Sampling Theory

ELEC 3004: **Systems**: Signals & Controls
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Lecture 5

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Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	24-Mar	Convolution Review
	28-Mar	Holiday
	31-Mar	
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Feedback
9	3-May	Introduction to Feedback Control
	5-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	12-May	Digitial Control
11	16-May	Digital Control Design
	19-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	31-May	Summary and Course Review

Signal: A carrier of (desired) information [1]

- Need **NOT** be electrical:
  - Thermometer
  - Clock hands
  - Automobile speedometer

- Need **NOT** always being given
  - "Abnormal" sounds/operations
  - Ex: "pitch" or "engine hum" during machining as an indicator for feeds and speeds

Signal: A carrier of (desired) information [2]

- Electrical signals
  - Voltage
  - Current

- **Digital signals**
  - **Convert analog electrical signals to an appropriate digital electrical message**
  - **Processing by a microcontroller or microprocessor**

Transduction (sensor to an electrical signal)

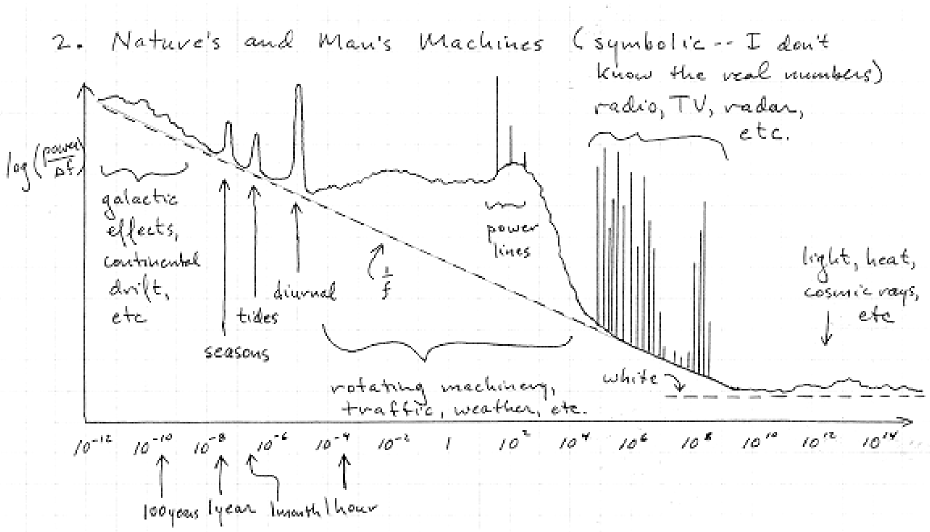
- Sensor reacts to environment (physics)

- Turn this into an electrical signal:
  - V: voltage source
  - I: current source

- **Measure this signal**
  - Resistance
  - Capacitance
  - Inductance

Noise!

BUT there is Noise …



Note: this picture illustrates the concepts but it is not quantitatively precise

Source: Prof. M. Siegel, CMU

Noise: "Unwanted" Signals Carrying Errant Information

- Cross-coupled measurements
- Cross-talk (at a restaurant or even a lecture)
- A bright sunny day obstructing picture subject
- Strong radio station near weak one
- observation-to-observation variation
  - Measurement fluctuates (ex: student)
  - Instrument fluctuates (ex: quiz !)
- Unanticipated effects / variation (Temperature)
- **One man's noise might be another man's signal**

Noise: Fundamental Natural Sources

- Voltage (EMF) – Capacitive & Inductive Pickup
- Johnson Noise – thermal / Brownian
- 1/f ($V_j = \sqrt{4 k_b T R}$
- Shot noise (interval-to-interval statistical count)

$$V_f = \sqrt{\frac{\alpha V_R^2}{N f}}$$

Digital
Signals & Systems

Why?

Digital Signal

- Representation of a signal against a discrete set
- The set is fixed in by computing hardware

$$s \in \mathbb{Z}$$

- Can be scaled or normalized … but is limited

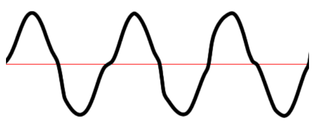
$$s \in \mathbb{Z}(0, \ldots, 2^{16})$$

- Time is also discretized

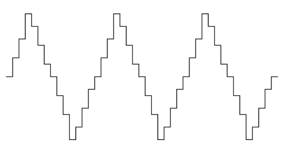
$$s' \in \frac{\mathbb{Z}(0,\ldots,2^{16})}{2^{16}}$$

Analog vs Digital

- *Analog Signal*: An analog or analogue signal is any variable signal **continuous** in both time and amplitude



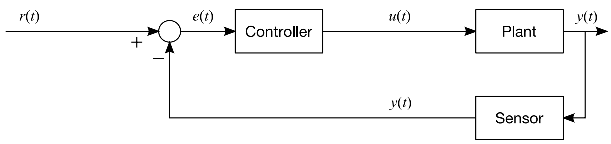
- *Digital Signal*: A digital signal is a signal that is both **discrete** and quantized



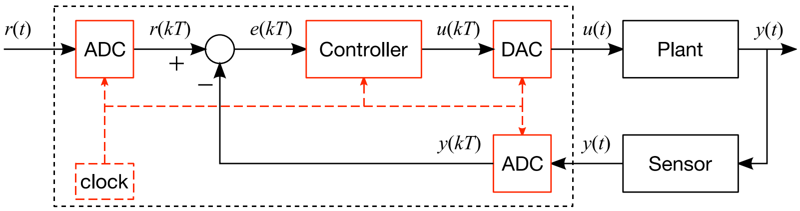
E.g. Music stored in a CD: 44,100 Samples per second and 16 bits to represent amplitude

Digital Systems

- Continuous:



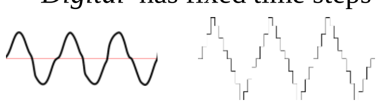
- Digital:



➔ Digital Systems ∵

Better SNR

- We trade-off "**certainty in time**" for "**signal noise/uncertainty**"
- Analog: ∞ time resolution
  - *Digital* has fixed time steps



- This avoids the noise and uncertainty in component values that affect analogue signal processing.

Better Processing

- **Digital microprocessors** are in a range of objects, from obvious (e.g. phone) to disposable (e.g. Go cards). (what doesn't have one?)

Compared to antilog computing (op-amp):

- **Accuracy**: digital signals are usually represented using 12 bits or more.
- **Reliability**: The ALU is stable over time.
- **Flexibility**: limited only programming ability!
- **Cost**: advances in technology make microcontrollers economical even for small, low cost applications. (Raspberry Pi 3: US$35)

SNR : Signal to Noise Ratio

$$V = V_s + V_n$$

Magnitude: $\bar{V}^2 = \bar{V}_s^2 + \bar{V}_n^2 + \bar{V_s V_n}$

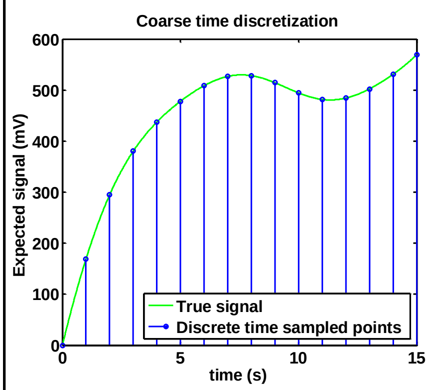
$$\frac{S}{N} = \frac{V_s^2}{V_n^2}$$

in dB: $10 \log \left(\frac{\bar{V_s}^2}{\bar{V_n}^2} \right) = 20 \log \left(\frac{V_s^{rms}}{V_n^{rms}} \right)$

