http://elec3004.com

Applications in Industry – (Extended) Kalman Filter

ELEC 3004: **Systems**: Signals & Controls
Dr. Surya Singh

Lecture 22
(with material from B. Lathi and Jur van den Berg)

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May 26, 2016

Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	24-Mar	Convolution Review
	28-Mar	Holiday
	31-Mar	
5	4-Apr	Frequency Response & Filter Analysis
	7-Apr	Filters
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Introduction to Feedback Control
9	3-May	Holiday
	5-May	Feedback Control & Regulation
10	9-May	Servoregulation/PID
	12-May	Introduction to (Digital) Control
11	16-May	Digital Control Design & State-Space
	19-May	Observability, Controllability & Stability of Digital Systems
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	2-Jun	Summary and Course Review

Announcements

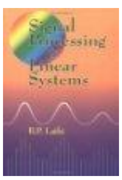
PS 3 Peer Review Competition

- The PS 3 Review with the highest Likert Score
- Deadline for reviews: June 3 (11:59 pm)
- Good reviews discussed June 2nd Last Lecture
- Reward: 3004¢

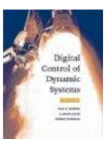
ELEC 3004 "Review Lab" ("Lab 5"):

- Redo any aspects of any of the labs
- Review course
- Review 2015 Final exam (which we will do on June 2nd also)

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998



G. Franklin,
J. Powell,
M. Workman
Digital Control of Dynamic Systems
1990

TJ216.F72 1990
[Available as UQ Ebook]

Today

➔ **State-space** 🡐

- FPW
  - Chapter 6 - Design of Digital Control Systems Using State-Space Methods

- Lathi Ch. 2 (?)
  - § 2.7-6 Time Constant and Rate of Information Transmission
- Information Theory!

Next Time

</assessable>

- Nothing beyond this point is on the exam.
  (except for the exam review ☺)
- Do not pay attention.
- Do not attempt to learn.

Example 0:
`tf2ss`

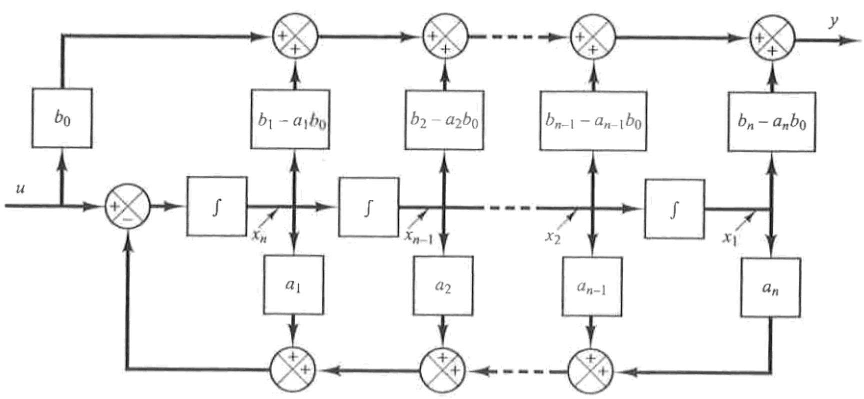
TF 2 SS – Control Canonical Form)

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Control Canonical Form as a Block Diagram



Modal Form

- CCF is not the only way to tf2ss
- Partial-fraction expansion of the system

➔ System poles appear as diagonals of Am

- Two issues:
  - The elements of matrix maybe complex if the poles are complex
  - It is non-diagonal with repeated poles

Modal Form

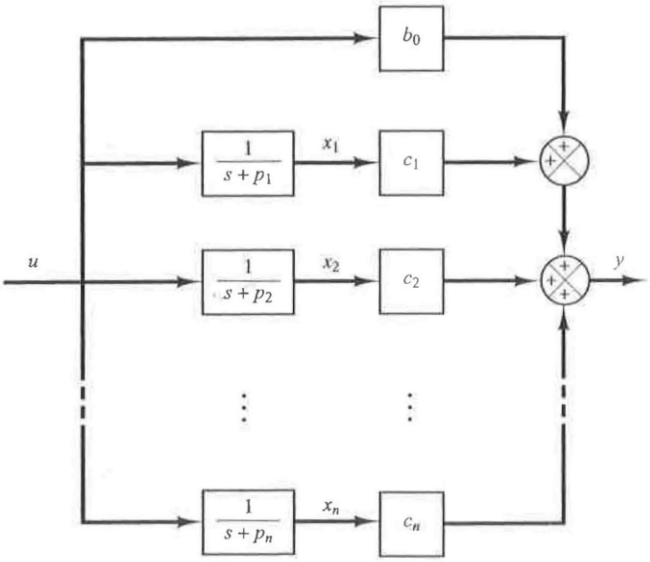
$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

$$= b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$+ \quad y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Modal Form Block Diagram



