consider the case where the DSP performs no filtering operations
  - i.e., only passes $x_c(t)$ to the reconstruction filter
- To understand we need to look at the frequency domain
- Sampling: we know
  - multiplication in time ≡ convolution in frequency
  - F{x(t)} = X(w)
  - F{δT(t)} = ∑δ(w - 2πn/Δt),
  - i.e., an impulse train in the frequency domain

Frequency Domain Analysis of Sampling

- In the frequency domain we have

$$X_c(w) = \frac{1}{2\pi}\left(X(w) * \frac{2\pi}{\Delta t}\sum_n \delta\left(w - \frac{2\pi n}{\Delta t}\right)\right)$$
$$= \frac{1}{\Delta t}\sum_n X\left(w - \frac{2\pi n}{\Delta t}\right)$$

Remember convolution with an impulse? Same idea for an impulse train

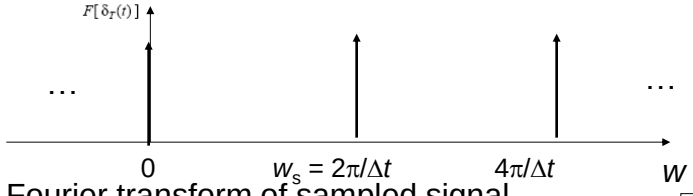
  - Let's look at an example
    - where X(w) is triangular function
    - with maximum frequency w_m rad/s
    - being sampled by an impulse train, of frequency w_s rad/s

Sampling Frequency

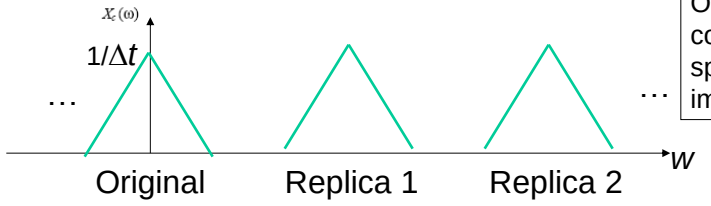
- In this example it was possible to recover the original signal from the discrete-time samples
- But is this always the case?
- Consider an example where the sampling frequency w_s is reduced
  - i.e., Δt is increased

Fourier transform of original signal X(ω) (signal spectrum)

Fourier transform of impulse train $\delta_T(\omega/2\pi)$ (sampling signal)

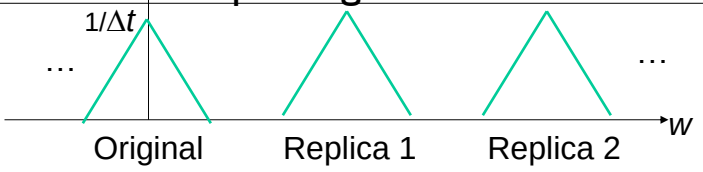


Fourier transform of sampled signal

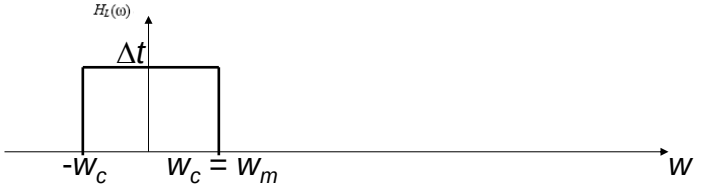


Original spectrum convolved with spectrum of impulse train

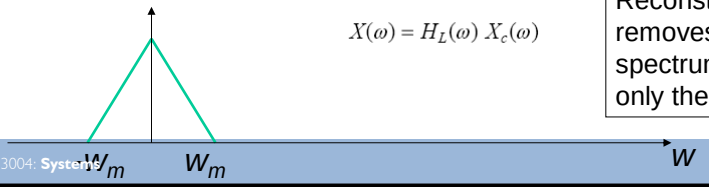
Spectrum of sampled signal



Reconstruction filter (ideal lowpass filter)

