Past exams in detail

Problem 2 – 2015 exam

- I have a column vector , $x \epsilon R^n$. Is the matrix $x \cdot x^T$ invertible?

What defines an invertible matrix

- Must be square (although for non-square matrices a psudeo-inverse may exist)
- Must be Full Rank
- So is this matrix square?
  - Yes! It has dimension $n \times n$
- Is the matrix full rank?

What is the Rank of $x \cdot x^T$?

- Well, $x \cdot x^T$ is the matrix $\begin{bmatrix} x_1 \cdot x_1 & \cdots & x_1 \cdot x_n \\ \vdots & \ddots & \vdots \\ x_n \cdot x_1 & \cdots & x_n \cdot x_n \end{bmatrix}$=$[\, x_1 \cdot x \quad \cdots \quad x_n \cdot x]$
- It can be seen that columns 2 to n, are simply linear scaling's of column 1.
- Therefore the matrix is rank 1, which will NOT be full rank for any matrix that is non trivial (n>1)
- This argument can be extended to show that $x \cdot y^T$ for any two vectors is Rank deficient as well.

$$x \cdot y^T = [\, y_1 \cdot x \quad \cdots \quad y_n \cdot x]$$

Why do we care?

- A matrix multiplied by its inverse is the identity matrix.
- A matrix can be used as a model of a linear system. A system, B, driven by an input s(t) will produce an output

$$y(t) = C \; \cdot s(t)$$

- if we can compute B^{-1} , it can be used to invert the effect of B, recovering s exactly.

$$C^{-1} \cdot y(t) = s(t)$$

- But, if the B is rank deficient, s(t) cannot be recovered, as B^{-1} does not exist.

Lets do it in Matlab

- Lets take a vector signal, and use matlab to visualize what happens when it is acted upon by the terribly degenerate matrix, $x \cdot x^T$
- I'm choosing x to be $[2\ 3\ 1]$

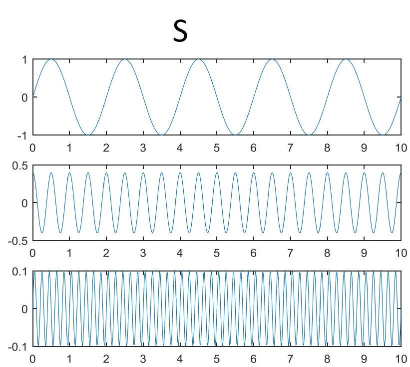
$$x \cdot x^T = C = \begin{bmatrix} 4 & 6 & 2 \\ 6 & 9 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

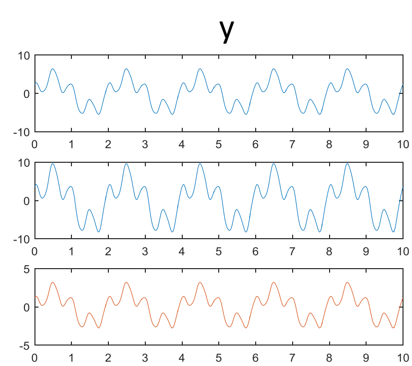
- The eigenvalues of C in matlab are 14,E^{-14},E^{-15}
- The only significant eigenvector is $0.236 \cdot x$

Lets do it in Matlab II

- My vector signal, $s(t) = \begin{bmatrix} \sin(\pi t) \\ 0.4 \cdot \sin(4\pi t) \\ 0.1 \cdot \sin(10\pi t) \end{bmatrix}$
- My result is $y(t) \; = C \cdot s(t)$

The result

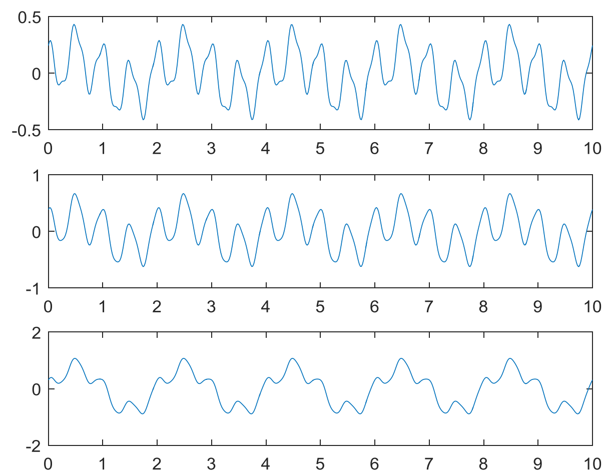




$$y_1(t) = g_1 \cdot y_2(t) = g_2 \cdot y_3(t)$$

There is no linear combination of the signals in X which recovers a signal in S, however if C had been full rank all of S could have been recovered