Matlab's tf2ss

- Given: $\frac{Y(s)}{U(s)} = \frac{25.04s+5.008}{s^3+5.03247s^2+25.1026s+5.008}$

  Get a state space representation of this system

- Matlab:

```
num = [25.04 5.008];
den = [1 5.03247 25.1026 5.008];
[A,B,C,D] = tf2ss(num/den);
```

- Answer:

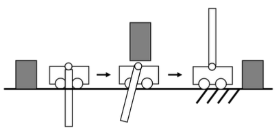
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -5.0325 & -25.1026 & -5.008 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

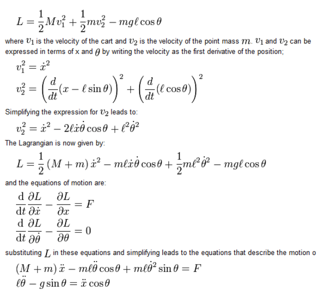
$$y = [0 \quad 25.04 \quad 5.008] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Example 1:
Inverted Pendulum

Digital Control

Wikipedia,
Cart and pole





$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$

$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

and the equations of motion are:

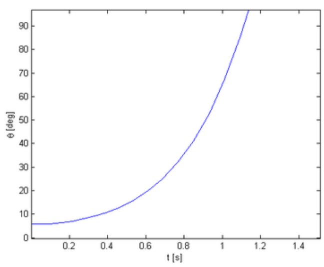
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

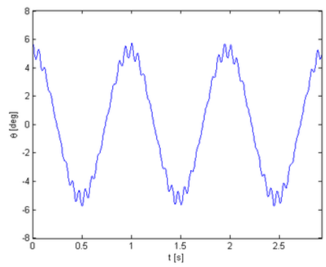
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

substituting L in these equations and simplifying leads to the equations that describe the motion o

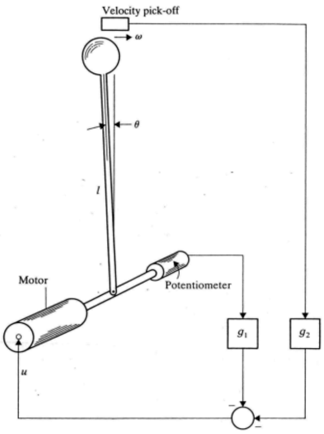
$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$

$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$





Inverted Pendulum



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

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and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

substituting L in these equations and simplifying leads to the equations that describe the motion o

$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$
$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$

Inverted Pendulum – Equations of Motion

- The equations of motion of an inverted pendulum (under a small angle approximation) may be linearized as:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \ddot{\theta} = Q^2\theta + Pu$$

Where:

$$Q^2 = \left(\frac{M+m}{Ml}\right)g$$
$$P = \frac{1}{Ml}.$$

If we further assume unity ***Ml*** ($Ml \approx 1$), then $P \approx 1$

Inverted Pendulum –State Space

- We then select a state-vector as:

$$\boldsymbol{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}, \text{ hence } \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix}$$

- Hence giving a state-space model as:

$$A = \begin{bmatrix} 0 & 1 \\ Q^2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The resolvent of which is:

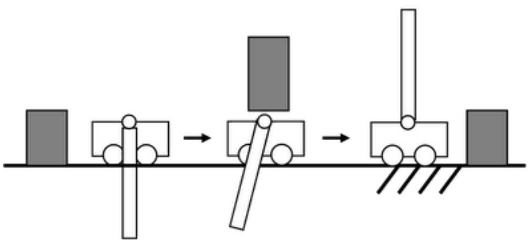
$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ -Q^2 & s \end{bmatrix}^{-1} = \frac{1}{s^2 - Q^2} \begin{bmatrix} s & 1 \\ Q^2 & s \end{bmatrix}$$

- And a state-transition matrix as:

$$\Phi(t) = \begin{bmatrix} \cosh Qt & \dfrac{\sinh Qt}{Q} \\ Q\ sinh\ Qt & \cosh Qt \end{bmatrix}$$

Cart & Pole in State-Space With Obstacles?