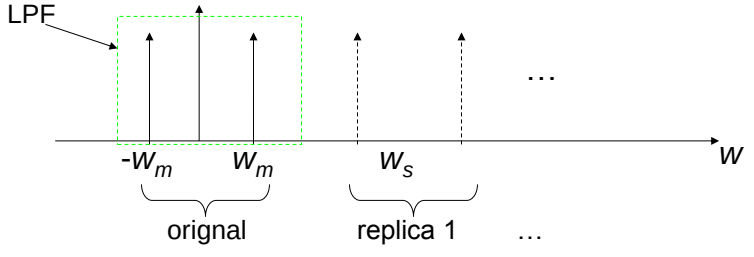


Spectrum of reconstructed signal



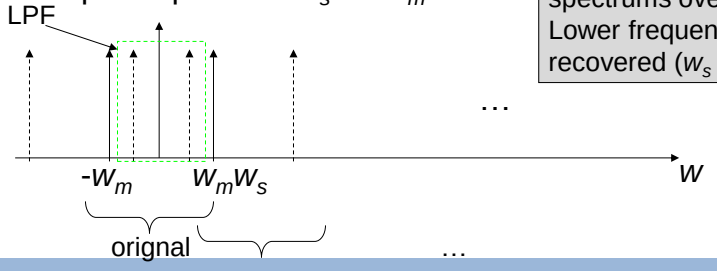
Reconstruction filter removes the replica spectrums & leaves only the original

Sampled Spectrum $w_s > 2wm$



original freq recovered

Sampled Spectrum $w_s < 2w_m$



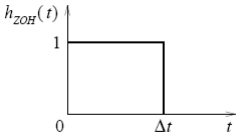
Original and replica spectrums overlap
Lower frequency recovered ($w_s - w_m$)

Sampling and Reconstruction Theory and Practice

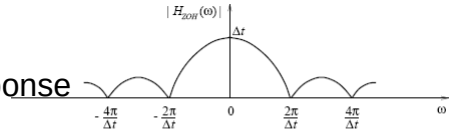
- Signal is bandlimited to bandwidth WB
  - Problem: real signals are not bandlimited
    - Therefore, require (non-ideal) anti-aliasing filter
- Signal multiplied by ideal impulse train
  - problems: sample pulses have finite width
  - and not $\otimes$ in practice, but sample & hold circuit
- Samples discrete-time, continuous valued
  - Problem: require discrete values for DSP
    - Therefore, require A/D converter (quantisation)
- Ideal lowpass reconstruction ('sinc' interpolation)
  - problems: ideal lowpass filter not available
    - Therefore, use D/A converter and practical lowpass filter

Zero Order Hold (ZOH)

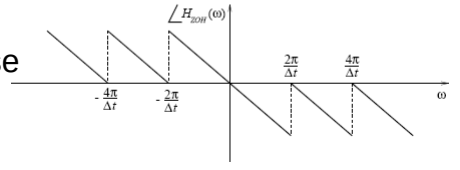
ZOH impulse response



ZOH amplitude response



ZOH phase response



Time Domain Analysis of Reconstruction

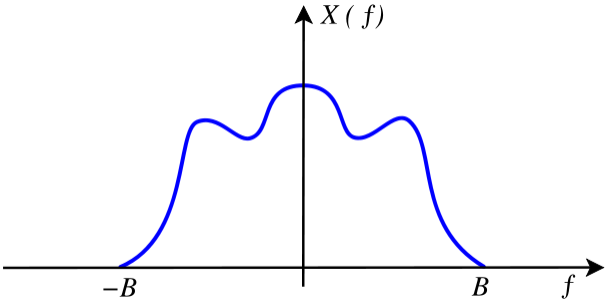
- Frequency domain: multiply by ideal LPF
  - ideal LPF: 'rect' function (gain Δt, cut off w_c)
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with 'sinc' function
  - as $F^{-1}\{\Delta t \ \text{rect}(w/w_c)\} = \Delta t \, w_c \, \text{sinc}(w_c t/\pi)$
  - i.e., weighted sinc on every sample
- Normally, $w_c = w_s/2$

