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Shaping the Dynamic Response

ELEC 3004: **Systems**: Signals & Controls
Dr. Surya Singh

Lecture 21
(with material from FPW and Lathi)

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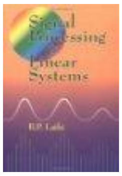
May 23, 2016

Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	24-Mar	Convolution Review
	28-Mar	Holiday
	31-Mar	
5	4-Apr	Frequency Response & Filter Analysis
	7-Apr	Filters
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Introduction to Feedback Control
9	3-May	Holiday
	5-May	Feedback Control & Regulation
10	9-May	Servoregulation/PID
	12-May	Introduction to (Digital) Control
11	16-May	Digital Control Design & State-Space
	19-May	Observability, Controllability & Stability of Digital Systems
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	2-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi

Signal processing and linear systems

1998

TK5102.9.L38 1998

G. Franklin,
J. Powell,
M. Workman

Digital Control of Dynamic Systems
1990

TJ216.F72 1990
[Available as UQ Ebook]

Today

➔ State-space 🡰

- FPW
  - Chapter 6 - Design of Digital Control Systems Using State-Space Methods

- Lathi Ch. 13
  - § 13.2 Systematic Procedure for Determining State Equations
  - § 13.3 Solution of State Equations
  - Using State-Space Methods

Next Time

More Online Reading Materials

CHAPTER
SIX

SHAPING THE DYNAMIC RESPONSE

6.1 INTRODUCTION

At last we have arrived at the point of using state-space methods for control system design. In this chapter we will develop a simple method of designing a control system for a process in which all the state variables are accessible for measurement—the method known as *pole-placement*. We will find that in a controllable system, with all the state variables accessible for measurement, it is possible to place the closed-loop poles anywhere we wish in the complex s plane. This means that we can, in principle, completely specify the closed-loop dynamic performance of the system. In principle, we can start with a sluggish open-loop system and force it to behave with alacrity; in principle, we can start with a system that has very little open-loop damping and provide any amount of damping desired. Unfortunately, however, what can be attained in principle may not be attainable in practice. Speeding the response of a sluggish system requires the use of large control signals which the actuator (or power supply) may not be capable of delivering. The consequence is generally that the actuator saturates at the largest signal that it can supply. In some instances the system behavior may be acceptable in spite of the saturation. But in other cases the effect of saturation is to make the closed-loop system unstable. It is usually not possible to alter open-loop dynamic behavior very drastically without creating practical difficulties.

Adding a great deal of damping to a system having poles near the imaginary axis is also problematic, not only because of the magnitude of the control signals needed, but also because the control system gains are very sensitive to the location of the open-loop poles. Slight changes in the open-loop pole

222

CHAPTER
THREE

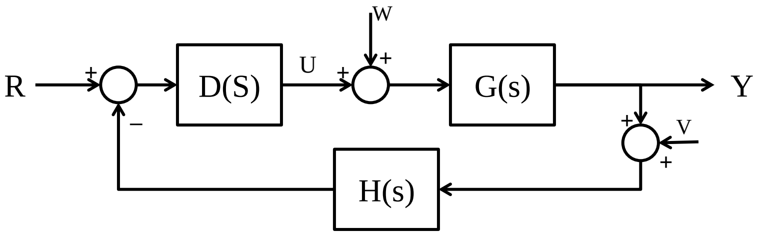
DYNAMICS OF LINEAR SYSTEMS

3.1 DIFFERENTIAL EQUATIONS REVISITED

- Friedland, Control System Design Ch. 6 and 3

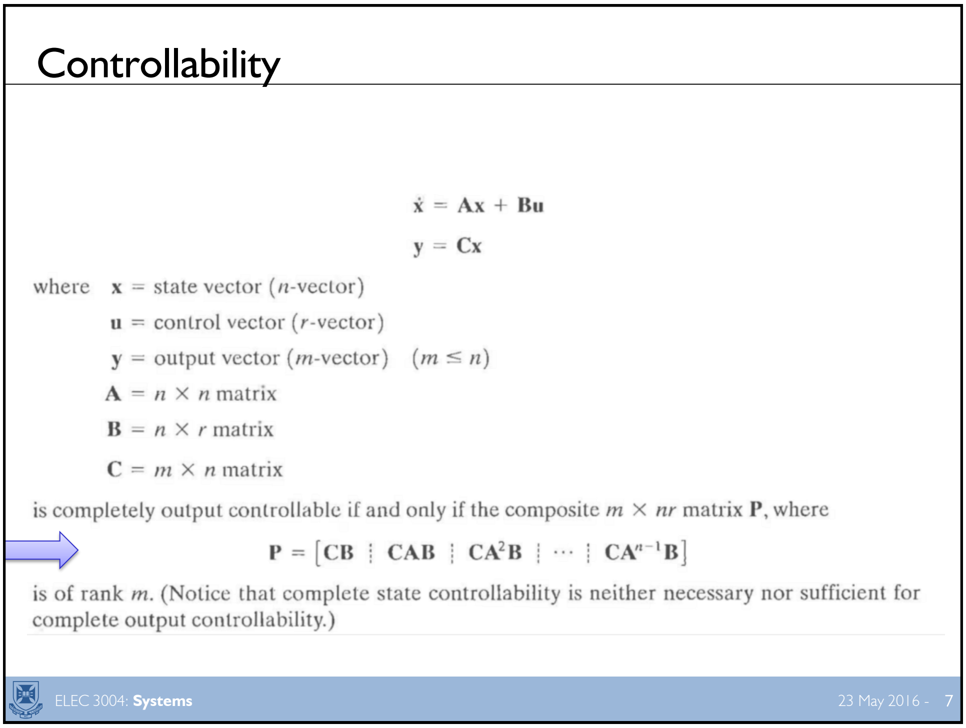
➔ **http://robotics.itee.uq.edu.au/~elec3004/tutes.html**

Basic Closed-loop Block Diagram



$$TF(s) = \frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1 + D(s)G(s)H(s)} = \frac{DG}{1 + DGH}$$

Controlabilty



Controllability

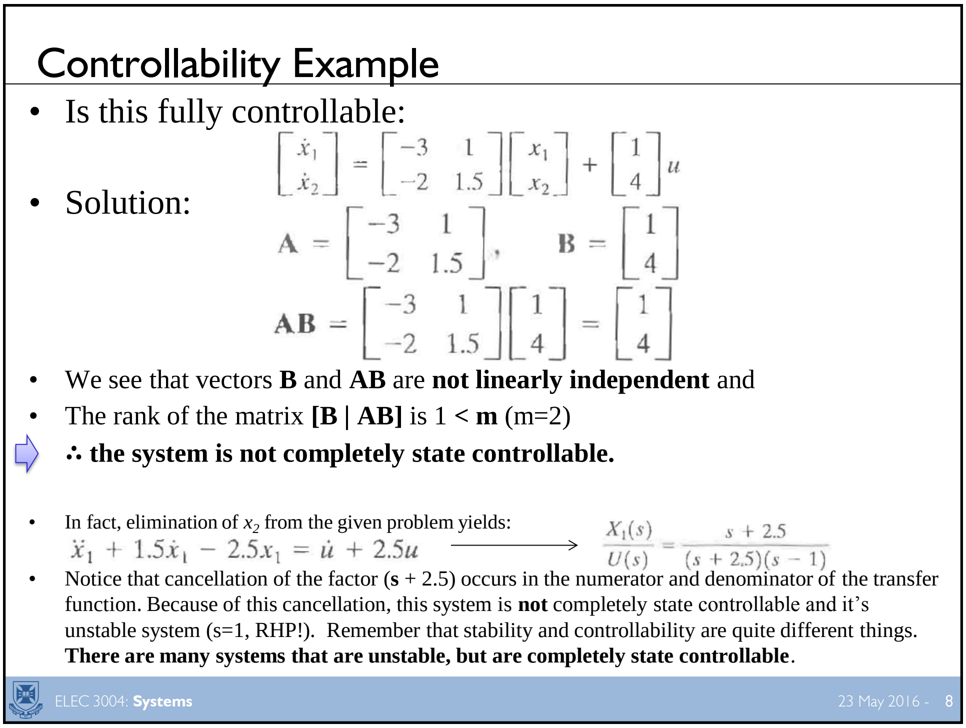
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where $\mathbf{x}$ = state vector (n-vector)
$\mathbf{u}$ = control vector (r-vector)
$\mathbf{y}$ = output vector (m-vector) $(m \leq n)$
$\mathbf{A}$ = $n \times n$ matrix
$\mathbf{B}$ = $n \times r$ matrix
$\mathbf{C}$ = $m \times n$ matrix