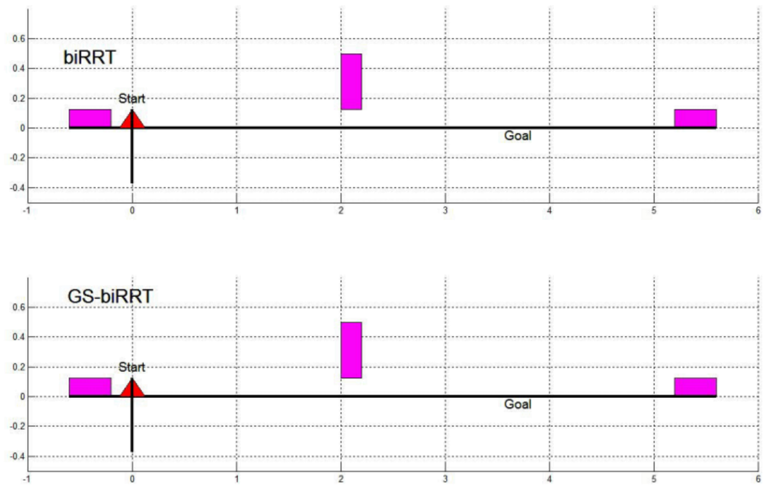
Swing-up is a little more than stabilization…



See also: METR4202 – Tutorial 11:
http://robotics.itee.uq.edu.au/~metr4202/tpl/t11-Week11-pendulum.pdf

Cart & Pole in State-Space

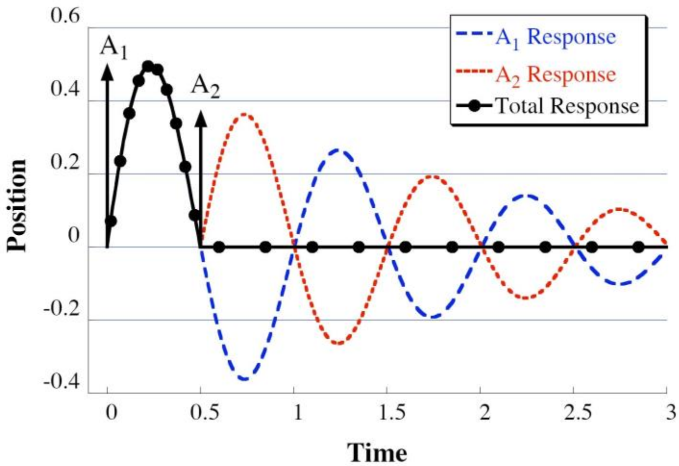
Swing-up is a little more than stabilization…



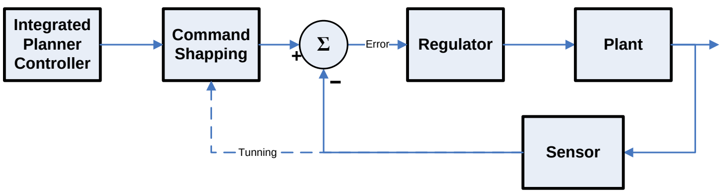
Example 2: Command Shaping

Experiments: Scanning Over Obstacle

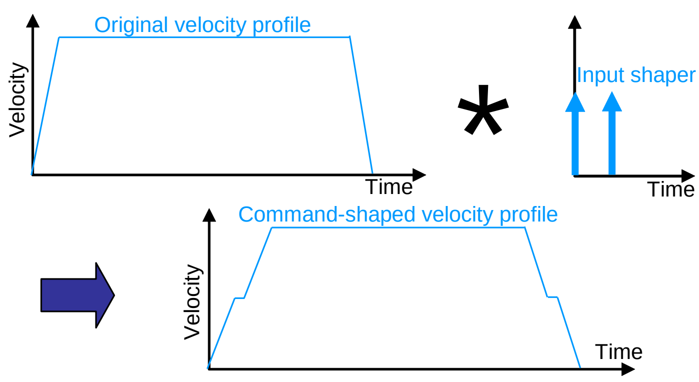
Command Shaping



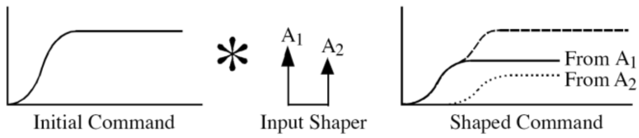
Robust Control:
Command Shaping for Vibration Reduction



Command Shaping



Command Shaping



- Zero Vibration (ZV)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{1+K} & \frac{K}{1+K} \\ 0 & \frac{T_d}{2} \end{bmatrix} \qquad K = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

