Shaping the Dynamic Response

ELEC 3004: **Systems**: Signals & Controls
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Lecture 21
(with material from FPW and Lathi)

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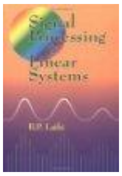
Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	24-Mar	Convolution Review
	28-Mar	Holiday
	31-Mar	
5	4-Apr	Frequency Response & Filter Analysis
	7-Apr	Filters
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Introduction to Feedback Control
9	3-May	Holiday
	5-May	Feedback Control & Regulation
10	9-May	Servoregulation/PID
	12-May	Introduction to (Digital) Control
11	16-May	Digital Control Design & State-Space
	19-May	Observability, Controllability & Stability of Digital Systems
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	2-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
[TK5102.9.L38 1998]

G. Franklin,
J. Powell,
M. Workman
Digital Control of Dynamic Systems
1990

[TJ216.F72 1990]
[Available as UQ Ebook]

Today

➔ **State-space** ➔

- FPW
  - Chapter 6 - Design of Digital Control Systems Using State-Space Methods

- Lathi Ch. 13
  - § 13.2 Systematic Procedure for Determining State Equations
  - § 13.3 Solution of State Equations
  - Using State-Space Methods

Next Time

More Online Reading Materials


CHAPTER
SIX
SHAPING THE DYNAMIC RESPONSE

6.1 INTRODUCTION

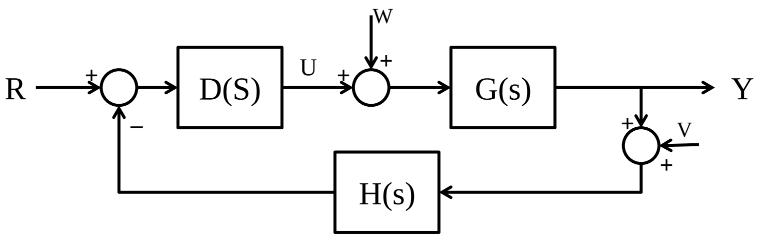

CHAPTER
THREE
DYNAMICS OF LINEAR SYSTEMS

3.1 DIFFERENTIAL EQUATIONS REVISITED

- Friedland, Control System Design Ch. 6 and 3

➔ http://robotics.itee.uq.edu.au/~elec3004/tutes.html

Basic Closed-loop Block Diagram



$$TF(s) = \frac{Y(s)}{R(s)} = \frac{D(s)G(s)}{1 + D(s)G(s)H(s)} = \frac{DG}{1 + DGH}$$

Controlabilty

Controllability

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where $\mathbf{x}$ = state vector (n-vector)

$\mathbf{u}$ = control vector (r-vector)

$\mathbf{y}$ = output vector (m-vector) $(m \leq n)$

$\mathbf{A} = n \times n$ matrix

$\mathbf{B} = n \times r$ matrix

$\mathbf{C} = m \times n$ matrix

is completely output controllable if and only if the composite $m \times nr$ matrix $\mathbf{P}$, where

$$\mathbf{P} = [\mathbf{CB} \mid \mathbf{CAB} \mid \mathbf{CA}^2\mathbf{B} \mid \cdots \mid \mathbf{CA}^{n-1}\mathbf{B}]$$

is of rank m. (Notice that complete state controllability is neither necessary nor sufficient for complete output controllability.)

Controllability Example

- Is this fully controllable:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 4 \end{bmatrix} u$$

- Solution:

$$\mathbf{A} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} -3 & 1 \\ -2 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

- We see that vectors **B** and **AB** are **not linearly independent** and
- The rank of the matrix **[B | AB]** is 1 < **m** (m=2)
- ∴ **the system is not completely state controllable.**

- In fact, elimination of x_2 from the given problem yields:
  $\ddot{x}_1 + 1.5\dot{x}_1 - 2.5x_1 = \dot{u} + 2.5u \longrightarrow \frac{X_1(s)}{U(s)} = \frac{s + 2.5}{(s + 2.5)(s - 1)}$
- Notice that cancellation of the factor (**s** + 2.5) occurs in the numerator and denominator of the transfer function. Because of this cancellation, this system is **not** completely state controllable and it's unstable system (s=1, RHP!). Remember that stability and controllability are quite different things. **There are many systems that are unstable, but are completely state controllable**.

Controllability Example II

- TF → CCF

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -0.4 & -1.3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y = [0.8 \quad 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$\longrightarrow$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -0.4\\ 1 & -1.3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0.8\\ 1\end{bmatrix}u$$

$$y = [0 \quad 1]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Solution. Consider the system defined by Equations (9–120) and (9–121). The rank of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix}0 & 1\\ 1 & -1.3\end{bmatrix}$$

is 2. Hence, the system is completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix}0.8 & -0.4\\ 1 & -0.5\end{bmatrix}$$

is 1. Hence the system is not observable.

Next consider the system defined by Equations (9–122) and (9–123). The rank of the controllability matrix

$$[\mathbf{B} \mid \mathbf{AB}] = \begin{bmatrix}0.8 & -0.4\\ 1 & -0.5\end{bmatrix}$$

is 1. Hence, the system is not completely state controllable. The rank of the observability matrix

$$[\mathbf{C}^* \mid \mathbf{A}^*\mathbf{C}^*] = \begin{bmatrix}0 & 1\\ 1 & -1.3\end{bmatrix}$$

is 2. Hence, the system is observable.