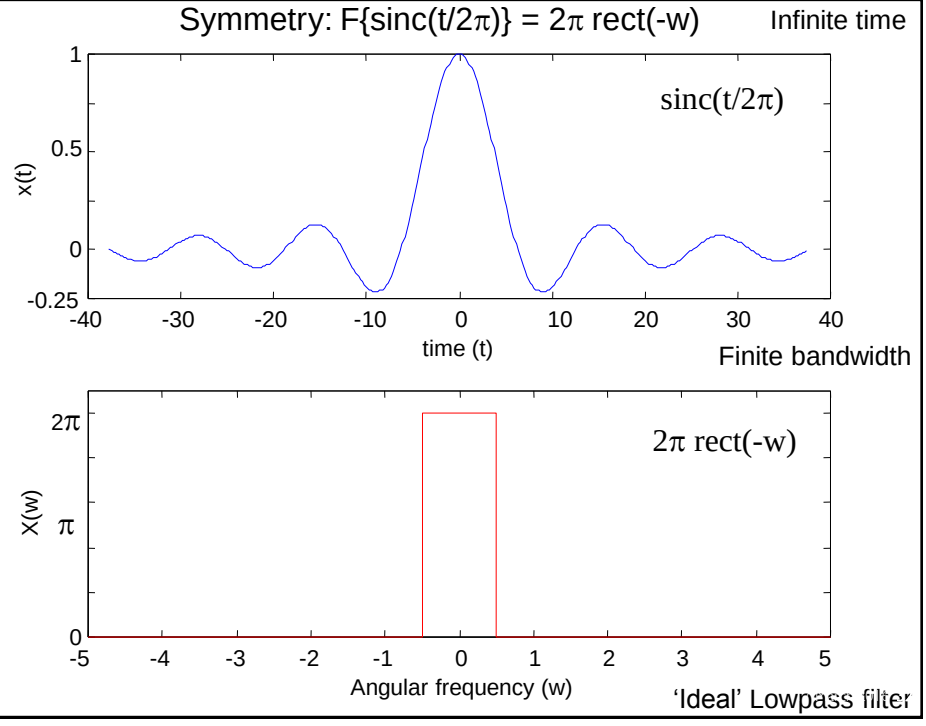
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \, \text{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$