is completely output controllable if and only if the composite $m \times nr$ matrix $\mathbf{P}$, where

$$\mathbf{P} = [\mathbf{CB} \mid \mathbf{CAB} \mid \mathbf{CA}^2\mathbf{B} \mid \cdots \mid \mathbf{CA}^{n-1}\mathbf{B}]$$

is of rank m. (Notice that complete state controllability is neither necessary nor sufficient for complete output controllability.)



Controllability Example

- Is this fully controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- We see that vectors **B** and **AB** are **not linearly independent** and
- The rank of the matrix **[B | AB]** is 1 **< m** (m=2)
- **∴ the system is not completely state controllable.**

- In fact, elimination of x_2 from the given problem yields:

$$\ddot{x}_1 + 1.5\dot{x}_1 - 2.5x_1 = \dot{u} + 2.5u \longrightarrow \frac{X_1(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$$

- Notice that cancellation of the factor (**s** + 2.5) occurs in the numerator and denominator of the transfer function. Because of this cancellation, this system is **not** completely state controllable and it's unstable system (s=1, RHP!). Remember that stability and controllability are quite different things. **There are many systems that are unstable, but are completely state controllable**.

Controllability Example II

- TF → CCF

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -0.4 & -1.3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}0.8 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\longrightarrow$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -0.4\\ 1 & -1.3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0.8\\ 1\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Solution. Consider the system defined by Equations (9–120) and (9–121). The rank of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix}0 & 1\\ 1 & -1.3\end{bmatrix}$$

is 2. Hence, the system is completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix}0.8 & -0.4\\ 1 & -0.5\end{bmatrix}$$

is 1. Hence the system is not observable.

Next consider the system defined by Equations (9–122) and (9–123). The rank of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix}0.8 & -0.4\\ 1 & -0.5\end{bmatrix}$$

is 1. Hence, the system is not completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix}0 & 1\\ 1 & -1.3\end{bmatrix}$$

is 2. Hence, the system is observable.

The apparent difference in the controllability and observability of the same system is caused by the fact that the original system has a pole-zero cancellation in the transfer function. Referring to Equation (2–29), for $D = 0$ we have

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

If we use Equations (9–120) and (9–121), then

$$\begin{aligned} G(s) &= \begin{bmatrix}0.8 & 1\end{bmatrix}\begin{bmatrix}s & -1\\ 0.4 & s+1.3\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix} \\ &= \frac{1}{s^2 + 1.3s + 0.4}\begin{bmatrix}0.8 & 1\end{bmatrix}\begin{bmatrix}s+1.3 & 1\\ -0.4 & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} \\ &= \frac{s+0.8}{(s+0.8)(s+0.5)} \end{aligned}$$

[Note that the same transfer function can be obtained by using Equations (9–122) and (9–123).] Clearly, cancellation occurs in this transfer function.

Stability

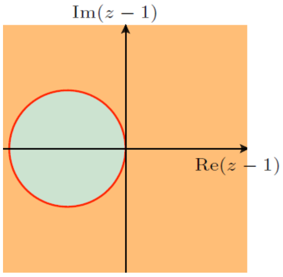
Fast sampling revisited

- For small T:

$$z = e^{sT} = 1 + sT + \frac{(sT)^2}{2} + \ldots \approx 1 + sT$$

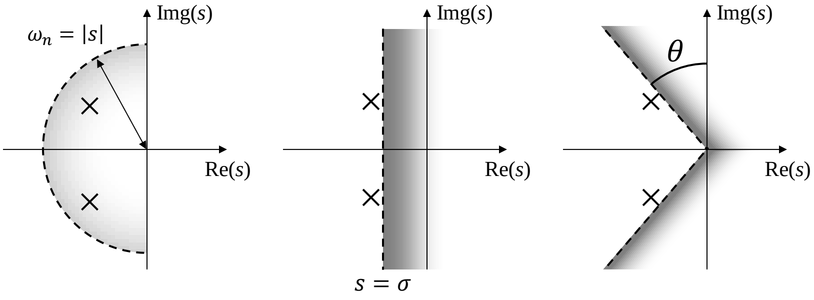
$$\rightarrow z \approx 1 + sT \rightarrow s = \frac{z-1}{T}$$

- Hence, the unit circle under the map from z to s-plane becomes:



Specification bounds

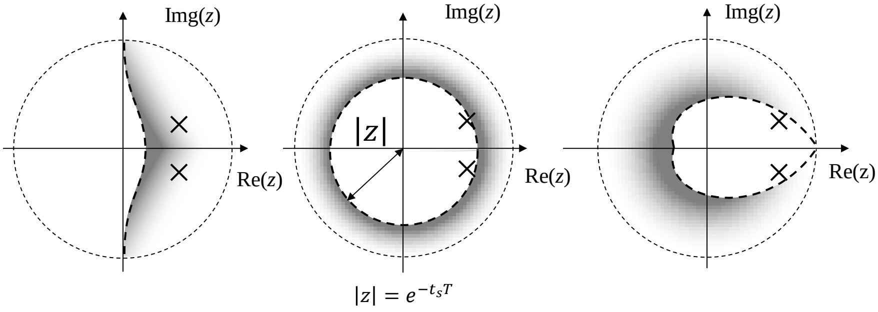
- Recall in the continuous domain, response performance metrics map to the s-plane:



$$|s| = \frac{1.8}{t_r} \qquad s = \frac{4.6}{t_s} \qquad \theta = \sin^{-1}\zeta$$

Discrete bounds

- These map to the discrete domain:



In practice, you'd use Matlab to plot these, and check that the spec is satisfied

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Example Code:

```
%% Input System Model G
numg=5; deng=[1 20 0]; sysg=tf(numg, deng);

%% Approximate the ZOH (1-e^-sT)/(s)
[nd, dd]=pade(1,2);  %pade gives us the "hold" or -e^-sT of a ZOH
sysp=tf(nd, dd); sysi=tf([1],[1,0]); %Now we need the "1/s" portion
sys1=series(1-sysp, sysi); % Approximation as a series

%% Open loop response
syso=series(sys1, sysg); % computer the open loop G with the ZOH
sys=feedback(syso,1); % Computer the unity feedback response
step(sys) % Display the step response
```

Obtaining a Time Response

From SS to Time Response — Impulse Functions

- Given: $\dot{x} = Ax + Bu$
- Solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Substituting $t_0 = 0$ into this:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Write the impulse as: $\mathbf{u}(t) = \delta(t)\mathbf{w}$
  - where **w** is a vector whose components are the magnitudes of **r** impulse functions applied at t=0

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)\mathbf{w}\,d\tau$$

⇒

$$= e^{\mathbf{A}t}\mathbf{x}(0-) + e^{\mathbf{A}t}\mathbf{B}\mathbf{w}$$

From SS to Time Response — Step Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = \boldsymbol{k}$
  Where **k** is a vector whose components are the magnitudes of *r* step functions applied at t=0.