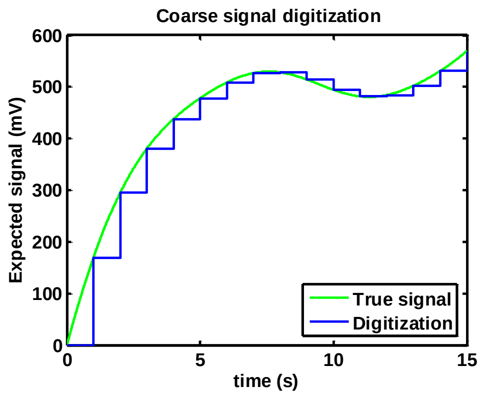
Data Acquisition

Representation of Signal

- Time Discretization



- Digitization



Quantisation

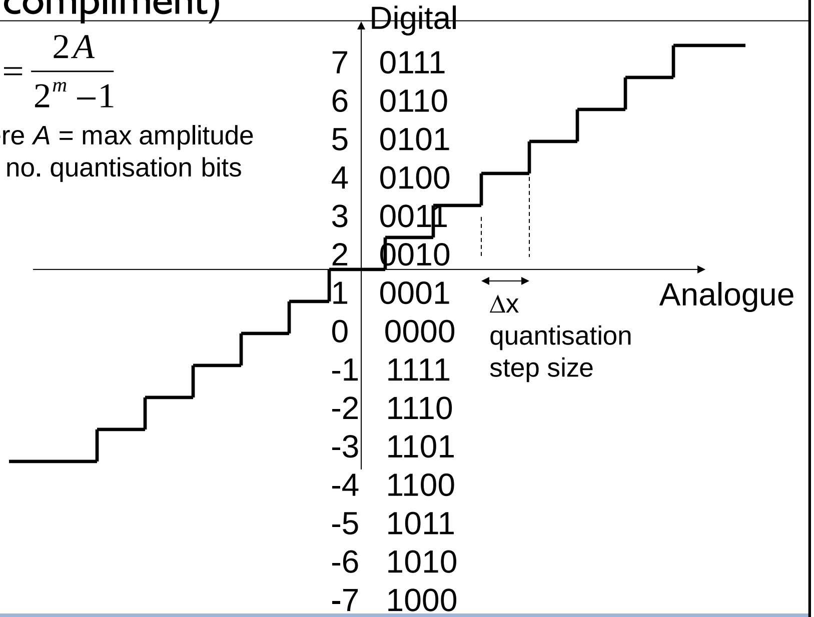
- Analogue to digital converter (A/D)
  - Calculates nearest binary number to $x(n\Delta t)$
    - $x_q[n] = \mathrm{q}(x(n\Delta t))$, where q() is non-linear rounding fctn
  - output modeled as $x_q[n] = x(n\Delta t) + e[n]$
- Approximation process
  - therefore, loss of information (unrecoverable)
  - known as ‘quantisation noise’ ($e[n]$)
  - error reduced as number of bits in A/D increased
    - i.e., Δx, quantisation step size reduces

$$|e[n]| \leq \frac{\Delta x}{2}$$

Input-output for 4-bit quantiser (two's compliment)

$$\Delta x = \frac{2A}{2^m - 1}$$

where A = max amplitude
m = no. quantisation bits



Signal to Quantisation Noise

- To estimate SQNR we assume
  - $e[n]$ is uncorrelated to signal and is a
  - uniform random process
- assumptions not always correct!
  - not the only assumptions we could make…
- Also known a 'Dynamic range' (R_D)
  - expressed in decibels (dB)
  - ratio of power of largest signal to smallest (noise)

$$R_D = 10\log_{10}\left(\frac{P_{signal}}{P_{noise}}\right)$$

Dynamic Range

Need to estimate:

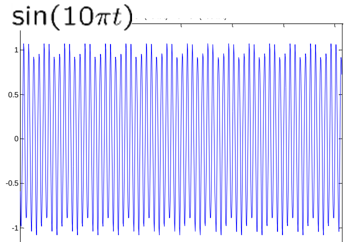
1. Noise power
   – uniform random process: $P_{noise} = \Delta x^2/12$
2. Signal power
   – (at least) two possible assumptions
   1. sinusoidal: $P_{signal} = A^2/2$
   2. zero mean Gaussian process: $P_{signal} = \sigma^2$
      - Note: as $\sigma \approx A/3$: $P_{signal} \approx A^2/9$
      - where σ^2 = variance, A = signal amplitude

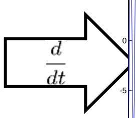
1 extra bit halves Δx
i.e., 20log10(1/2) = 6dB

Regardless of assumptions: R_D increases by 6dB for every bit that is added to the quantiser

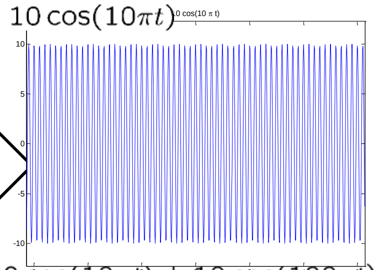
Derivatives magnify noise!