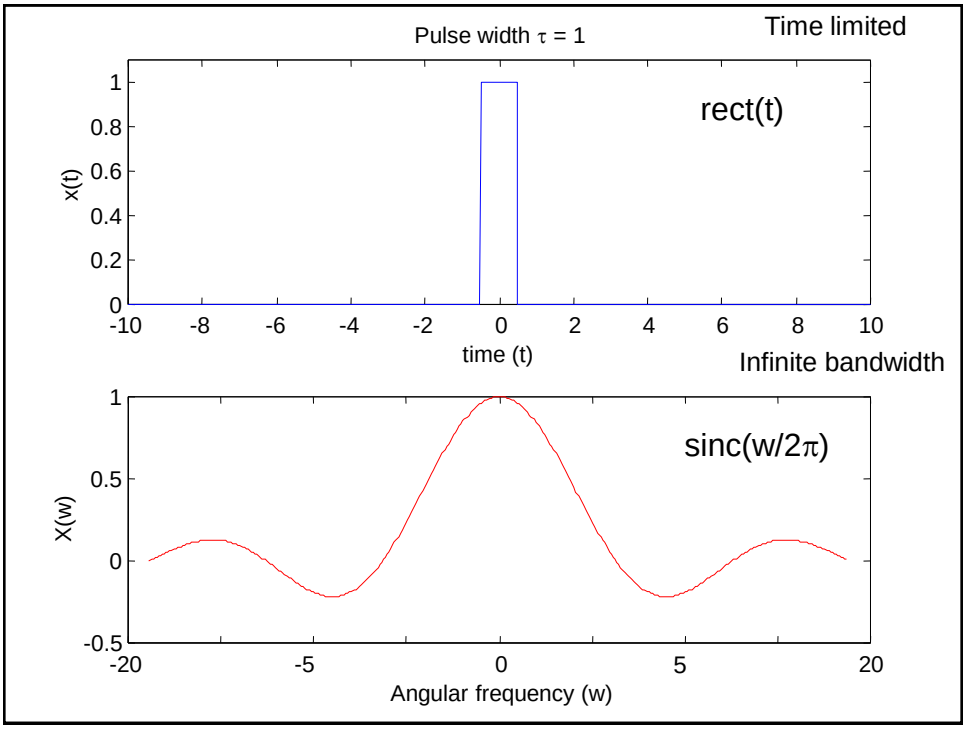
Reconstruction

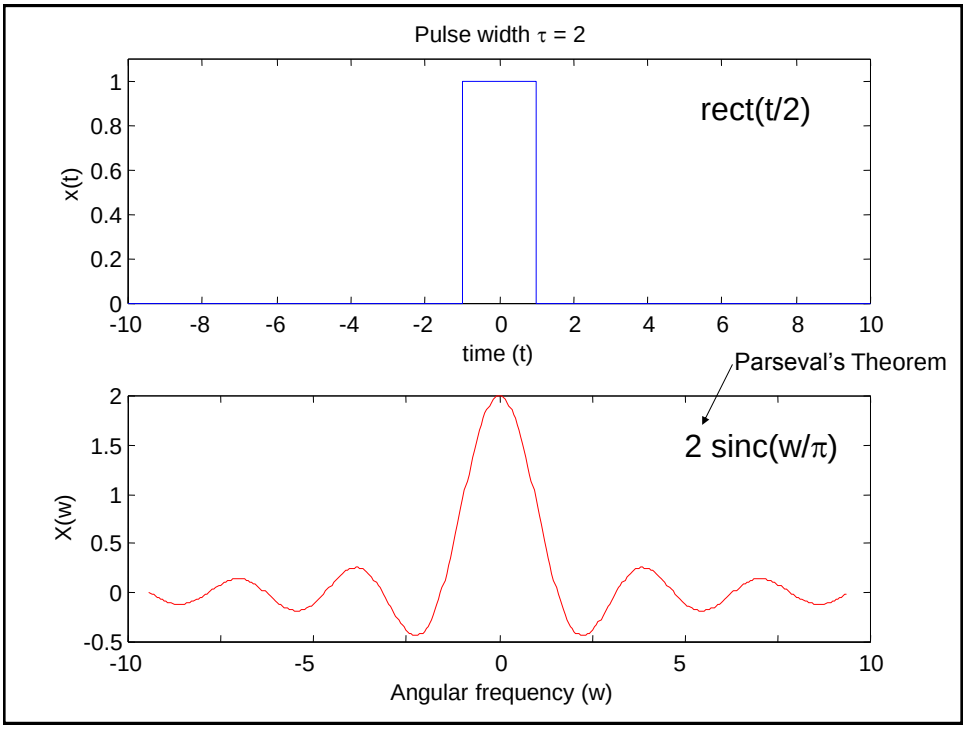
- Whittaker–Shannon interpolation formula