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{k}\,d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[\int_0^t \left(\mathbf{I} - \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} - \cdots\right)d\tau\right]\mathbf{B}\mathbf{k} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{\mathbf{A}^2t^3}{3!} - \cdots\right)\mathbf{B}\mathbf{k}\end{aligned}$$

  - Assume A is non-singular

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[-(\mathbf{A}^{-1})(e^{-\mathbf{A}t} - \mathbf{I})\right]\mathbf{B}\mathbf{k} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}\mathbf{k}\end{aligned}$$

From SS to Time Response — Ramp Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = t\boldsymbol{v}$
  Where **v** is a vector whose components are magnitudes of ramp functions applied at *t* = 0

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\tau\mathbf{v}\,d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\tau\,d\tau\,\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\frac{\mathbf{I}}{2}t^2 - \frac{2\mathbf{A}}{3!}t^3 + \frac{3\mathbf{A}^2}{4!}t^4 - \frac{4\mathbf{A}^3}{5!}t^5 + \cdots\right)\mathbf{B}\mathbf{v}\end{aligned}$$

  - Assume A is non-singular

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + (\mathbf{A}^{-2})(e^{\mathbf{A}t} - \mathbf{I} - \mathbf{A}t)\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \left[\mathbf{A}^{-2}(e^{\mathbf{A}t} - \mathbf{I}) - \mathbf{A}^{-1}t\right]\mathbf{B}\mathbf{v}\end{aligned}$$

Example: Obtain the Step Response

- Given: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_1 \end{bmatrix} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} u, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u(t) = 1(t)$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s+1 & 0.5 \\ -1 & s \end{bmatrix}^{-1} = \frac{1}{s^2 + s + 0.5}\begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s + 0.5 - 0.5}{(s+0.5)^2 + 0.5^2} & \dfrac{-0.5}{(s+0.5)^2 + 0.5^2} \\ \dfrac{1}{(s+0.5)^2 + 0.5^2} & \dfrac{s + 0.5 + 0.5}{(s+0.5)^2 + 0.5^2} \end{bmatrix}$$

⇒

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \begin{bmatrix} e^{-0.5t}(\cos 0.5t - \sin 0.5t) & -e^{-0.5t}\sin 0.5t \\ 2e^{-0.5t}\sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t) \end{bmatrix}$$

  - Set k=1, x(0)=0:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}k$$

$$= \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5 \\ e^{-0.5t}\sin 0.5t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-0.5t}\sin 0.5t \\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1 \end{bmatrix}$$

⇒

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 = e^{-0.5t}\sin 0.5t$$

Example II: Obtain the Step Response

- Given:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad u(t) = 1(t)$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

⇒

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathscr{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

⇒

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1]\,d\tau$$

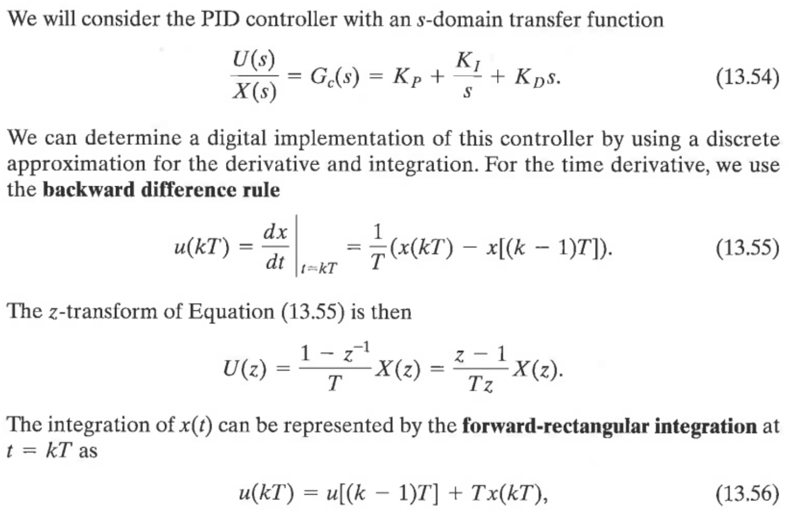
$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

  - Assume x(0)=0:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

Digital PID Controls (Magic PID Made Easy Equations)

Implementation of Digital PID Controllers



We will consider the PID controller with an s-domain transfer function

$$\frac{U(s)}{X(s)} = G_c(s) = K_P + \frac{K_I}{s} + K_D s. \tag{13.54}$$

We can determine a digital implementation of this controller by using a discrete approximation for the derivative and integration. For the time derivative, we use the **backward difference rule**

$$u(kT) = \left.\frac{dx}{dt}\right|_{t=kT} = \frac{1}{T}(x(kT) - x[(k-1)T]). \tag{13.55}$$

The z-transform of Equation (13.55) is then

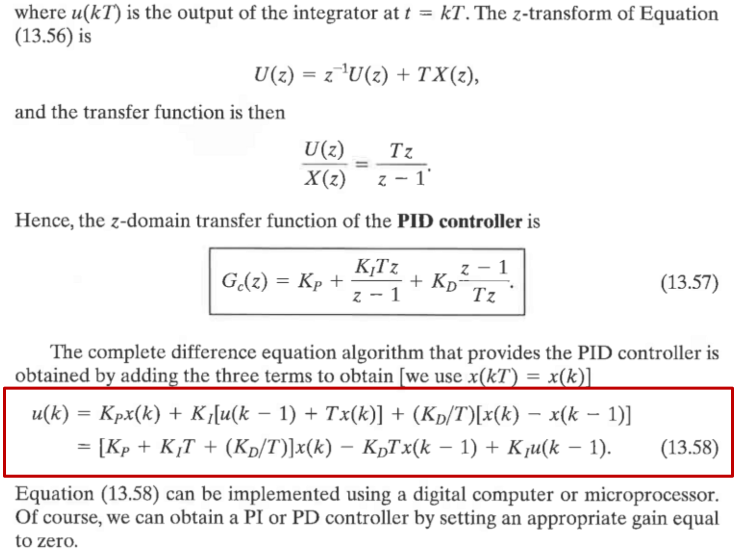
$$U(z) = \frac{1 - z^{-1}}{T}X(z) = \frac{z-1}{Tz}X(z).$$

The integration of $x(t)$ can be represented by the **forward-rectangular integration** at $t = kT$ as

$$u(kT) = u[(k-1)T] + Tx(kT), \tag{13.56}$$

Source: Dorf & Bishop, Modern Control Systems, §13.9, pp. 1030-1

Implementation of Digital PID Controllers (2)



where $u(kT)$ is the output of the integrator at $t = kT$. The z-transform of Equation (13.56) is

$$U(z) = z^{-1}U(z) + TX(z),$$

and the transfer function is then

$$\frac{U(z)}{X(z)} = \frac{Tz}{z-1}.$$

Hence, the z-domain transfer function of the **PID controller** is

$$\boxed{G_c(z) = K_P + \frac{K_I Tz}{z-1} + K_D\frac{z-1}{Tz}.} \qquad (13.57)$$

The complete difference equation algorithm that provides the PID controller is obtained by adding the three terms to obtain [we use $x(kT) = x(k)$]

$$\begin{aligned} u(k) &= K_P x(k) + K_I[u(k-1) + Tx(k)] + (K_D/T)[x(k) - x(k-1)] \\ &= [K_P + K_I T + (K_D/T)]x(k) - K_D T x(k-1) + K_I u(k-1). \qquad (13.58) \end{aligned}$$

Equation (13.58) can be implemented using a digital computer or microprocessor. Of course, we can obtain a PI or PD controller by setting an appropriate gain equal to zero.