- $\sin(10\pi t)$

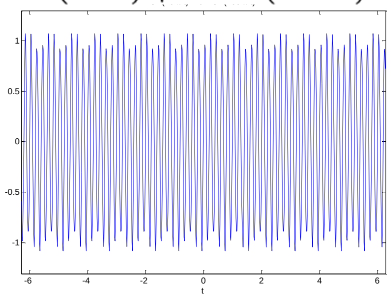




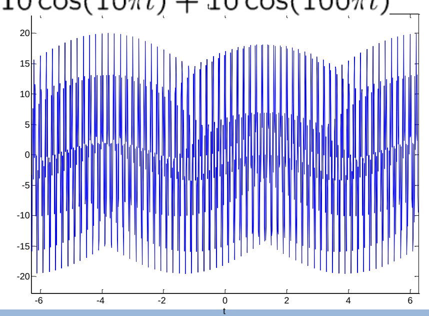
- $10\cos(10\pi t)$



- $\sin(10\pi t) + 0.1\sin(100\pi t)$

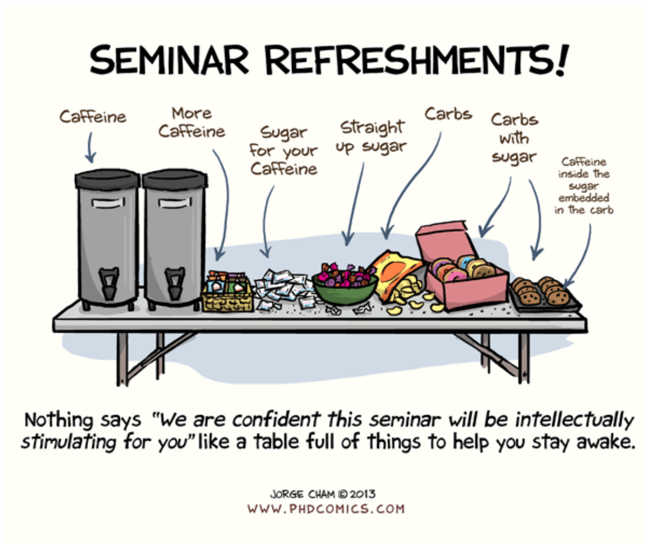


- $10\cos(10\pi t) + 10\cos(100\pi t)$

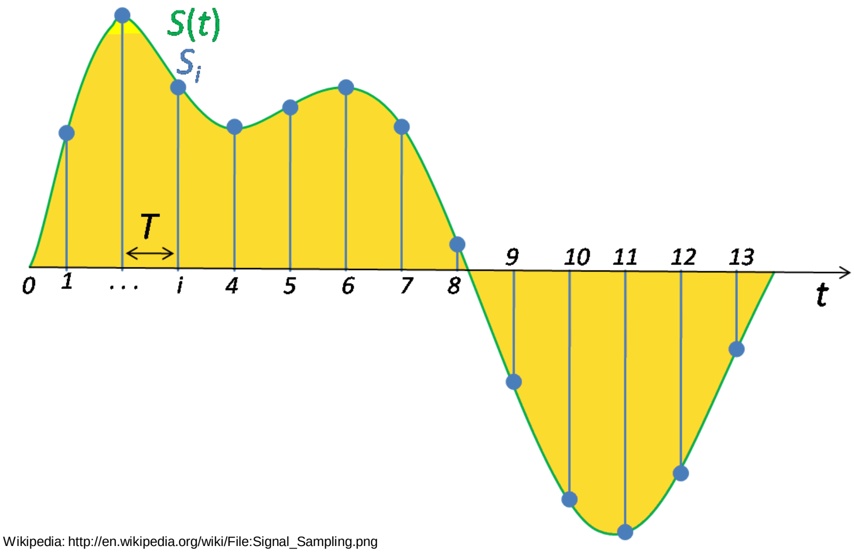


Sampling!

Not this type of sampling … ☺



This type of sampling…



Source: Wikipedia: http://en.wikipedia.org/wiki/File:Signal_Sampling.png

Sampling Theorem

- The Nyquist criterion states:

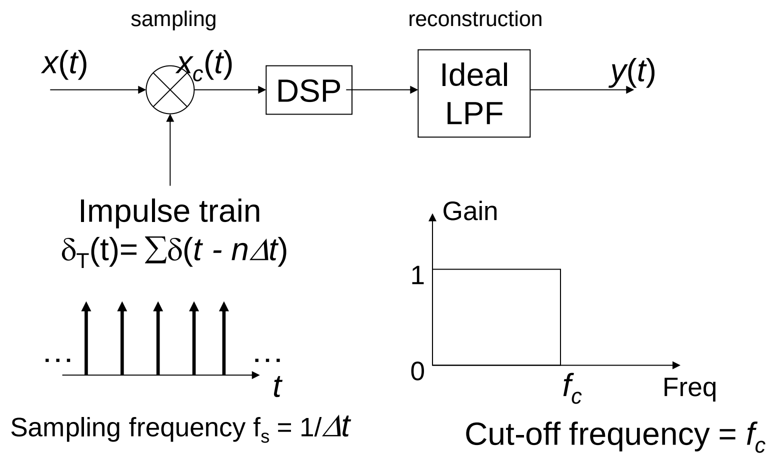
To prevent aliasing, a ***bandlimited*** *signal of bandwidth* w_B *rad/s must be sampled at a rate greater than* $2w_B$ *rad/s*

$$w_s > 2w_B$$

Note: this is a > sign not a ≥

Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

Mathematics of Sampling and Reconstruction

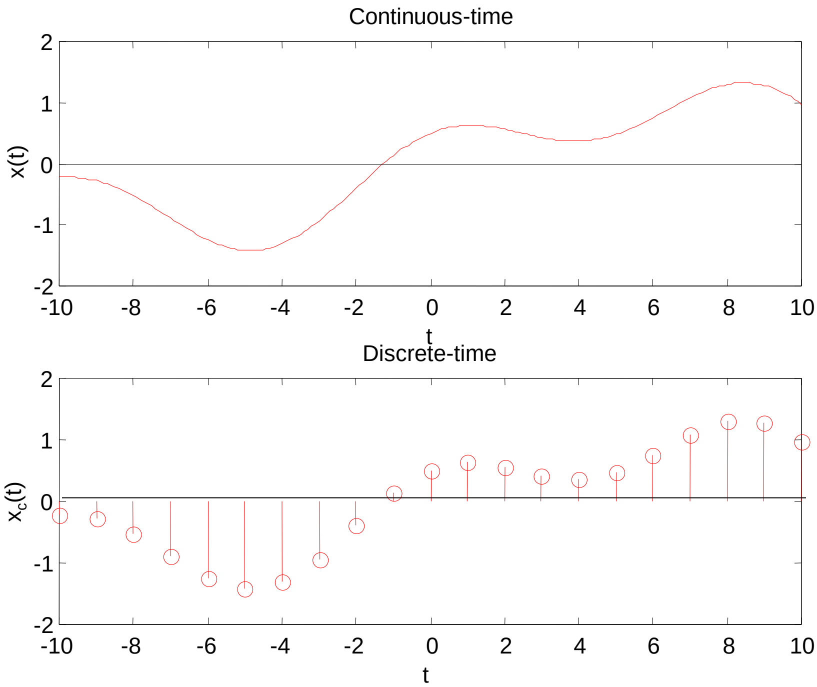


Mathematical Model of Sampling

- x(t) multiplied by impulse train δT(t)

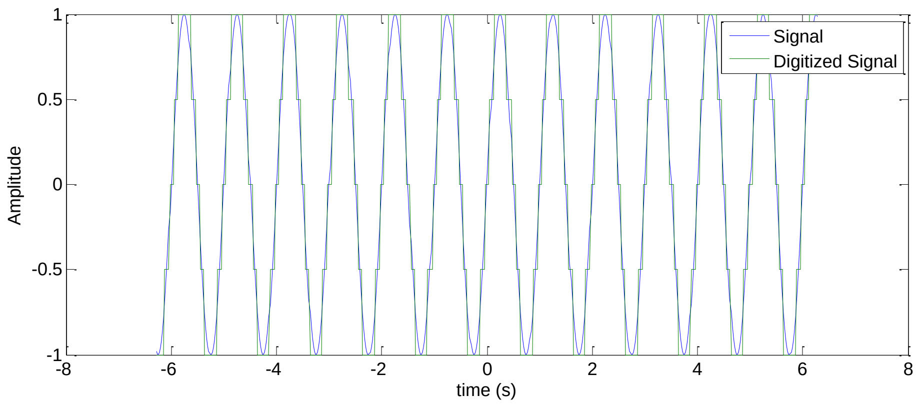
$$
\begin{aligned}
x_c(t) &= x(t)\delta_T(t) \\
&= x(t)\left[\delta(t)+\delta(t-\Delta t)+\delta(t-2\Delta t)+\cdots\right] \\
&= \sum_n x(n\Delta t)\delta(t-n\Delta t)
\end{aligned}
$$