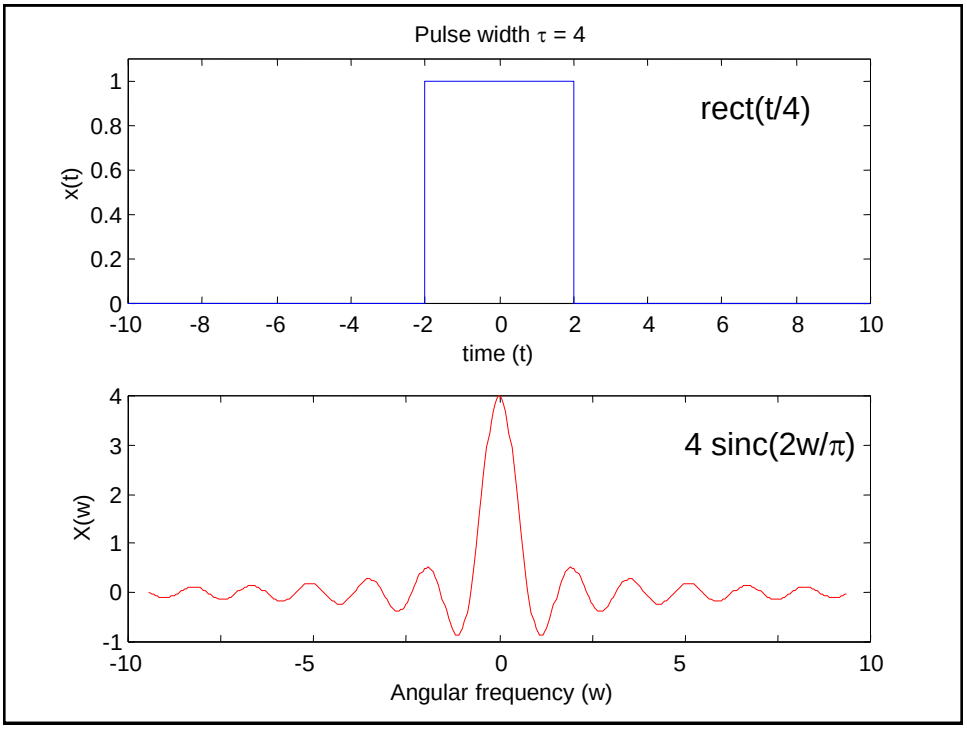
$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