- Zero Vibration and Derivative (ZVD)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{(1+K)^2} & \frac{2K}{(1+K)^2} & \frac{K^2}{(1+K)^2} \\ 0 & \frac{T_d}{2} & T_d \end{bmatrix}$$

Experiments: Command Shaping

Estimation:
Yet another way
to beat the noise

Along multiple dimensions

State Space

- We collect our set of uncertain variables into a vector …
  $\mathbf{x} = [x_1, x_2, \ldots, x_N]^T$
- The set of values that **x** might take on is termed the *state space*
- There is a *single* true value for **x**,
  but it is unknown

State Space Dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$H(s) = C(sI - A)^{-1}B$$

Measured versus True

- Measurement errors are inevitable
- So, add Noise to State...
  - State Dynamics becomes:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + w$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + v$$

- Can represent this as a "Normal" Distribution

$$\mathcal{N}(x;\mu,\sigma) = \frac{1}{\left(\sqrt{2\pi}\right)\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Recovering The Truth

- Numerous methods
- Termed "Estimation" because we are trying to estimate the truth from the signal
- A strategy discovered by Gauss
- Least Squares in Matrix Representation

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} n & \sum_1^n t_i \\ \sum_1^n t_i & \sum_1^n t_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_1^n z_i \\ \sum_1^n t_i z_i \end{bmatrix}$$

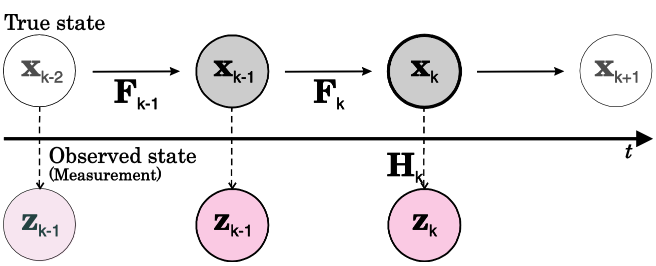
Recovering the Truth: Terminology

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u} + \mathbf{w}$$

$$\mathbf{z} = \mathbf{H}\mathbf{x} + \mathbf{v}$$

- $\mathbf{x}$: the state vector
- $\mathbf{x}_{A|B}$: the state of $\mathbf{x}$ at time A based on data taken up to time B
- $\hat{\mathbf{x}}$: estimate of the true state vector
- $\mathbf{F}$: system dynamics matrix in continuous time (equivalent to $\mathbf{A}$ in Eq. 1)
- $\mathbf{G}$: system control matrix relating deterministic input, $\mathbf{u}$, to the state (equivalent to $\mathbf{B}$ in Eq. 1)
- $\mathbf{H}$: measurement matrix in continuous time (equivalent to $\mathbf{C}$ in Eq. 2)
- $\mathbf{F}_i$: system model in **discrete** time at $t = t_i$
- $\mathbf{H}_i$: measurement model in **discrete** time at $t = t_i$
- $\mathbf{P}_i$: estimate covariance in **discrete** time at $t = t_i$
- $\mathbf{w}$: process uncertainty (noise) vector (of type $\mathcal{N}(0, s)$)
- $\mathbf{Q}$: process noise matrix, $\mathbf{Q} = E\left[ww^{\mathsf{T}}\right]$
- $\mathbf{Q}_i$: $\mathbf{Q}$ in discrete time at $t = t_i$
- $\mathbf{v}$: measurement noise vectors (of type $\mathcal{N}(0, \sigma)$)
- $\mathbf{R}_i$: the measurement variance matrix, $\mathbf{R} = E\left[vv^{\mathsf{T}}\right]$, in discrete time at $t = t_i$