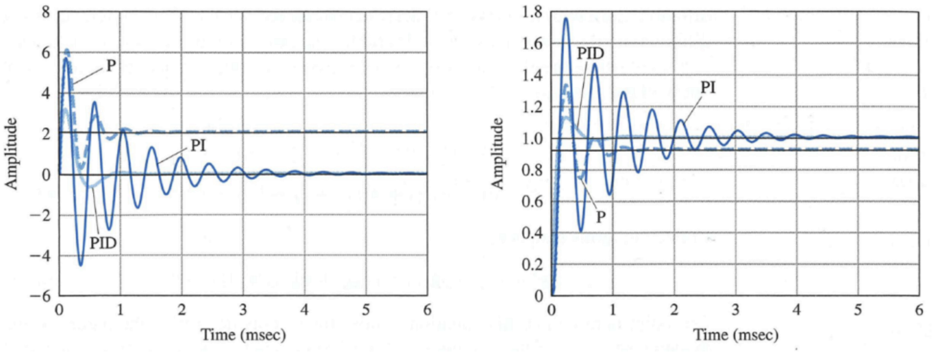
Source: Dorf & Bishop, Modern Control Systems, §13.9, pp. 1030-1

PID Intuition

Effects of increasing a parameter independently					
Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability
K_p	↓	⇑	Minimal change	↓	↓
K_I	↓	⇑	⇑	Eliminate	↓
K_D	Minor change	↓	↓	No effect / minimal change	Improve (if K_D small)

PID Intuition: P and PI and PID

- Responses of P, PI, and PID control to



(a) step disturbance input

(b) step reference input

Shaping the Dynamic Response: Pole Placement

Pole Placement (Following FPW – Chapter 6)

- FPW has a slightly different notation:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u,$$
$$y = \mathbf{H}\mathbf{x}.$$

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k),$$
$$y(k) = \mathbf{H}\mathbf{x}(k),$$

$$\mathbf{\Phi} = e^{\mathbf{F}T},$$

$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G},$$

Pole Placement

- Start with a simple feedback control law (“controller”)

$$u = -\mathbf{K}\mathbf{x} = -[K_1 K_2 \ldots] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

- It’s actually a regulator
  ∵ it does not allow for a reference input to the system.
  (there is no “reference” **r** (r = 0))

- Substitute in the difference equation

$$x(k+1) = \Phi x(k) - \Gamma K x(k)$$

- $\mathcal{Z}$ Transform:

$$(zI - \Phi + \Gamma K) X(z) = 0$$

➔Characteristic Eqn:

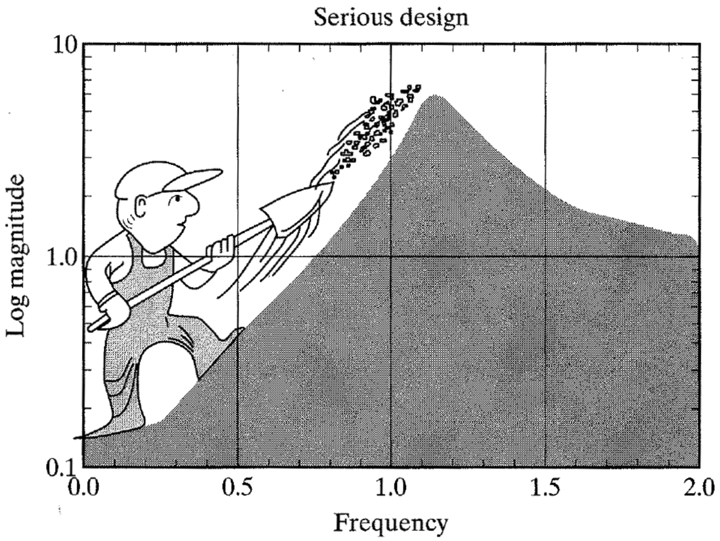
$$\det|zI - \Phi + \Gamma K| = 0$$

Pole Placement

Pole placement: Big idea:

- Arbitrarily select the desired root locations of the closed-loop system and see if the approach will work.
- AKA: full state feedback
  ∵ enough parameters to influence all the closed-loop poles
- Finding the elements of K so that the roots are in the desired locations. Unlike classical design, where we iterated on parameters in the compensator (hoping) to find acceptable root locations, the full state feedback, pole-placement approach guarantees success and allows us to arbitrarily pick any root locations, providing that *n* roots are specified for an n^{th}-order system.

Meaning…



Back to Pole Placement

- Given:

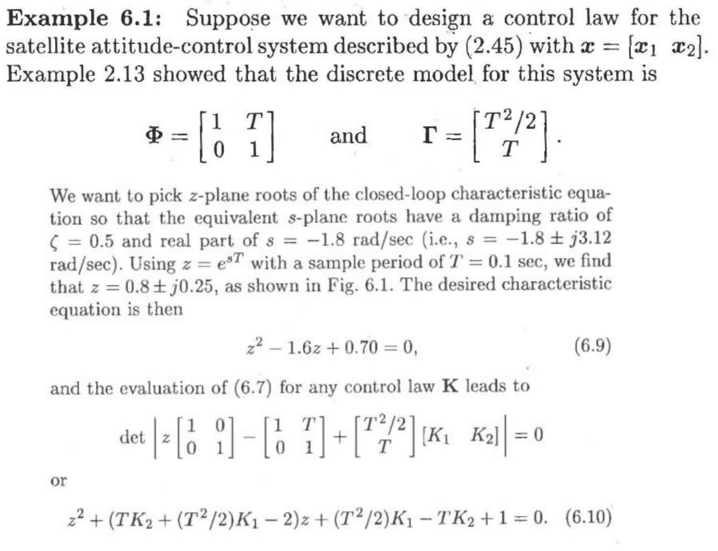
$$z_i = \beta_1, \beta_2, \beta_3, \ldots$$

- This gives the desired control-characteristic equation as:

$$a_c(z) = (z-\beta_1)(z-\beta_2)(z-\beta_3)\ldots =$$

- Now we "just solve" for **K** and "bingo"

Pole Placement Example (FPW p. 241)



Example 6.1: Suppose we want to design a control law for the satellite attitude-control system described by (2.45) with $\boldsymbol{x} = [\boldsymbol{x}_1 \;\; \boldsymbol{x}_2]$. Example 2.13 showed that the discrete model for this system is

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Gamma} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}.$$