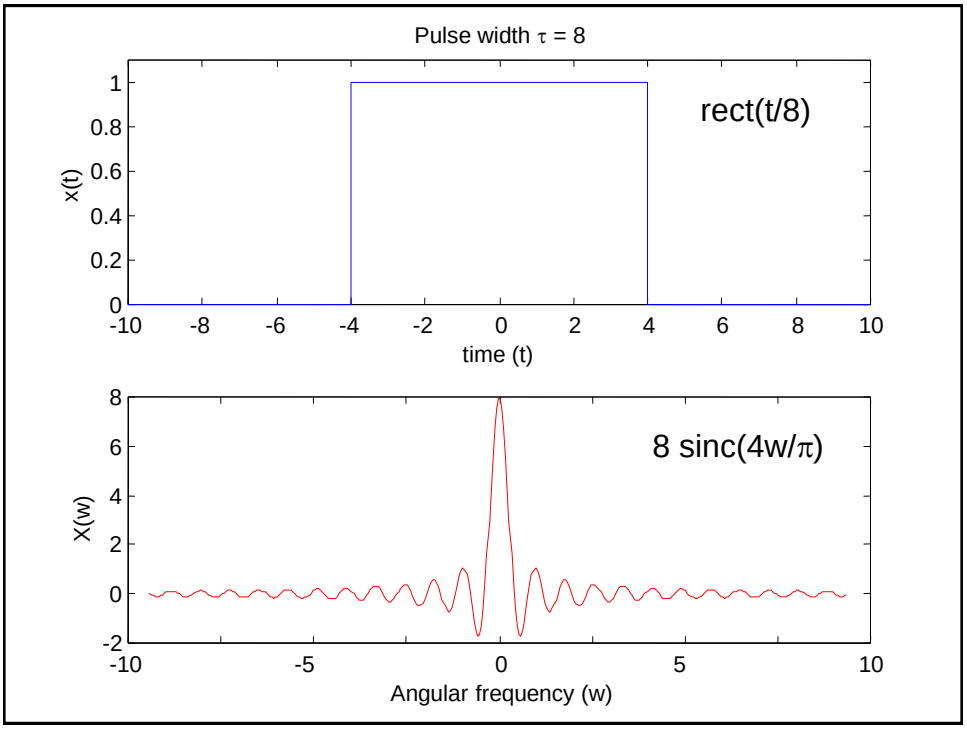


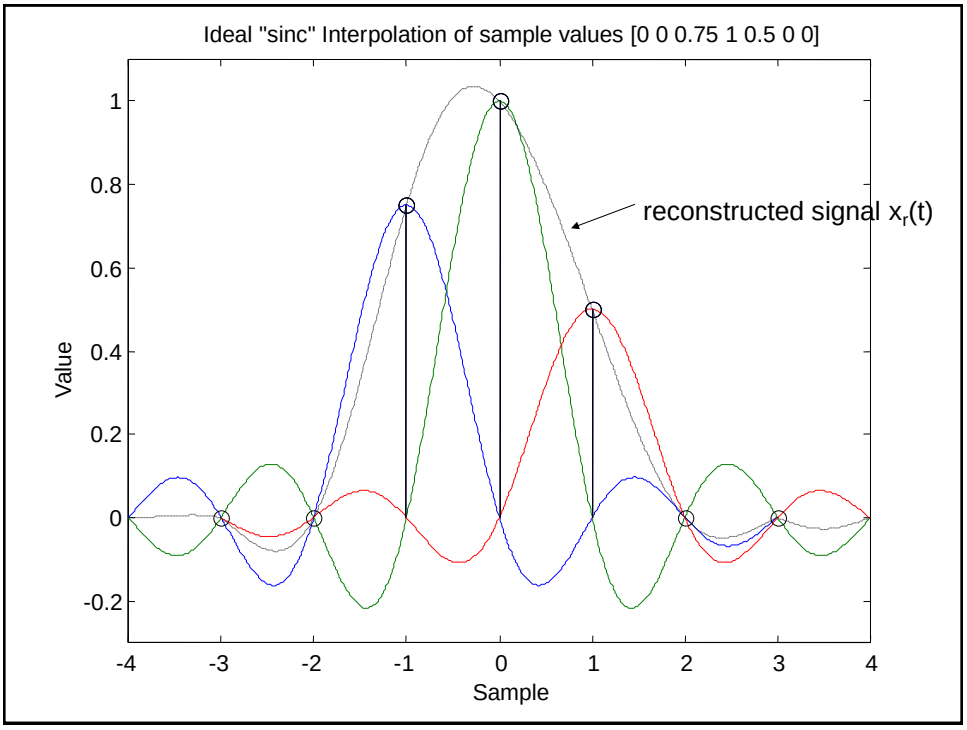


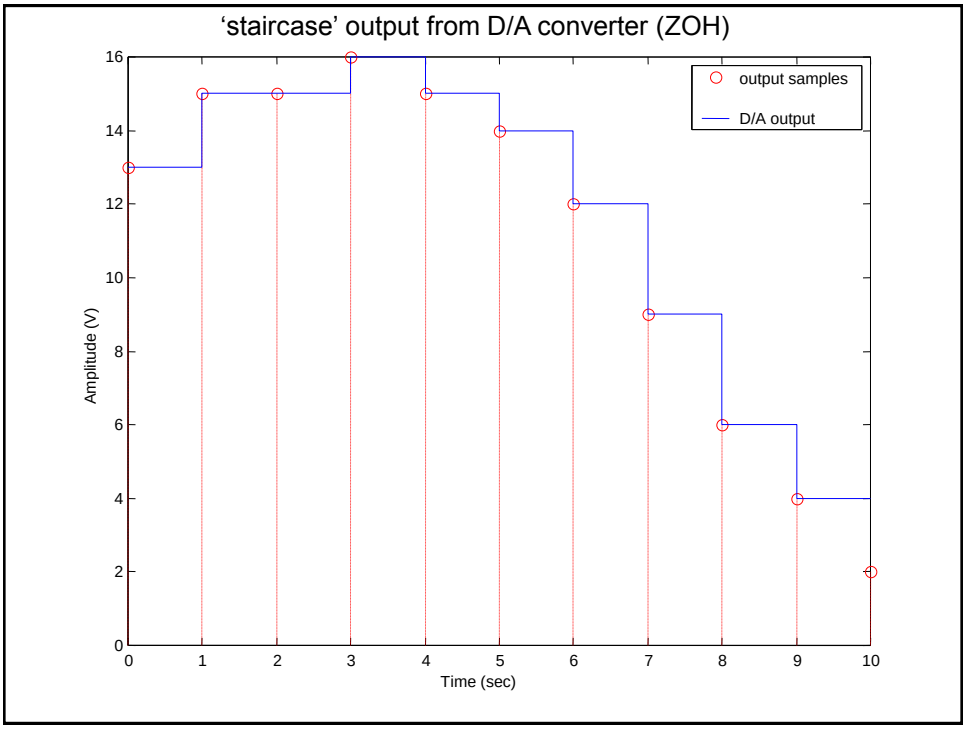

Pulse width τ = 1
Time limited
rect(t)
x(t)
time (t)
Infinite bandwidth
sinc(w/2π)
X(w)
Angular frequency (w)


Pulse width τ = 2
rect(t/2)
x(t)