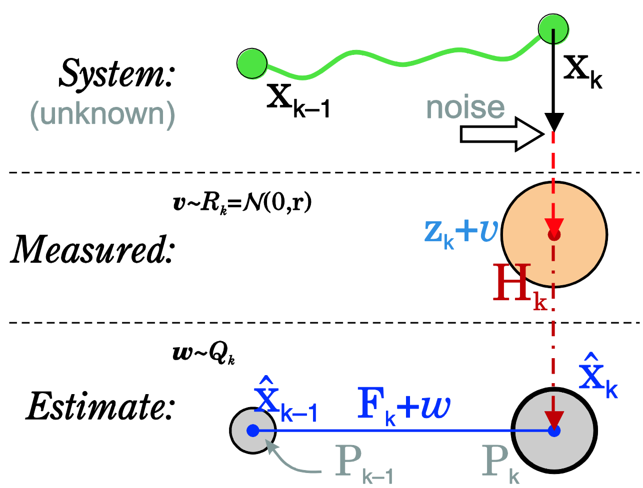
General Problem…



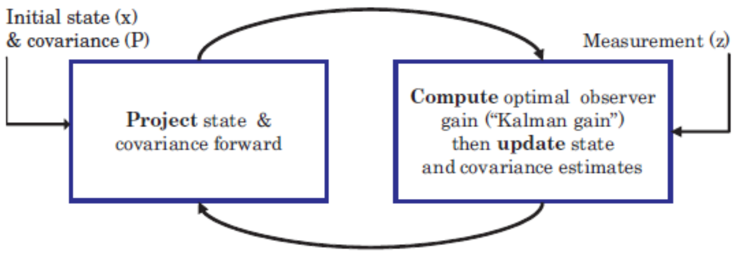
Duals and Dual Terminology

	Estimation		Control
Model:	$\dot{x} = Fx$ (discrete: $x = F_k x$)	$\leftrightarrow$	$\dot{x} = Ax$, $A = F^T$
Regulates:	P (covariance)	$\leftrightarrow$	M (performance matrix)
Minimized function:	Q (or GQG^T)	$\leftrightarrow$	V
Optimal Gain:	K	$\leftrightarrow$	G
Completeness law:	Observability	$\leftrightarrow$	Controllability

Estimation Process in Pictures



Kalman Filter Process



KF Process in Equations

Prediction: $\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1|k-1},$ (state prediction)

$\mathbf{P}_{k|k-1} = \mathbf{Q}_{k-1} + \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\mathsf{T}}{}_{k-1},$ (covariance prediction)

Kalman Gain: $\mathbf{K}_k = \mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}}[\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_k]^{-1},$

Update: $\mathbf{P}_{k|k} = [\mathbf{I} - \mathbf{K}_k\mathbf{H}]\mathbf{P}_{k|k-1},$ (covariance update)

$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\hat{\mathbf{x}}_{k|k-1})$ (state update)

KF Considerations

$$\underbrace{\hat{\mathbf{x}}_{k|k-1}}_{n\times 1} = \underbrace{\mathbf{F}_{k-1}}_{n\times n}\hat{\mathbf{x}}_{k-1|k-1} + \underbrace{\mathbf{G}_{k-1}}_{n\times j}\underbrace{\mathbf{u}_{k-1}}_{j\times 1}$$

$$\underbrace{\mathbf{P}_{k|k-1}}_{n\times n} = \underbrace{\mathbf{Q}_{k-1}}_{n\times n} + \mathbf{F}_{k-1}\mathbf{P}_{k-1|k-1}\mathbf{F}^{\mathsf{T}}{}_{k-1}$$

$$\underbrace{\mathbf{K}_{k}}_{n\times m} = \mathbf{P}_{k|k-1}\underbrace{\mathbf{H}^{\mathsf{T}}}_{n\times m}\underbrace{\left[\mathbf{H}\mathbf{P}_{k|k-1}\mathbf{H}^{\mathsf{T}} + \mathbf{R}_{k}\right]^{-1}}_{m\times m}$$

$$\mathbf{P}_{k|k} = [\mathbf{I} - \mathbf{K}_{k}\mathbf{H}]\mathbf{P}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_{k}(\underbrace{\mathbf{z}_{k}}_{m\times 1} - \underbrace{\mathbf{H}}_{m\times n}\hat{\mathbf{x}}_{k|k-1} - \mathbf{H}\mathbf{G}_{k}\mathbf{u}_{k-1})$$

Ex: Kinematic KF: Tracking

- Consider a System with Constant Acceleration

$$\ddot{y} = -g$$
$$\dot{y} = gt + p_1$$
$$y = p_0 + p_1 t + \frac{gt^2}{2}$$

$$\begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ g \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \mathbf{F_k} = \begin{bmatrix} 0 & t_s & \frac{t_s{}^2}{2} \\ 0 & 0 & t_s \\ 0 & 0 & 0 \end{bmatrix}$$

$$\hat{\mathbf{x}}_k = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{K}_k(\mathbf{z}_k - \mathbf{H}\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1})$$