  - $x_c(t)$ is a train of impulses of height $x(t)|_{t=n\Delta t}$



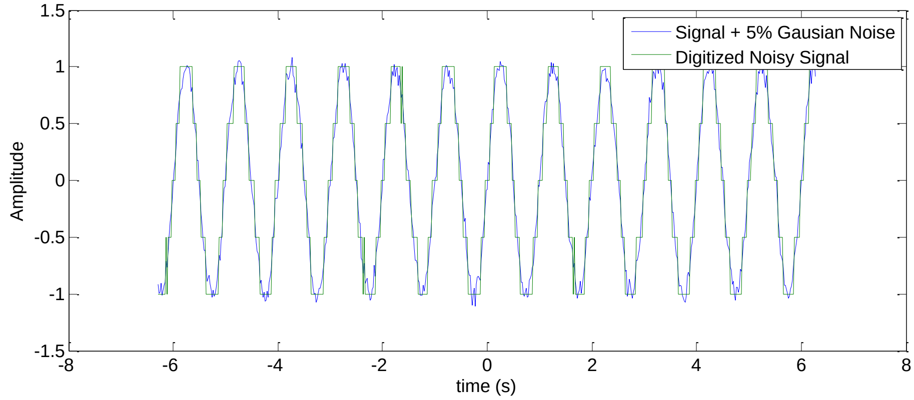
Discrete Time Signal

- Image a signal…



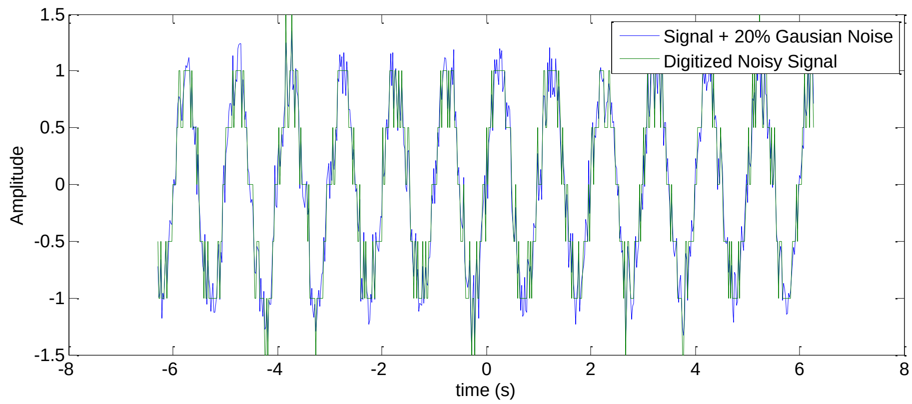
Discrete Time Signals

- Digitization helps beat the Noise!



Discrete Time Signals

- But only so much…



Signal Manipulations

- Shifting

$$y\left(n\right)=x\left(n-n_0\right)$$

- Reversal

$$y(n)=x(-n)$$

- Time Scaling

  (Down Sampling)

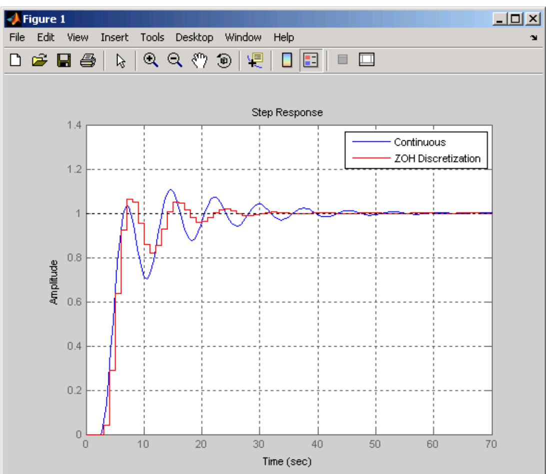
$$y(M)=x(Mn)$$

  (Up Sampling)

$$y\left(n\right)=x\left(\frac{n}{N}\right)$$

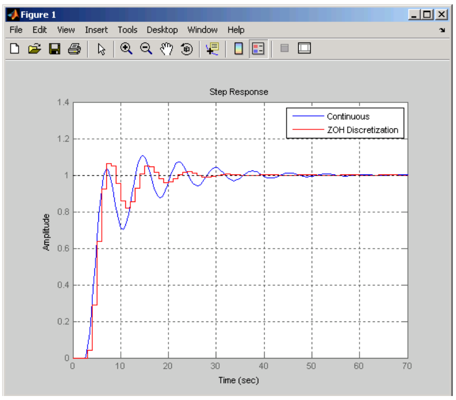
Discrete Time Signals

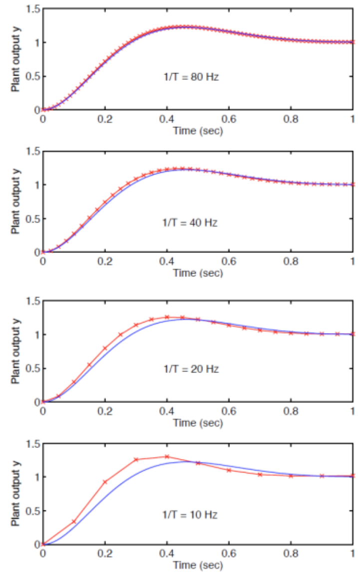
- Can make control tricky!



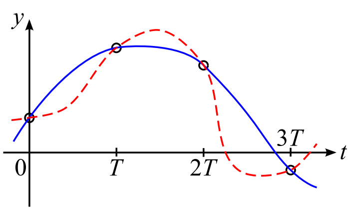
Discrete Time Signals

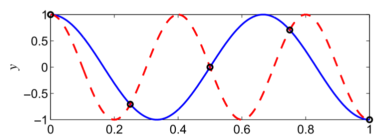
- Can make control tricky!





Nyquist Sampling Theorem and Aliasing





- A signal $y(t)$ is uniquely defined by its samples $y(kT)$ if the sampling frequency is more than twice the bandwidth of $y(t)$.

Sampling Theorem

- The Nyquist criterion states:

To prevent aliasing, a ***bandlimited*** *signal of bandwidth w_B rad/s must be sampled at a rate greater than $2w_B$ rad/s*

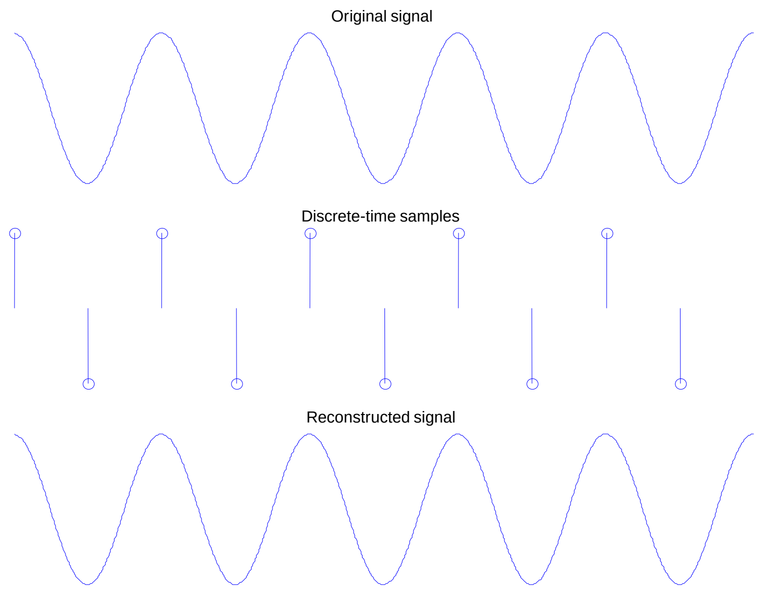
$$-w_s > 2w_B$$

Note: this is a > sign not a ≥

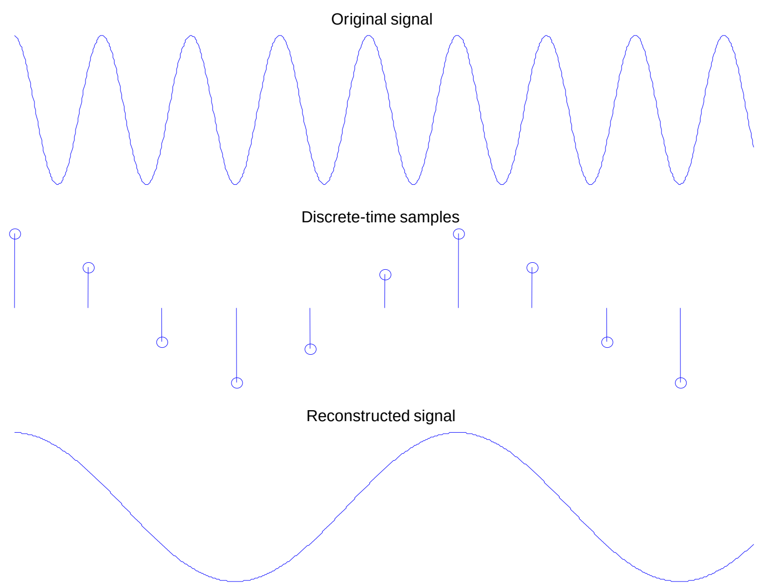
Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