time (t)
Parseval's Theorem
2 sinc(w/π)
X(w)
Angular frequency (w)

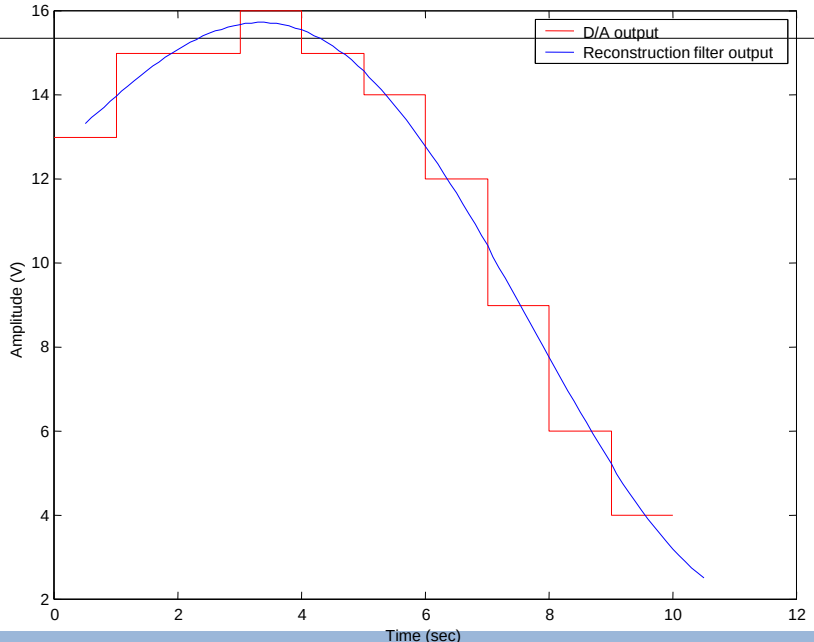

Pulse width τ = 4
rect(t/4)
x(t)
time (t)
4 sinc(2w/π)
X(w)
Angular frequency (w)


Pulse width τ = 8
rect(t/8)
x(t)
time (t)
8 sinc(4w/π)
X(w)
Angular frequency (w)