We want to pick z-plane roots of the closed-loop characteristic equation so that the equivalent s-plane roots have a damping ratio of $\zeta = 0.5$ and real part of $s = -1.8$ rad/sec (i.e., $s = -1.8 \pm j3.12$ rad/sec). Using $z = e^{sT}$ with a sample period of $T = 0.1$ sec, we find that $z = 0.8 \pm j0.25$, as shown in Fig. 6.1. The desired characteristic equation is then

$$z^2 - 1.6z + 0.70 = 0, \qquad (6.9)$$

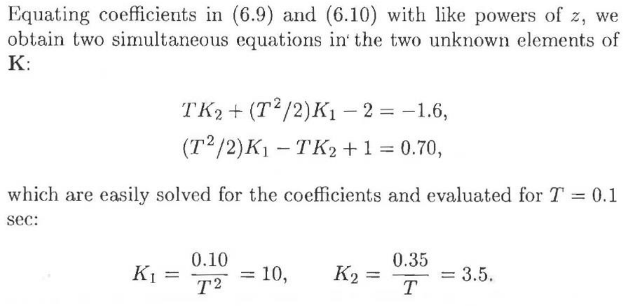
and the evaluation of (6.7) for any control law $\mathbf{K}$ leads to

$$\det \left| z\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} [K_1 \quad K_2] \right| = 0$$

or

$$z^2 + (TK_2 + (T^2/2)K_1 - 2)z + (T^2/2)K_1 - TK_2 + 1 = 0. \qquad (6.10)$$

Pole Placement Example (FPW p. 241)



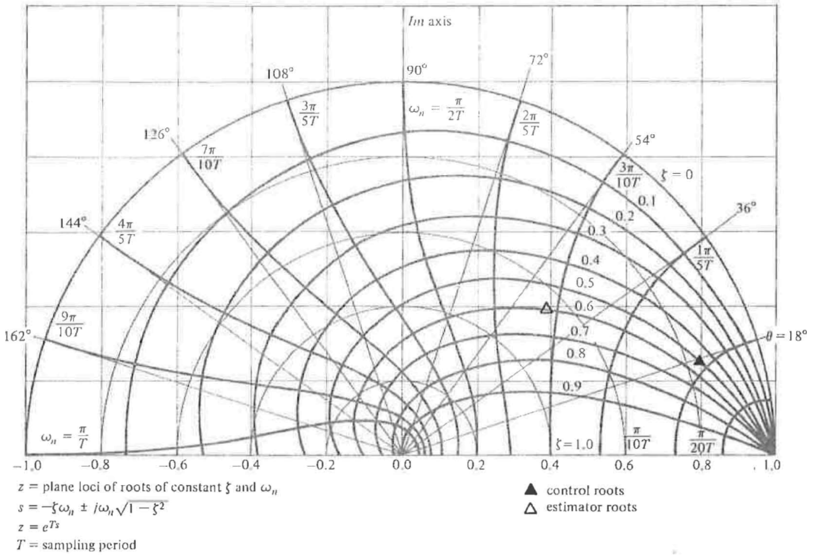
Equating coefficients in (6.9) and (6.10) with like powers of z, we obtain two simultaneous equations in the two unknown elements of $\mathbf{K}$:

$$TK_2 + (T^2/2)K_1 - 2 = -1.6,$$
$$(T^2/2)K_1 - TK_2 + 1 = 0.70,$$

which are easily solved for the coefficients and evaluated for $T = 0.1$ sec:

$$K_1 = \frac{0.10}{T^2} = 10, \qquad K_2 = \frac{0.35}{T} = 3.5.$$

Pole Placement Example (FPW p. 241)



Ackermann's Formula (FPW p. 245)

- Gains maybe approximated with:

$$\boxed{\mathbf{K} = [0 \ldots 0 \quad 1][\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \quad \mathbf{\Phi}^2\mathbf{\Gamma} \ldots \mathbf{\Phi}^{n-1}\mathbf{\Gamma}]^{-1}\alpha_c(\mathbf{\Phi}),}$$

- Where: C = controllability matrix, n is the order of the system (or number of state elements) and α_c:

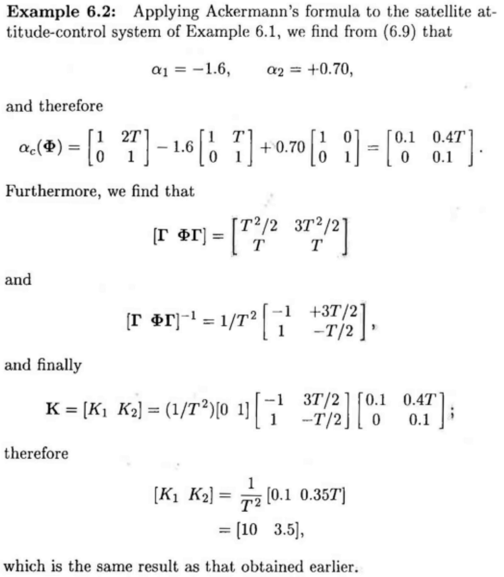
$$\mathcal{C} = [\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \ldots]$$

$$\alpha_c(\mathbf{\Phi}) = \mathbf{\Phi}^n + \alpha_1\mathbf{\Phi}^{n-1} + \alpha_2\mathbf{\Phi}^{n-2} + \cdots + \alpha_n\mathbf{I},$$

  - α_i: coefficients of the desired characteristic equation

$$\alpha_c(z) = |\, z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma}\mathbf{K} \,| = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n.$$

Ackermann's Formula Example (FPW p.246)



Example 6.2: Applying Ackermann's formula to the satellite attitude-control system of Example 6.1, we find from (6.9) that

$$\alpha_1 = -1.6, \qquad \alpha_2 = +0.70,$$

and therefore

$$\alpha_c(\mathbf{\Phi}) = \begin{bmatrix} 1 & 2T \\ 0 & 1 \end{bmatrix} - 1.6\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + 0.70\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix}.$$

Furthermore, we find that

$$[\mathbf{\Gamma} \ \ \mathbf{\Phi\Gamma}] = \begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix}$$

and

$$[\mathbf{\Gamma} \ \ \mathbf{\Phi\Gamma}]^{-1} = 1/T^2\begin{bmatrix} -1 & +3T/2 \\ 1 & -T/2 \end{bmatrix},$$

and finally

$$\mathbf{K} = [K_1 \ \ K_2] = (1/T^2)[0 \ \ 1]\begin{bmatrix} -1 & 3T/2 \\ 1 & -T/2 \end{bmatrix}\begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix};$$

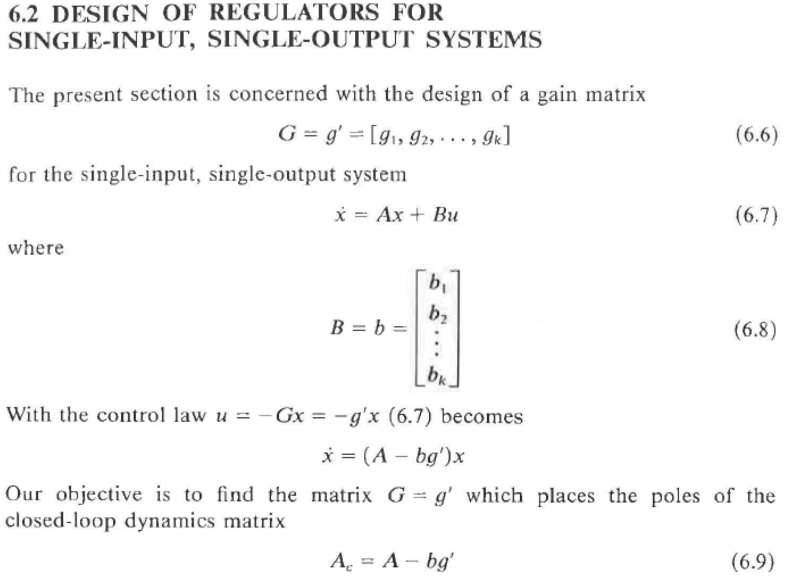
therefore

$$[K_1 \ \ K_2] = \frac{1}{T^2}[0.1 \ \ 0.35T]$$
$$= [10 \quad 3.5],$$

which is the same result as that obtained earlier.