In Summary

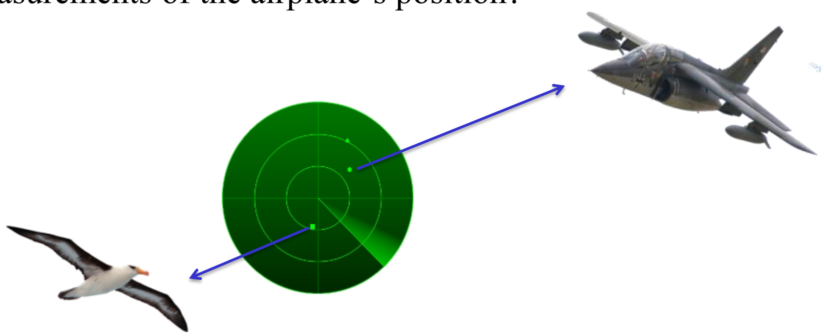
- KF:
  - The true state (*x*) is separate from the measured (*z*)
  - Lets you **combine** prior controls knowledge with measurements to filter signals and find the truth
  - It **regulates** the covariance (P)
    - As P is the scatter between *z* and *x*
    - So, if P → 0, then *z* → *x* (measurements → truth)
- EKF:
  - Takes a Taylor series approximation to get a local "F" (and "G" and "H")

Estimation:
"Bayesian Perspective"

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Kalman Filtering

- (Optimal) estimation of the (hidden) state of a linear dynamic process of which we obtain noisy (partial) measurements
- Example: radar tracking of an airplane.
  What is the state of an airplane given noisy radar measurements of the airplane's position?



Model

- Discrete time steps, continuous state-space
- (Hidden) state: $\mathbf{x}_t$, measurement: $\mathbf{y}_t$
- Airplane example:
- Position, speed and acceleration

$$\mathbf{x}_t = \begin{pmatrix} x_t \\ \dot{x}_t \\ \ddot{x}_t \end{pmatrix}, \quad \mathbf{y}_t = \left(\tilde{x}_t\right)$$

Dynamics and Observation model

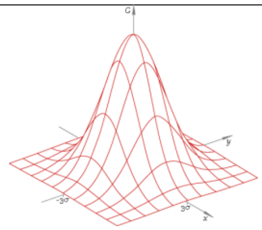
- Linear dynamics **model** describes relation between the state and the next state, and the observation:
- Airplane example (if process has time-step δ):

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim W_t = N(\mathbf{0}, Q) \\ \mathbf{y}_t &= C\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim V_t = N(\mathbf{0}, R) \end{aligned}$$

$$A = \begin{pmatrix} 1 & \delta & \frac{1}{2}\delta^2 \\ 0 & 1 & \delta \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Normal distributions

- Let X_0 be a normal distribution of the initial state $\mathbf{x}_0$
- Then, every X_t is a normal distribution of hidden state $\mathbf{x}_t$. Recursive definition:



- And every Y_t is a normal distribution of observation $\mathbf{y}_t$. Definition:

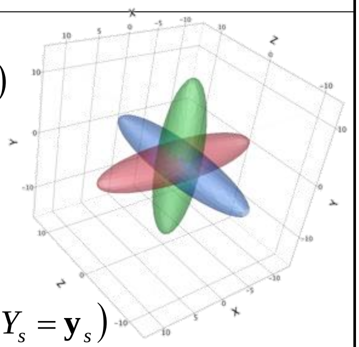
$$X_{t+1} = AX_t + W_t$$

- **Goal of filtering**: compute conditional distribution

$$Y_t = CX_t + V_t$$

$$\left(X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_t = \mathbf{y}_t\right)$$

Normal distribution



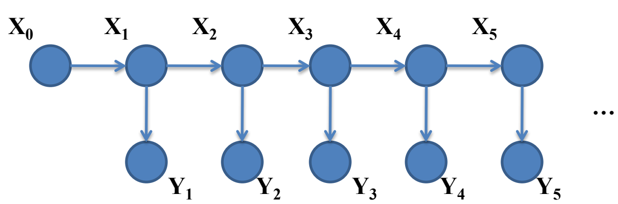
- Because X_t's and Y_t's are normal distributions, $(X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_t = \mathbf{y}_t)$ is also a normal distribution
- Normal distribution is fully specified by mean and covariance
- We denote:

$$
\begin{aligned}
X_{t|s} &= (X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_s = \mathbf{y}_s) \\
&= N\left(\mathrm{E}(X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_s = \mathbf{y}_s), \mathrm{Var}(X_t \mid Y_0 = \mathbf{y}_0, \ldots, Y_s = \mathbf{y}_s)\right) \\
&= N\left(\hat{\mathbf{x}}_{t|s}, P_{t|s}\right)
\end{aligned}
$$