If C is full rank



A physical analog

- I have a car, and I am interested in three variables:

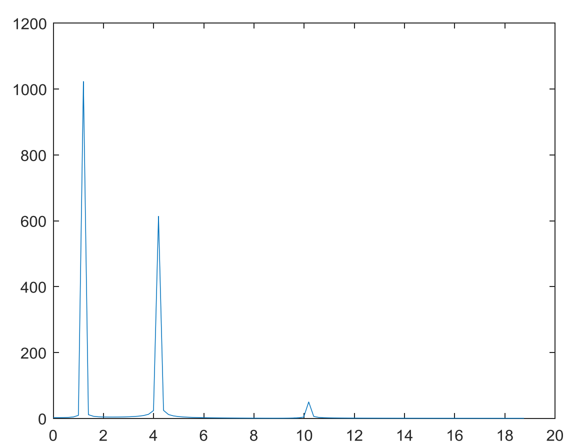
$$s(t) = \begin{bmatrix} Pos_{throttle}\ (t) \\ \theta_{steering}\ (t) \\ \theta_{incline}(t) \end{bmatrix}$$

- I can only use 3 sensors ($$$)
  - If I choose to sense engine RPM, driveshaft RPM and wheel RPM, I will get linearly dependent measurements X, and recovering S is impossible.
  - Choosing to sense engine RPM, throttle position and lateral acceleration, the matrix B could be invertible, and if so can be used to recover s(t) completely

Bonus

- Does anyone know a technique by which we might recover at least some information about the signals within S from Y?
  - (the hint is they are all sine waves)

Fourier!

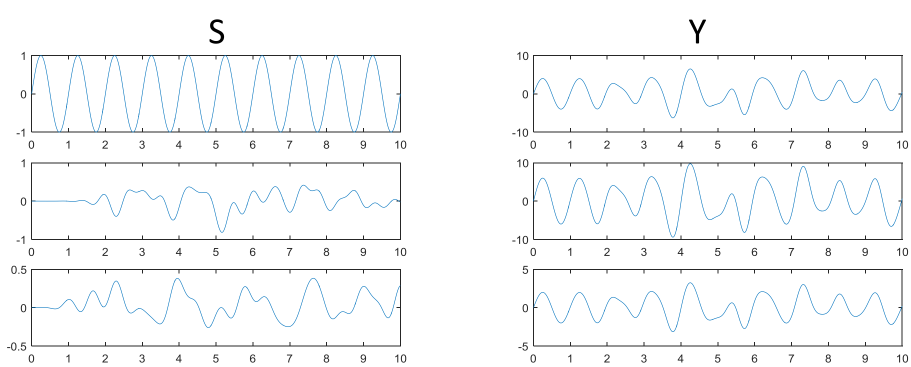


But I will show you why in the real world Fourier won't always save you

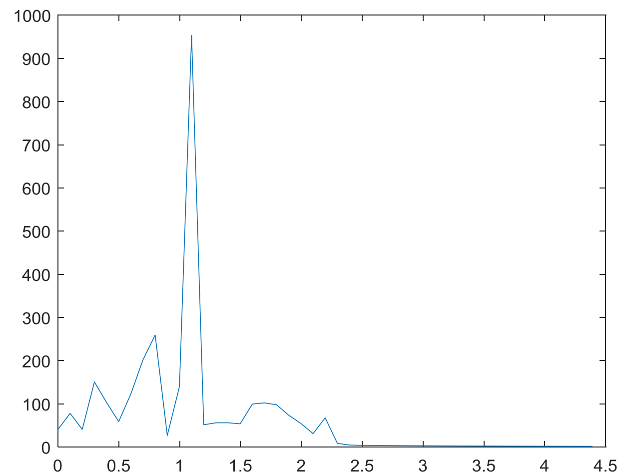
When Fourier wont save you

- Lets take a look at signals that are stochastic
- I'm going to generate noise, and shape it with a filter
- $$S(t) = [\sin(3t) \quad ; \quad filt_1(t) * \eta(t, \sigma^1) \quad ; \quad filt_2(t) * \eta(t, \sigma^2) \quad]$$

The results



In the frequency domain



Problem 2 2014 exam(quickly)

Does a linear system imply invertiblilty?

Key points

- The matrix of xx^T is not invertible
  - It may be square, but its rank deficient
- A vector signal, S multiplied by a Rank deficient matrix, will result in linearly dependent vectors Y
- The result is that in general, S is unrecoverable from Y
- Linearity does NOT imply invertibilty, and a great example is the dynamic system driven by $A = xx^T$

Questions?

A sampler may be described as a continuous time system with by the function

$$y(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT)$$

A. Is the sampling operation

i. Causal?