Shaping the Dynamic Response: SISO

Design of regulators for single-input, single-output systems



6.2 DESIGN OF REGULATORS FOR SINGLE-INPUT, SINGLE-OUTPUT SYSTEMS

The present section is concerned with the design of a gain matrix

$$G = g' = [g_1, g_2, \ldots, g_k] \tag{6.6}$$

for the single-input, single-output system

$$\dot{x} = Ax + Bu \tag{6.7}$$

where

$$B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \tag{6.8}$$

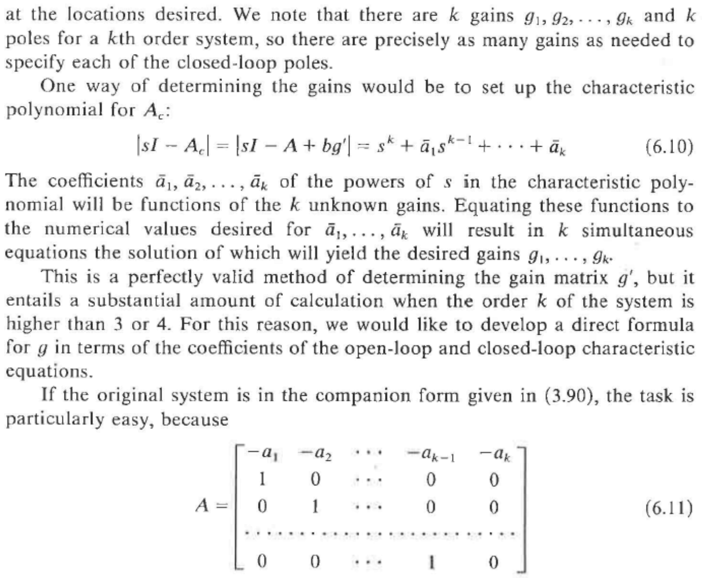
With the control law $u = -Gx = -g'x$ (6.7) becomes

$$\dot{x} = (A - bg')x$$

Our objective is to find the matrix $G = g'$ which places the poles of the closed-loop dynamics matrix

$$A_c = A - bg' \tag{6.9}$$

Design of regulators for single-input, single-output systems



at the locations desired. We note that there are k gains $g_1, g_2, \ldots, g_k$ and k poles for a kth order system, so there are precisely as many gains as needed to specify each of the closed-loop poles.

One way of determining the gains would be to set up the characteristic polynomial for A_c:

$$|sI - A_c| = |sI - A + bg'| = s^k + \bar{a}_1 s^{k-1} + \cdots + \bar{a}_k \tag{6.10}$$

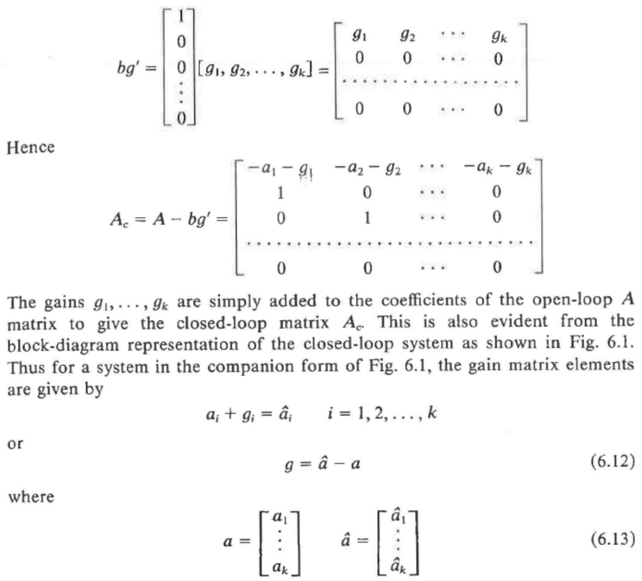
The coefficients $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$ of the powers of s in the characteristic polynomial will be functions of the k unknown gains. Equating these functions to the numerical values desired for $\bar{a}_1, \ldots, \bar{a}_k$ will result in k simultaneous equations the solution of which will yield the desired gains $g_1, \ldots, g_k$.

This is a perfectly valid method of determining the gain matrix g', but it entails a substantial amount of calculation when the order k of the system is higher than 3 or 4. For this reason, we would like to develop a direct formula for g in terms of the coefficients of the open-loop and closed-loop characteristic equations.

If the original system is in the companion form given in (3.90), the task is particularly easy, because

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6.11}$$

Design of regulators for single-input, single-output systems



$$bg' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [g_1, g_2, \ldots, g_k] = \begin{bmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence

$$A_c = A - bg' = \begin{bmatrix} -a_1 - g_1 & -a_2 - g_2 & \cdots & -a_k - g_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The gains $g_1, \ldots, g_k$ are simply added to the coefficients of the open-loop A matrix to give the closed-loop matrix A_c. This is also evident from the block-diagram representation of the closed-loop system as shown in Fig. 6.1. Thus for a system in the companion form of Fig. 6.1, the gain matrix elements are given by

$$a_i + g_i = \hat{a}_i \qquad i = 1, 2, \ldots, k$$

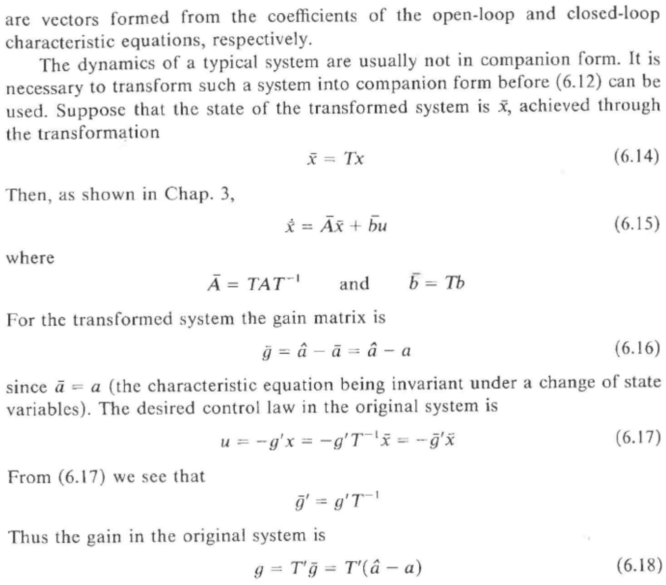
or

$$g = \hat{a} - a \tag{6.12}$$

where

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \qquad \hat{a} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_k \end{bmatrix} \tag{6.13}$$

Design of regulators for single-input, single-output systems



are vectors formed from the coefficients of the open-loop and closed-loop characteristic equations, respectively.