Problem reduces to computing $\underline{\mathbf{x}}_{t|t}$ and $P_{t|t}$

Recursive update of state

- Kalman filtering algorithm: repeat…
  - Time update:
    from $X_{t|t}$, compute **a priori** distribution $X_{t+1|t}$
  - Measurement update:
    from $X_{t+1|t}$ (and given $\mathbf{y}_{t+1}$), compute
    a posteriori distribution $X_{t+1|t+1}$



Time update

- From $X_{t|t}$, compute **a priori** distribution $X_{t+1|t}$:

$$
\begin{aligned}
X_{t+1|t} &= AX_{t|t} + W_t \\
&= N\left(\mathrm{E}\left(AX_{t|t} + W_t\right), \mathrm{Var}\left(AX_{t|t} + W_t\right)\right) \\
&= N\left(A\,\mathrm{E}\left(X_{t|t}\right) + \mathrm{E}\left(W_t\right), A\,\mathrm{Var}\left(X_{t|t}\right)A^T + \mathrm{Var}\left(W_t\right)\right) \\
&= N\left(A\hat{\mathbf{x}}_{t|t}, AP_{t|t}A^T + Q\right)
\end{aligned}
$$

- So:

$$
\hat{\mathbf{x}}_{t+1|t} = A\hat{\mathbf{x}}_{t|t}
$$

$$
P_{t+1|t} = AP_{t|t}A^T + Q
$$

Measurement update

From $X_{t+1|t}$ (and given $\mathbf{y}_{t+1}$), compute $X_{t+1|t+1}$.

1. Compute **a priori** distribution of the observation $Y_{t+1|t}$ from $X_{t+1|t}$:

$$
\begin{aligned}
Y_{t+1|t} &= CX_{t+1|t} + V_{t+1} \\
&= N\left(\mathrm{E}\left(CX_{t+1|t} + V_{t+1}\right), \mathrm{Var}\left(CX_{t+1|t} + V_{t+1}\right)\right) \\
&= N\left(C\,\mathrm{E}\left(X_{t+1|t}\right) + \mathrm{E}\left(V_{t+1}\right), C\,\mathrm{Var}\left(X_{t+1|t}\right)C^T + \mathrm{Var}\left(V_{t+1}\right)\right) \\
&= N\left(C\hat{\mathbf{x}}_{t+1|t}, CP_{t+1|t}C^T + R\right)
\end{aligned}
$$

Measurement update (cont'd)

2. Look at joint distribution of $X_{t+1|t}$ and $Y_{t+1|t}$:

$$\begin{aligned}(X_{t+1|t}, Y_{t+1|t}) &= N\left(\begin{pmatrix} \mathrm{E}(X_{t+1|t}) \\ \mathrm{E}(Y_{t+1|t}) \end{pmatrix}, \begin{pmatrix} \mathrm{Var}(X_{t+1|t}) & \mathrm{Cov}(X_{t+1|t}, Y_{t+1|t}) \\ \mathrm{Cov}(Y_{t+1|t}, X_{t+1|t}) & \mathrm{Var}(Y_{t+1|t}) \end{pmatrix}\right) \\ &= N\left(\begin{pmatrix} \hat{\mathbf{x}}_{t+1|t} \\ C\hat{\mathbf{x}}_{t+1|t} \end{pmatrix}, \begin{pmatrix} P_{t+1|t} & P_{t+1|t}C^T \\ CP_{t+1|t} & CP_{t+1|t}C^T + R \end{pmatrix}\right)\end{aligned}$$

where

$$\begin{aligned}\mathrm{Cov}(Y_{t+1}, X_{t+1|t}) &= \mathrm{Cov}(CX_{t+1|t} + V_{t+1}, X_{t+1|t}) \\ &= C\,\mathrm{Cov}(X_{t+1|t}, X_{t+1|t}) + \mathrm{Cov}(V_{t+1}, X_{t+1|t}) \\ &= C\,\mathrm{Var}(X_{t+1|t}) \\ &= CP_{t+1|t}\end{aligned}$$

Measurement update (cont'd)

- Recall that if

$$(Z_1, Z_2) = N\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\right)$$

then

$$(Z_1 \mid Z_2 = \mathbf{z}_2) = N\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{z}_2 - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\right)$$