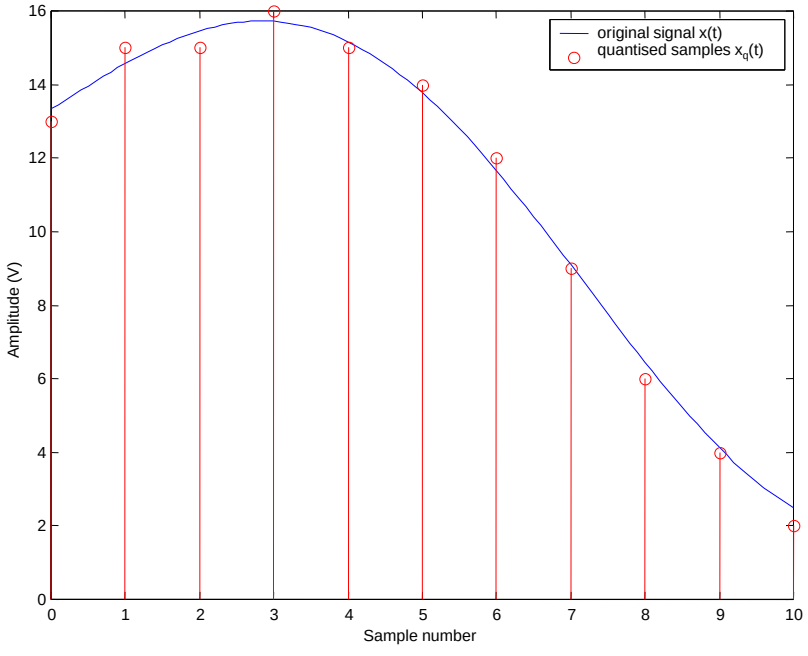

Ideal "sinc" Interpolation of sample values [0 0 0.75 1 0.5 0 0]
reconstructed signal $x_r(t)$
Value
Sample
1
0.8
0.6
0.4
0.2
0
-0.2
-4
-3
-2
-1
0
1
2
3
4


'staircase' output from D/A converter (ZOH)
output samples
D/A output
Amplitude (V)
Time (sec)
16
14
12
10
8
6
4
2
0
0
1
2
3
4
5
6
7
8
9
10

Smooth output from reconstruction filter

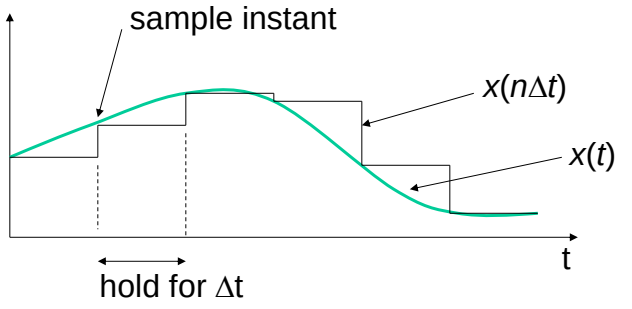


Example: error due to signal quantisation



Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every Δt seconds
  2. holds that value constant until next sample
- Produces 'staircase' waveform, $x(n\Delta t)$



Practical Reconstruction

Two stage process:

1. Digital to analogue converter (D/A)
   - zero order hold filter
   - produces 'staircase' analogue output
2. Reconstruction filter
   - non-ideal filter: $w_c = w_s/2$
   - further reduces replica spectrums
   - usually 4^{th} – 6^{th} order e.g., Butterworth
     - for acceptable phase response

D/A Converter

- Analogue output y(t) is
  - convolution of output samples y(nΔt) with $h_{ZOH}(t)$

$$y(t) = \sum_n y(n\Delta t) h_{ZOH}(t - n\Delta t)$$

$$h_{ZOH}(t) = \begin{cases} 1, & 0 \le t < \Delta t \\ 0, & \text{otherwise} \end{cases}$$

$$H_{ZOH}(w) = \Delta t \exp\left(\frac{-jw\Delta t}{2}\right) \frac{\sin(w\Delta t/2)}{w\Delta t/2}$$

D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional reconstruction filter required

Summary

- Theoretical model of Sampling
  - bandlimited signal (wB)
  - multiplication by ideal impulse train (ws > 2wB)
    - convolution of frequency spectrums (creates replicas)
  - Ideal lowpass filter to remove replica spectrums
    - wc = ws /2
    - Sinc interpolation
- Practical systems
  - Anti-aliasing filter (wc < ws /2)
  - A/D (S/H and quantisation)
  - D/A (ZOH)
  - Reconstruction filter (wc = ws /2)

Don't confuse theory and practice!

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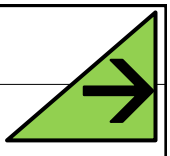
Next Time…



- **Digital Systems**
- Review:
  - Chapter 8 of Lathi
- A signal has many signals ☺
  [Unless it's bandlimited. Then there is the one ω]