The dynamics of a typical system are usually not in companion form. It is necessary to transform such a system into companion form before (6.12) can be used. Suppose that the state of the transformed system is $\bar{x}$, achieved through the transformation

$$\bar{x} = Tx \tag{6.14}$$

Then, as shown in Chap. 3,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \tag{6.15}$$

where

$$\bar{A} = TAT^{-1} \qquad \text{and} \qquad \bar{b} = Tb$$

For the transformed system the gain matrix is

$$\bar{g} = \hat{a} - \bar{a} = \hat{a} - a \tag{6.16}$$

since $\bar{a} = a$ (the characteristic equation being invariant under a change of state variables). The desired control law in the original system is

$$u = -g'x = -g'T^{-1}\bar{x} = -\bar{g}'\bar{x} \tag{6.17}$$

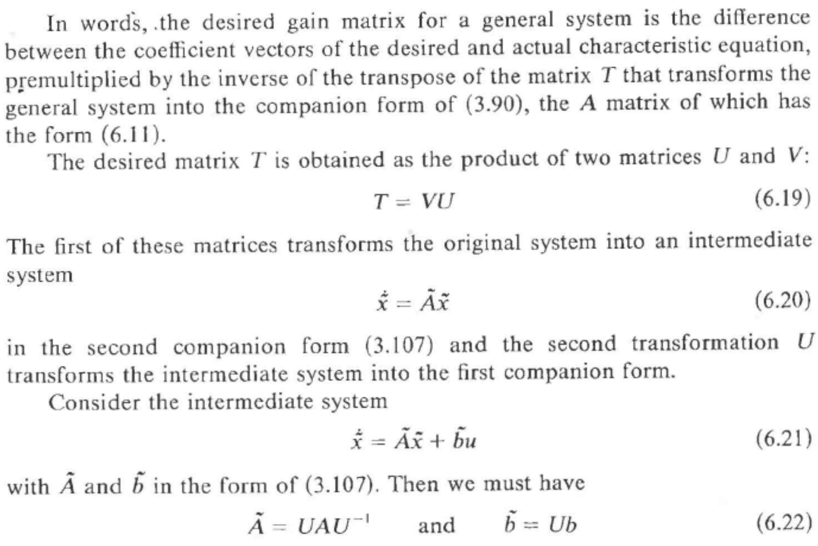
From (6.17) we see that

$$\bar{g}' = g'T^{-1}$$

Thus the gain in the original system is

$$g = T'\bar{g} = T'(\hat{a} - a) \tag{6.18}$$

Design of regulators for single-input, single-output systems



In words, the desired gain matrix for a general system is the difference between the coefficient vectors of the desired and actual characteristic equation, premultiplied by the inverse of the transpose of the matrix T that transforms the general system into the companion form of (3.90), the A matrix of which has the form (6.11).

The desired matrix T is obtained as the product of two matrices U and V:

$$T = VU \tag{6.19}$$

The first of these matrices transforms the original system into an intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} \tag{6.20}$$

in the second companion form (3.107) and the second transformation U transforms the intermediate system into the first companion form.

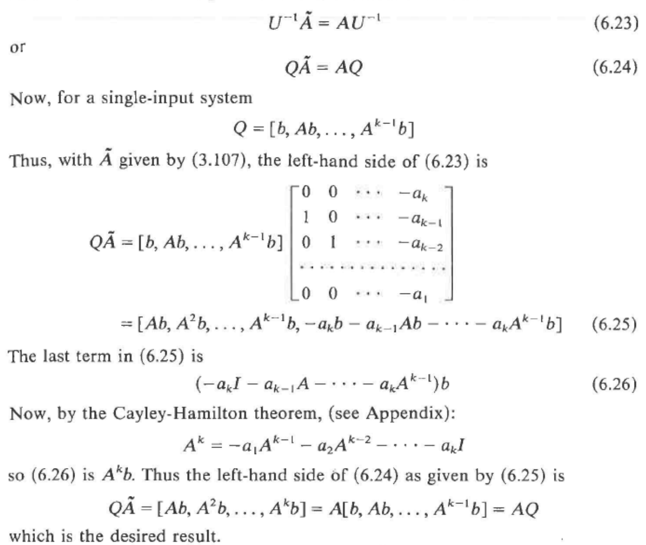
Consider the intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u \tag{6.21}$$

with $\tilde{A}$ and $\tilde{b}$ in the form of (3.107). Then we must have

$$\tilde{A} = UAU^{-1} \qquad \text{and} \qquad \tilde{b} = Ub \tag{6.22}$$

Design of regulators for single-input, single-output systems



The desired matrix U is precisely the inverse of the controllability test matrix Q of Sec. 5.4. To prove this fact, we must show that

$$U^{-1}\tilde{A} = AU^{-1} \tag{6.23}$$

or

$$Q\tilde{A} = AQ \tag{6.24}$$

Now, for a single-input system

$$Q = [b, Ab, \ldots, A^{k-1}b]$$

Thus, with $\tilde{A}$ given by (3.107), the left-hand side of (6.23) is

$$Q\tilde{A} = [b, Ab, \ldots, A^{k-1}b]\begin{bmatrix} 0 & 0 & \cdots & -a_k \\ 1 & 0 & \cdots & -a_{k-1} \\ 0 & 1 & \cdots & -a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_1 \end{bmatrix}$$

$$= [Ab, A^2b, \ldots, A^{k-1}b, -a_kb - a_{k-1}Ab - \cdots - a_kA^{k-1}b] \tag{6.25}$$

The last term in (6.25) is

$$(-a_kI - a_{k-1}A - \cdots - a_kA^{k-1})b \tag{6.26}$$

Now, by the Cayley-Hamilton theorem, (see Appendix):

$$A^k = -a_1A^{k-1} - a_2A^{k-2} - \cdots - a_kI$$

so (6.26) is A^kb. Thus the left-hand side of (6.24) as given by (6.25) is

$$Q\tilde{A} = [Ab, A^2b, \ldots, A^kb] = A[b, Ab, \ldots, A^{k-1}b] = AQ$$

which is the desired result.

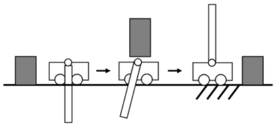
</assessable>

- Nothing beyond this point is on the exam.
  (except for the exam review ☺)
- Do not pay attention.
- Do not attempt to learn.