Time Domain Analysis of Sampling

- Frequency domain analysis of sampling is very useful to understand
  - sampling ($X(w)*\sum \delta(w - 2\pi n/\Delta t)$)
  - reconstruction (lowpass filter removes replicas)
  - aliasing (if $w_s \leq 2w_B$)
- Time domain analysis can also illustrate the concepts
  - sampling a sinewave of increasing frequency
  - sampling images of a rotating wheel

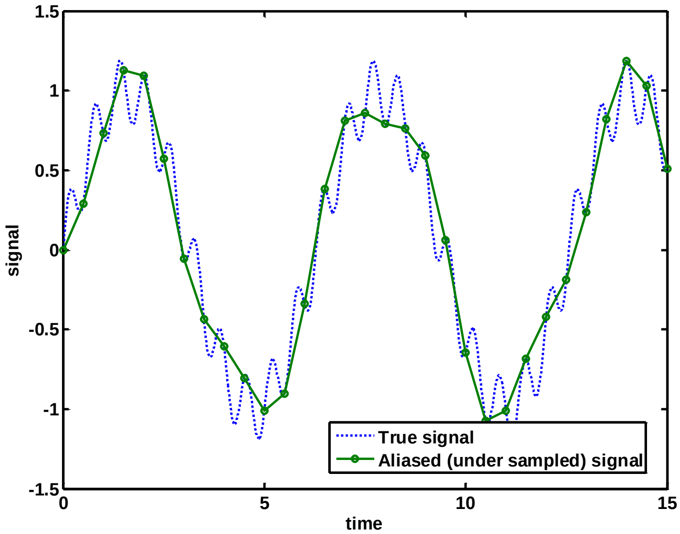


A signal of the original frequency is reconstructed

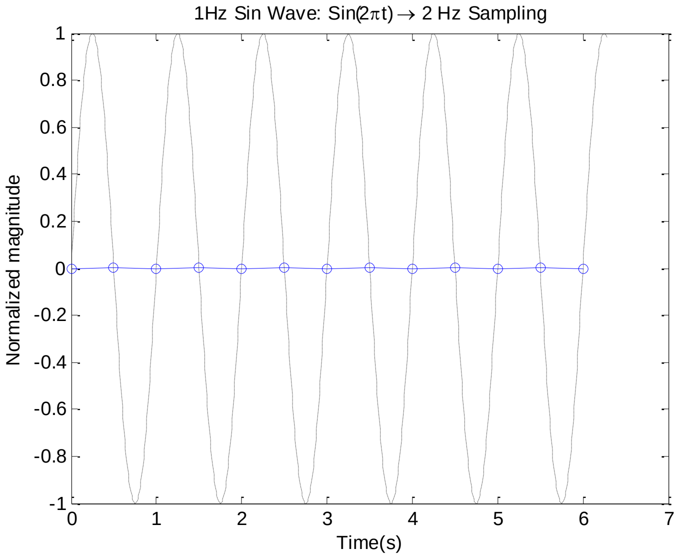


A signal with a reduced frequency is recovered, i.e., the signal is aliased to a lower frequency (we recover a replica)

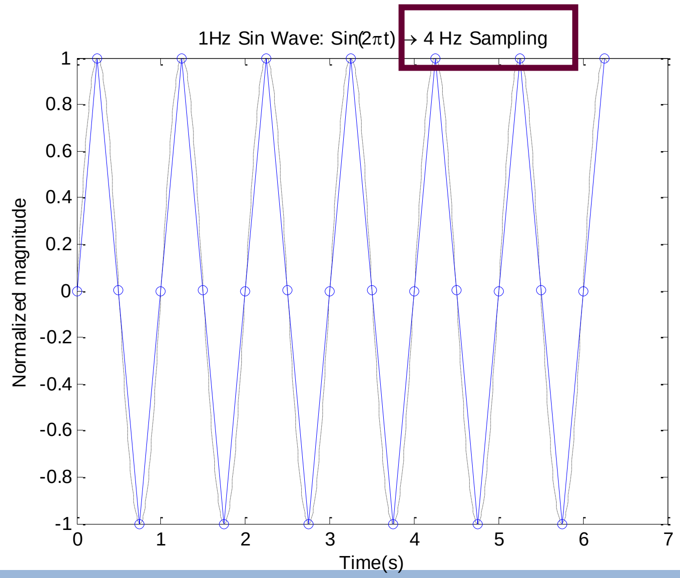
Sampling < Nyquist → Aliasing



Nyquist is not enough …



A little more than Nyquist is not enough …



Frequency Domain Analysis of Sampling

- Consider the case where the DSP performs no filtering operations
  - i.e., only passes xc(t) to the reconstruction filter
- To understand we need to look at the frequency domain
- Sampling: we know
  - multiplication in time ≡ convolution in frequency
  - F{x(t)} = X(w)
  - F{δT(t)} = ∑δ(w - 2πn/Δt),
  - i.e., an impulse train in the frequency domain

Frequency Domain Analysis of Sampling

- In the frequency domain we have

$$X_c(w) = \frac{1}{2\pi}\left(X(w) * \frac{2\pi}{\Delta t} \sum_n \delta\left(w - \frac{2\pi n}{\Delta t} \right) \right)$$
$$= \frac{1}{\Delta t} \sum_n X\left(w - \frac{2\pi n}{\Delta t} \right)$$

Remember convolution with an impulse? Same idea for an impulse train

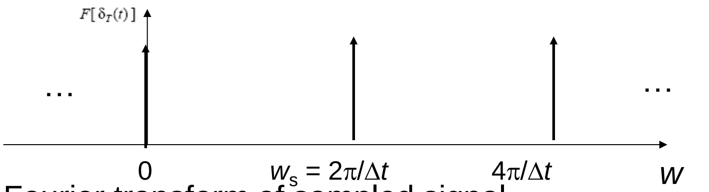
- Let's look at an example
  - where X(w) is triangular function
  - with maximum frequency w_m rad/s
  - being sampled by an impulse train, of frequency w_s rad/s

Sampling Frequency

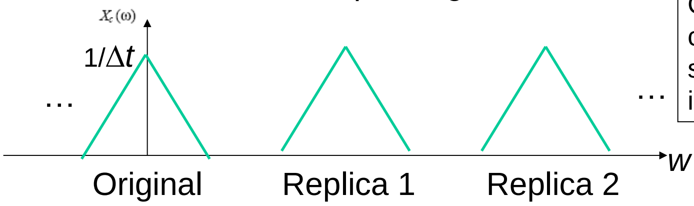
- In this example it was possible to recover the original signal from the discrete-time samples
- But is this always the case?
- Consider an example where the sampling frequency w_s is reduced
  - i.e., Δt is increased

Fourier transform of original signal X(ω) (signal spectrum)

Fourier transform of impulse train $\delta_T(\omega/2\pi)$ (sampling signal)