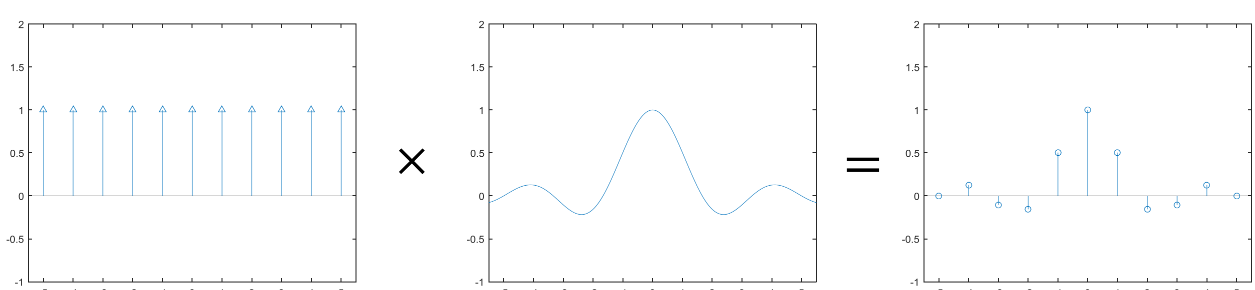
ii. Linear?

iii. Time invariant?

iv. Invertible?

B. If $\boldsymbol{x}(\boldsymbol{t})$ = $\mathbf{cos}(\boldsymbol{\pi}\ \mathbf{2}\ \boldsymbol{t})$, please sketch the sampled output $y(t)$ for T = 1. (Hint: What is the frequency of this signal in Hz?)

What the function is doing



Causal - Y(t) is non-zero for t<0, so it is not causal

Linear - Does it pass the superposition test?

Time invariance – does any time shifted input x(t+d) give a time shifted output y(t+d)

Invertible – No*

Linear

$$\alpha y_1(t) + \beta y_2(t) = H\left\{\alpha x_1(t) + \beta x_2(t)\right\}$$

$$\alpha y_1(t) = \alpha \sum_{k=-\infty}^{\infty} x_1(t)\delta(t - kT)$$

$$\beta y_2(t) = \beta \sum_{k=-\infty}^{\infty} x_2(t)\delta(t - kT)$$

$$y_{\alpha x_1(t)+\beta x_2(t)}(t) = \sum_{k=-\infty}^{\infty} \big(\alpha x_1(t) + \beta x_2(t)\big)\delta(t - kT)$$

$$= \sum_{k=-\infty}^{\infty} \alpha x_1(t)\delta(t - kT) + \beta x_2(t)\delta(t - kT)$$

$$= \sum_{k=-\infty}^{\infty} \alpha x_1(t)\delta(t - kT) + \sum_{k=-\infty}^{\infty} \beta x_2(t)\delta(t - kT)$$

$$= \alpha \sum_{k=-\infty}^{\infty} x_1(t)\delta(t - kT) + \beta \sum_{k=-\infty}^{\infty} x_2(t)\delta(t - kT)$$

Time invariance

$$y(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t - kT)$$

$$y(t-\tau) = \sum_{k=-\infty}^{\infty} x(t-\tau)\delta(t - \tau - kT)$$

$$y_{x(t-\tau)}(t) = \sum_{k=-\infty}^{\infty} x(t-\tau)\delta(t - kT)$$

$$y_{x(t-\tau)}(t) \neq y(t-\tau)$$

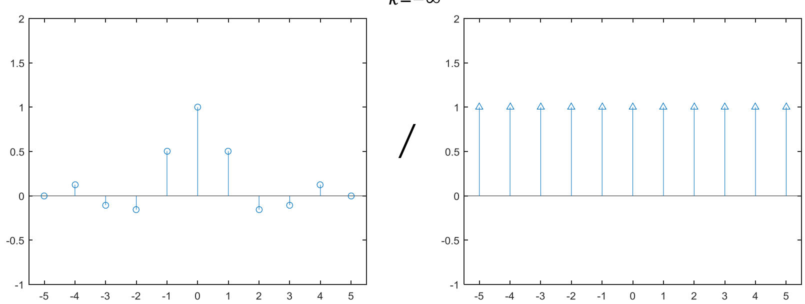
Invertibility

Dirac sampling is the pointwise product of a function x(t) and the Dirac sampling comb.

Inversion would be the pointwise Quotient of y(t) over the dirac sampling comb.

Between samples we have $\hat{x} = \frac{0}{0}$ which is undefined.

$$y^{-1}(t) = \sum_{k=-\infty}^{\infty} x(t) \frac{1}{\delta(t - kT)}$$



The *

- By meeting the Nyquist frequency, x(t) can be recovered exactly.
- From a mathematical perspective, we need Fourier and the frequency domain to prove it.