3. Compute $X_{t+1|t+1} = (X_{t+1|t} | Y_{t+1|t} = \mathbf{y}_{t+1})$:

$$\begin{aligned}X_{t+1|t+1} &= (X_{t+1|t} \mid Y_{t+1|t} = \mathbf{y}_{t+1}) \\ &= N\Big(\hat{\mathbf{x}}_{t+1|t} + P_{t+1|t}C^T(CP_{t+1|t}C^T + R)^{-1}(\mathbf{y}_{t+1} - C\hat{\mathbf{x}}_{t+1|t}), \\ &\qquad P_{t+1|t} - P_{t+1|t}C^T(CP_{t+1|t}C^T + R)^{-1}CP_{t+1|t}\Big)\end{aligned}$$

Measurement update (cont'd):

This can also (often) be written in terms of the **Kalman gain** matrix:

$$
\begin{aligned}
K_{t+1} &= P_{t+1|t}C^T\left(CP_{t+1|t}C^T + R\right)^{-1} \\
\hat{\mathbf{x}}_{t+1|t+1} &= \hat{\mathbf{x}}_{t+1|t} + K_{t+1}\left(\mathbf{y}_{t+1} - C\hat{\mathbf{x}}_{t+1|t}\right) \\
P_{t+1|t+1} &= P_{t+1|t} - K_{t+1}CP_{t+1|t}
\end{aligned}
$$

Initialization

- Choose distribution of initial state by picking $\underline{\mathbf{x}}_0$ and P_0
- Start with measurement update given measurement $\mathbf{y}_0$
- Choice for Q and R (identity)
  - small Q: dynamics "trusted" more
  - small R: measurements "trusted" more

(Bayesian) Kalman Filter Summary

I. Model:

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + \mathbf{w}_t, \quad \mathbf{w}_t \sim W_t = N(\mathbf{0}, Q)$$
$$\mathbf{y}_t = C\mathbf{x}_t + \mathbf{v}_t, \quad \mathbf{v}_t \sim V_t = N(\mathbf{0}, R)$$

II. Algorithm: Repeat…

– Time update:

$$\hat{\mathbf{x}}_{t+1|t} = A\hat{\mathbf{x}}_{t|t}$$
$$P_{t+1|t} = AP_{t|t}A^T + Q$$

– Measurement update:

$$K_{t+1} = P_{t+1|t}C^T\left(CP_{t+1|t}C^T + R\right)^{-1}$$
$$\hat{\mathbf{x}}_{t+1|t+1} = \hat{\mathbf{x}}_{t+1|t} + K_{t+1}\left(\mathbf{y}_{t+1} - C\hat{\mathbf{x}}_{t+1|t}\right)$$
$$P_{t+1|t+1} = P_{t+1|t} - K_{t+1}CP_{t+1|t}$$

(Bayesian) Kalman Filter Summary [II]

Take Aways:

- Kalman filter can be used in real time
- Use $\underline{\mathbf{x}}_{t|t}$'s as optimal estimate of state at time t, and use $P_{t|t}$ as a measure of uncertainty.

Extensions:

- Dynamic process with known **control input**
- **Non-linear** dynamic process
- **Kalman smoothing**: compute optimal estimate of state $\mathbf{x}_t$ given all data $\mathbf{y}_1, \ldots, \mathbf{y}_T$, with T > t (not real-time).
- Automatic parameter (Q and R) fitting using EM-algorithm

Next Time…

- **Information Theory & More!**

- Review:
  - Chapter 6 of FPW
  - Chapter 13 of Lathi

- Deeper Pondering??

Final Exam Review

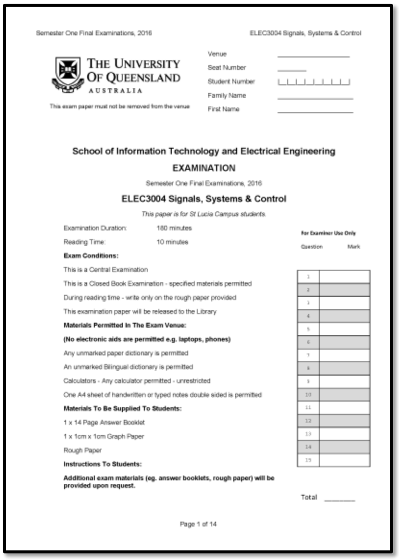
- June 10, 2016

- From: 2-4

- In: 45-204 (???)

- Some Review Notes (from Course Textbooks)

→ http://robotics.itee.uq.edu.au/~elec3004/tutes.html



Semester One Final Examinations, 2016 ELEC3004 Signals, Systems & Control

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