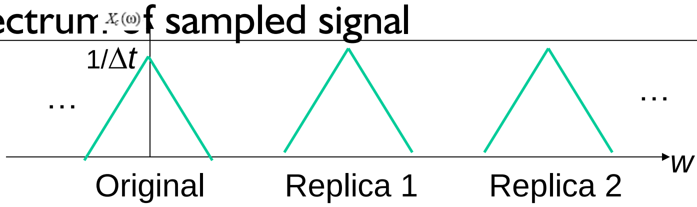


Fourier transform of sampled signal

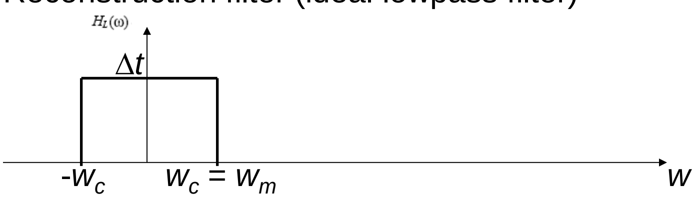


Original spectrum convolved with spectrum of impulse train

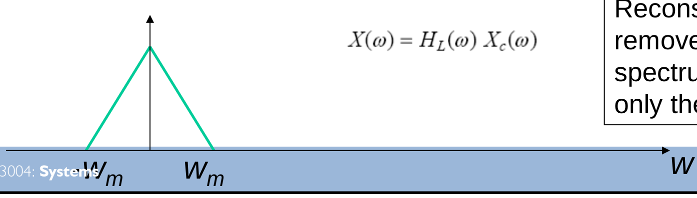
Spectrum of sampled signal



Reconstruction filter (ideal lowpass filter)



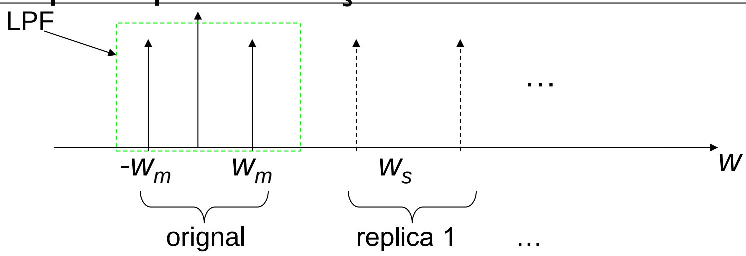
Spectrum of reconstructed signal



$$X(\omega) = H_L(\omega)\, X_c(\omega)$$

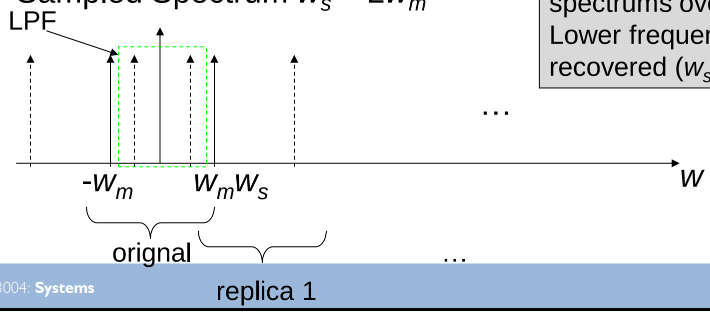
Reconstruction filter removes the replica spectrums & leaves only the original

Sampled Spectrum $w_s > 2wm$



original freq recovered

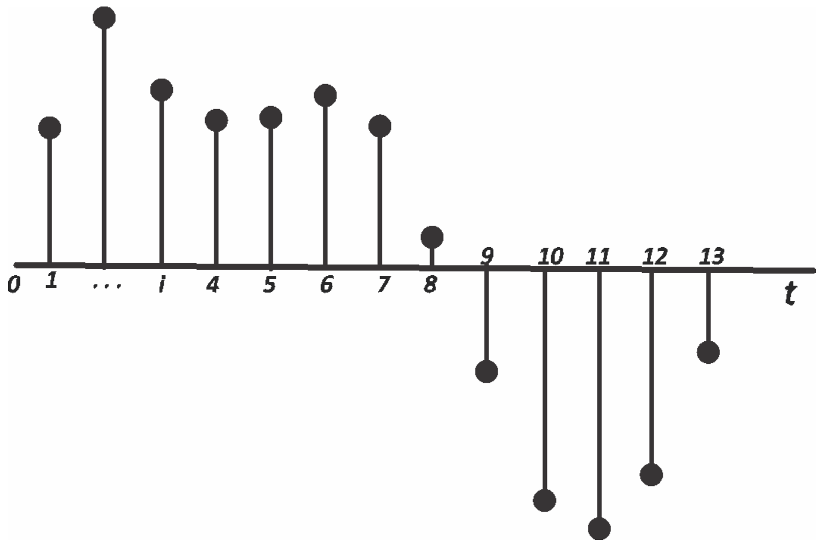
Sampled Spectrum $w_s < 2w_m$



Original and replica spectrums overlap
Lower frequency recovered ($w_s – w_m$)

RECONSTRUCTION

Reconstruction



Sampling and Reconstruction Theory and Practice

- Signal is bandlimited to bandwidth WB
  - Problem: real signals are not bandlimited
    - Therefore, require (non-ideal) anti-aliasing filter
- Signal multiplied by ideal impulse train
  - problems: sample pulses have finite width
  - and not ⊗ in practice, but sample & hold circuit
- Samples discrete-time, continuous valued
  - Problem: require discrete values for DSP
    - Therefore, require A/D converter (quantisation)
- Ideal lowpass reconstruction ('sinc' interpolation)
  - problems: ideal lowpass filter not available
    - Therefore, use D/A converter and practical lowpass filter

Time Domain Analysis of Reconstruction

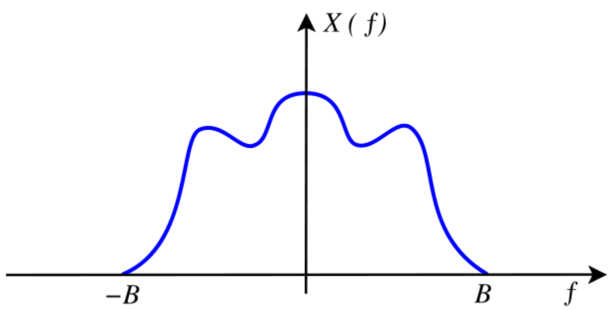
- Frequency domain: multiply by ideal LPF
  - ideal LPF: 'rect' function (gain Δt, cut off w_c)
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with 'sinc' function
  - as $F^{-1}\{\Delta t\ \text{rect}(w/w_c)\} = \Delta t\, w_c\, \text{sinc}(w_c t/\pi)$
  - i.e., weighted sinc on every sample
- Normally, $w_c = w_s/2$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \,\text{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$

Reconstruction

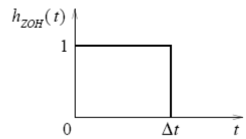
- Whittaker–Shannon interpolation formula

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \text{sinc}\left(\frac{t-nT}{T}\right)$$

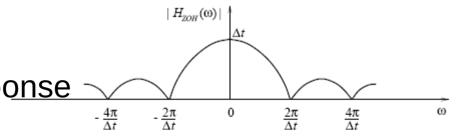


Zero Order Hold (ZOH)