Example 1:
Inverted Pendulum

Digital Control

Wikipedia,
Cart and pole



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$
$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

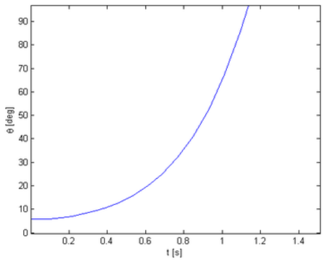
$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

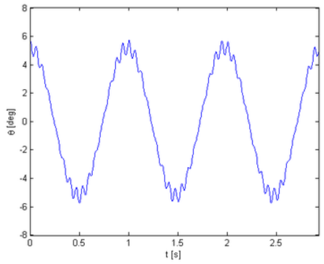
and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

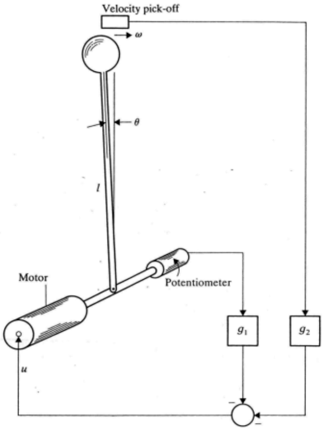
substituting L in these equations and simplifying leads to the equations that describe the motion o

$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$
$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$





Inverted Pendulum



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$
$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{x}} - \frac{\partial L}{\partial x} = F$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = 0$$

substituting L in these equations and simplifying leads to the equations that describe the motion o

$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$
$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$

Inverted Pendulum – Equations of Motion

- The equations of motion of an inverted pendulum (under a small angle approximation) may be linearized as:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \ddot{\theta} = Q^2\theta + Pu$$

Where:

$$Q^2 = \left(\frac{M+m}{Ml}\right)g$$
$$P = \frac{1}{Ml}.$$

If we further assume unity ***Ml*** ($Ml \approx 1$), then $P \approx 1$

Inverted Pendulum –State Space

- We then select a state-vector as:

$$\boldsymbol{x} = \begin{bmatrix}\theta\\ \omega\end{bmatrix}, \text{ hence } \dot{\boldsymbol{x}} = \begin{bmatrix}\dot{\theta}\\ \dot{\omega}\end{bmatrix} = \begin{bmatrix}\omega\\ \dot{\omega}\end{bmatrix}$$

- Hence giving a state-space model as:

$$A = \begin{bmatrix}0 & 1\\ Q^2 & 0\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix}$$

- The resolvent of which is:

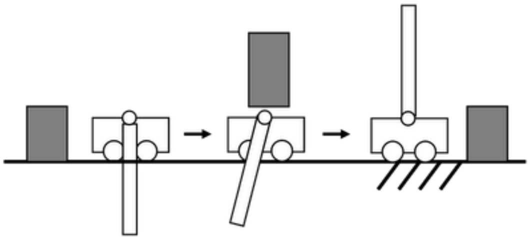
$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix}s & -1\\ -Q^2 & s\end{bmatrix}^{-1} = \frac{1}{s^2 - Q^2}\begin{bmatrix}s & 1\\ Q^2 & s\end{bmatrix}$$

- And a state-transition matrix as:

$$\Phi(t) = \begin{bmatrix}\cosh Qt & \dfrac{\sinh Qt}{Q}\\ Q\ sinh\ Qt & \cosh Qt\end{bmatrix}$$

Cart & Pole in State-Space With Obstacles?

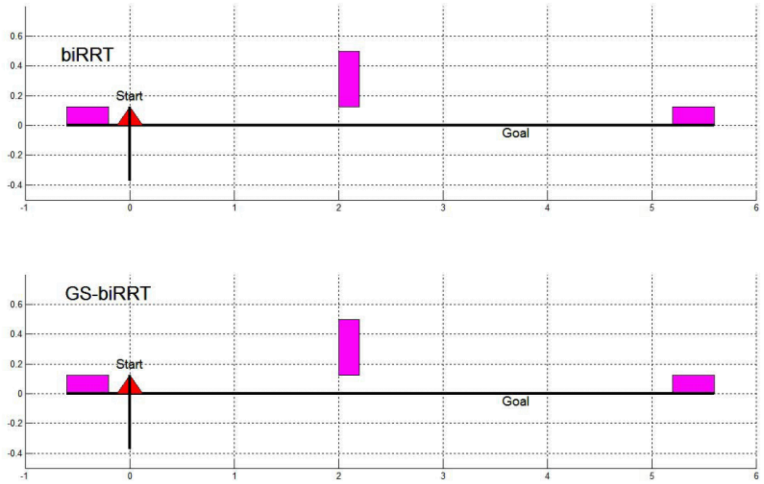
Swing-up is a little more than stabilization…



See also: METR4202 – Tutorial 11:
http://robotics.itee.uq.edu.au/~metr4202/tpl/t11-Week11-pendulum.pdf

Cart & Pole in State-Space

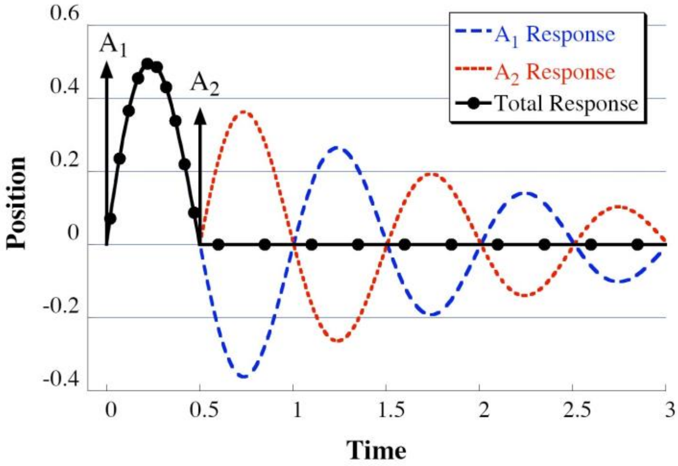
Swing-up is a little more than stabilization…



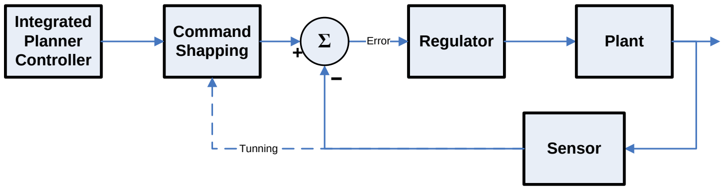
Example 2:
Command Shaping

Experiments: Scanning Over Obstacle

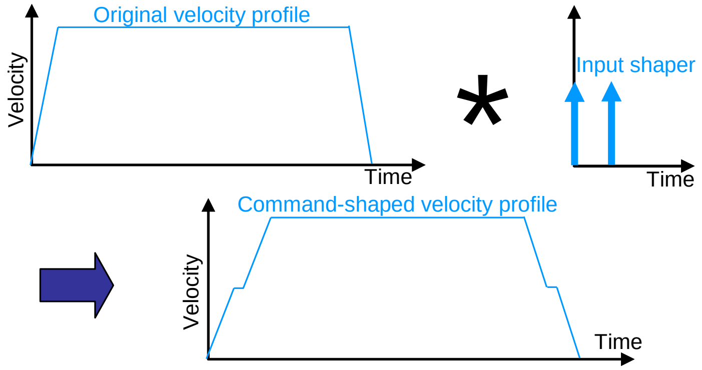
Command Shaping



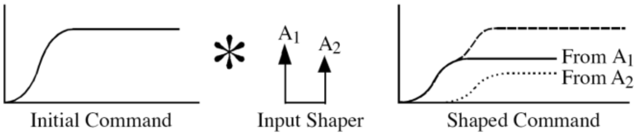
Robust Control:
Command Shaping for Vibration Reduction



Command Shaping



Command Shaping



- Zero Vibration (ZV)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{1+K} & \frac{K}{1+K} \\ 0 & \frac{T_d}{2} \end{bmatrix} \qquad K = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

- Zero Vibration and Derivative (ZVD)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{(1+K)^2} & \frac{2K}{(1+K)^2} & \frac{K^2}{(1+K)^2} \\ 0 & \frac{T_d}{2} & T_d \end{bmatrix}$$

Experiments: Command Shaping

Next Time…

- **Digital Control via Emulation!**

- Review:
  - Chapter 5 of FPW

- Deeper Pondering??