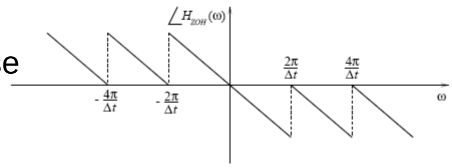
ZOH impulse response



ZOH amplitude response

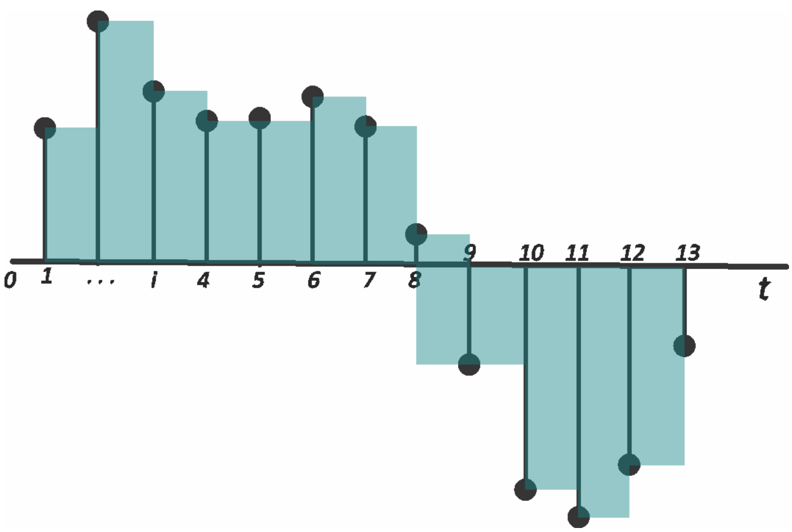


ZOH phase response



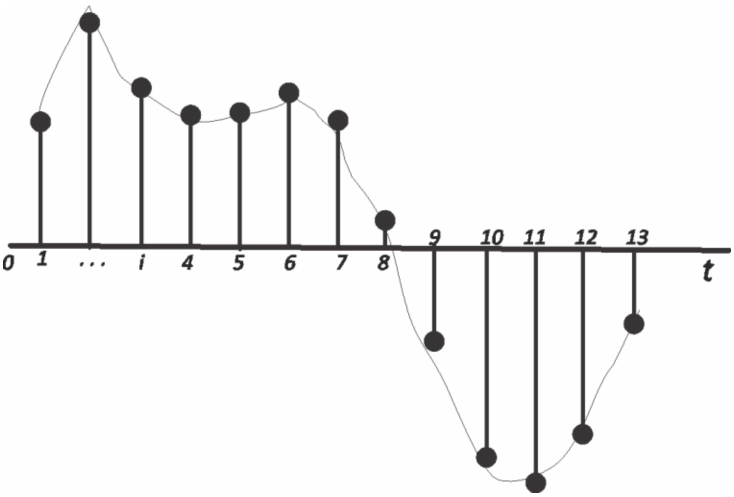
Reconstruction

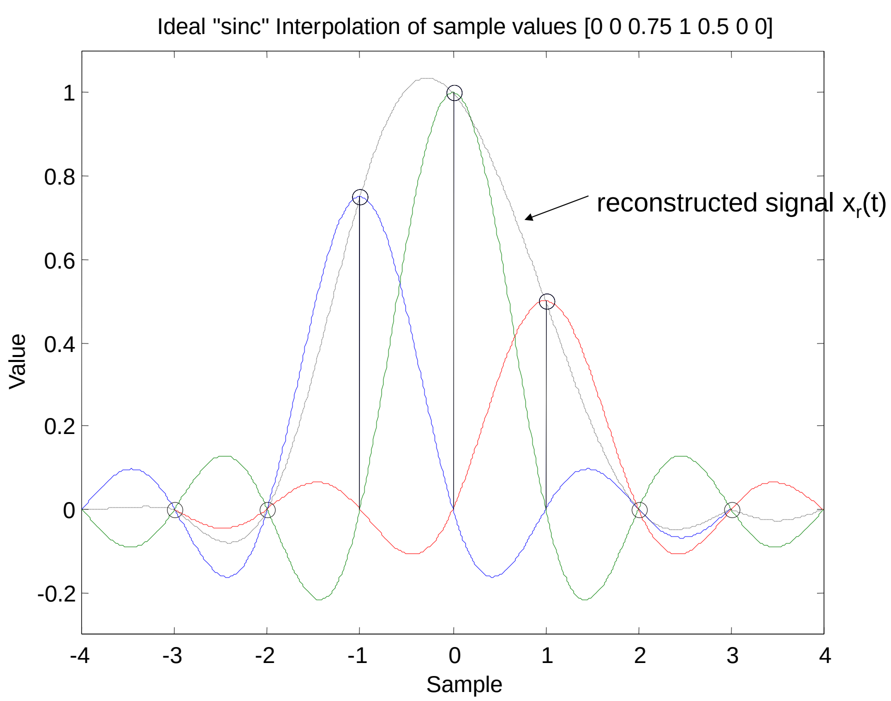
- Zero-Order Hold [ZOH]

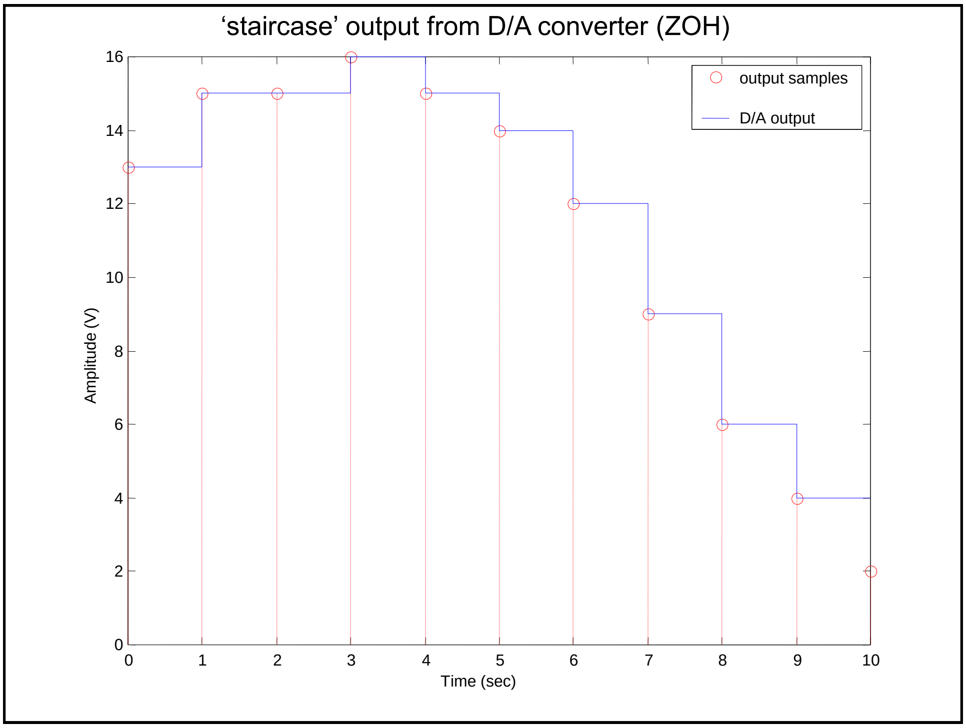


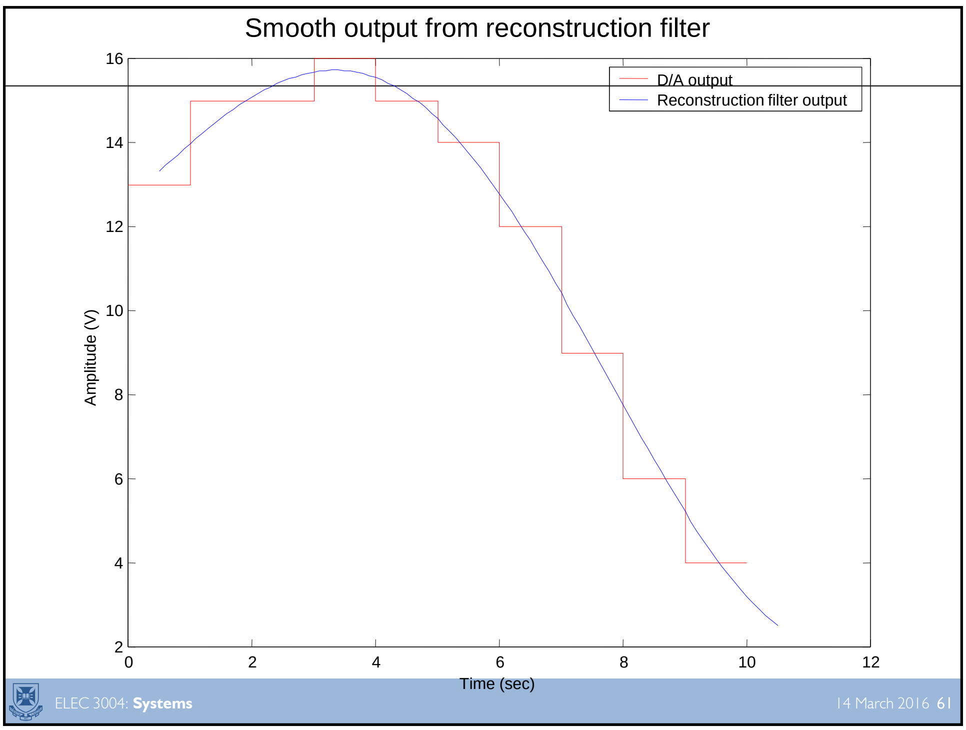
Reconstruction

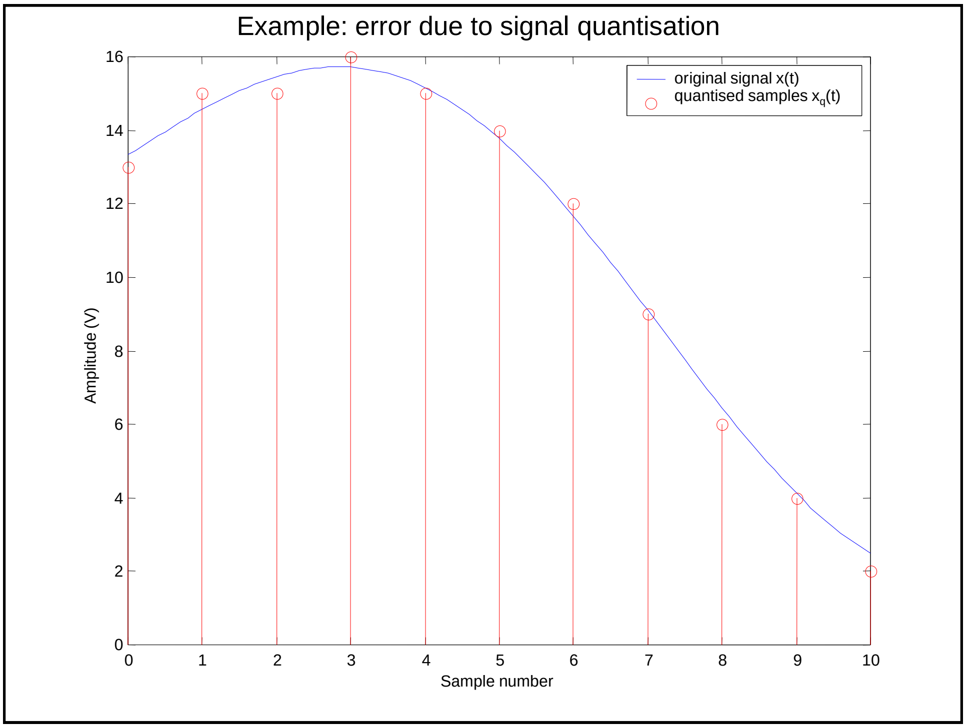
- Whittaker–Shannon interpolation formula






'staircase' output from D/A converter (ZOH)
output samples
D/A output
Amplitude (V)
Time (sec)
0
2
4
6
8
10
12
14
16
0
1
2
3
4
5
6
7
8
9
10


Smooth output from reconstruction filter
D/A output
Reconstruction filter output
Amplitude (V)
Time (sec)
2
4
6
8
10
12
14
16
0
2
4
6
8
10
12
ELEC 3004: Systems
14 March 2016 61

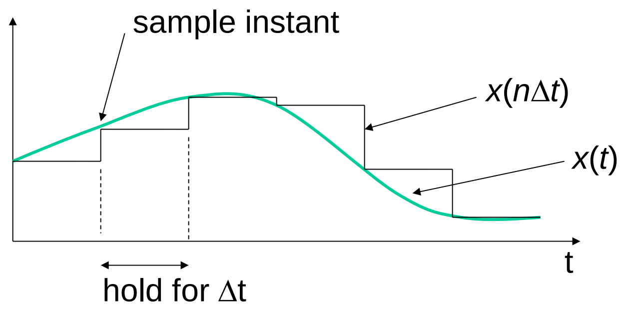


Finite Width Sampling

- Impulse train sampling not realisable
  - sample pulses have finite width (say nanosecs)
- This produces two effects,
- Impulse train has sinc envelope in frequency domain
  - impulse train is square wave with small duty cycle
  - Reduces amplitude of replica spectrums
    - smaller replicas to remove with reconstruction filter ☺
- Averaging of signal during sample time
  - effective low pass filter of original signal
    - can reduce aliasing, but can reduce fidelity ☹
    - negligible with most S/H ☺

Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every Δt seconds
  2. holds that value constant until next sample
- Produces 'staircase' waveform, $x(n\Delta t)$



Practical Reconstruction

Two stage process:

1. Digital to analogue converter (D/A)
   - zero order hold filter
   - produces 'staircase' analogue output
2. Reconstruction filter
   - non-ideal filter: $w_c = w_s/2$
   - further reduces replica spectrums
   - usually $4^{th} - 6^{th}$ order e.g., Butterworth
     - for acceptable phase response

D/A Converter

- Analogue output y(t) is
  - convolution of output samples y(nΔt) with $h_{ZOH}(t)$

$$y(t) = \sum_n y(n\Delta t) h_{ZOH}(t - n\Delta t)$$

$$h_{ZOH}(t) = \begin{cases} 1, & 0 \le t < \Delta t \\ 0, & \text{otherwise} \end{cases}$$

$$H_{ZOH}(w) = \Delta t \exp\left(\frac{-jw\Delta t}{2}\right) \frac{\sin(w\Delta t/2)}{w\Delta t/2}$$

D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional reconstruction filter required

Summary

- Theoretical model of Sampling
  - bandlimited signal (wB)
  - multiplication by ideal impulse train (ws > 2wB)
    - convolution of frequency spectrums (creates replicas)
  - Ideal lowpass filter to remove replica spectrums
    - wc = ws /2
    - Sinc interpolation
- Practical systems
  - Anti-aliasing filter (wc < ws /2)
  - A/D (S/H and quantisation)
  - D/A (ZOH)
  - Reconstruction filter (wc = ws /2)

Don't confuse theory and practice!

Next Time…

- **Aliasing and Anti-Aliasing**

- Review:
  - Chapter 5 of Lathi

- A signal has many signals ☺
  [Unless it's bandlimited]