The apparent difference in the controllability and observability of the same system is caused by the fact that the original system has a pole-zero cancellation in the transfer function. Referring to Equation (2–29), for $D = 0$ we have

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

If we use Equations (9–120) and (9–121), then

$$\begin{aligned} G(s) &= [0.8 \quad 1]\begin{bmatrix}s & -1\\ 0.4 & s+1.3\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{1}{s^2 + 1.3s + 0.4}[0.8 \quad 1]\begin{bmatrix}s+1.3 & 1\\ -0.4 & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\\ &= \frac{s+0.8}{(s+0.8)(s+0.5)}\end{aligned}$$

[Note that the same transfer function can be obtained by using Equations (9–122) and (9–123).] Clearly, cancellation occurs in this transfer function.

Stability

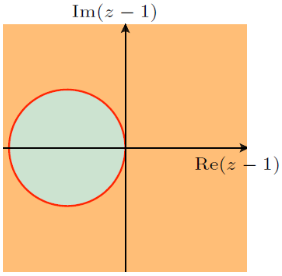
Fast sampling revisited

- For small T:

$$z = e^{sT} = 1 + sT + \frac{(sT)^2}{2} + \ldots \approx 1 + sT$$

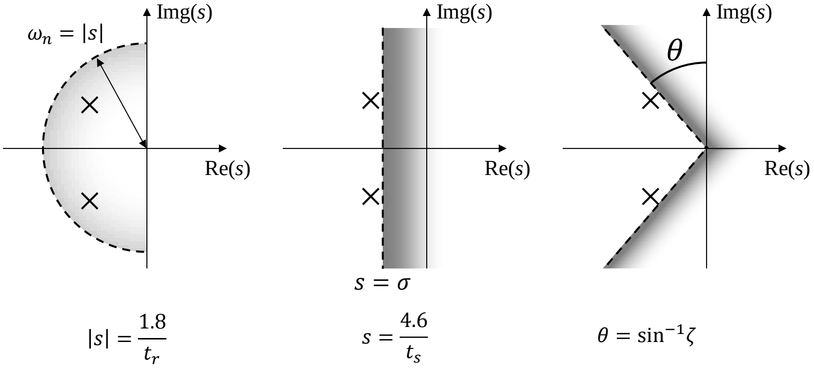
$$\rightarrow z \approx 1 + sT \rightarrow s = \frac{z-1}{T}$$

- Hence, the unit circle under the map from z to s-plane becomes:



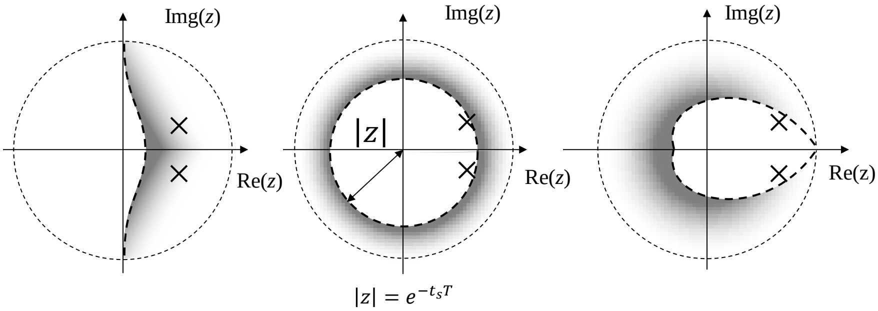
Specification bounds

- Recall in the continuous domain, response performance metrics map to the s-plane:



Discrete bounds

- These map to the discrete domain:



In practice, you'd use Matlab to plot these, and check that the spec is satisfied

Example Code:

```
%% Input System Model G
numg=5; deng=[1 20 0]; sysg=tf(numg, deng);

%% Approximate the ZOH (1-e^-sT)/(s)
[nd, dd]=pade(1,2);  %pade gives us the "hold" or -e^-sT of a ZOH
sysp=tf(nd, dd); sysi=tf([1],[1,0]); %Now we need the "1/s" portion
sys1=series(1-sysp, sysi); % Approximation as a series

%% Open loop response
syso=series(sys1, sysg); % computer the open loop G with the ZOH
sys=feedback(syso,1); % Computer the unity feedback response
step(sys) % Display the step response
```

Obtaining a Time Response

From SS to Time Response — Impulse Functions

- Given: $\dot{x} = Ax + Bu$
- Solution:

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Substituting $t_0 = 0$ into this:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

  - Write the impulse as: $\mathbf{u}(t) = \delta(t)\mathbf{w}$
  - where **w** is a vector whose components are the magnitudes of **r** impulse functions applied at t=0

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0-) + \int_{0-}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(\tau)\mathbf{w}\,d\tau$$

$$= e^{\mathbf{A}t}\mathbf{x}(0-) + e^{\mathbf{A}t}\mathbf{B}\mathbf{w}$$

From SS to Time Response — Step Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = \boldsymbol{k}$
  Where **k** is a vector whose components are the magnitudes of *r* step functions applied at t=0.

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{k}\,d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[\int_0^t\left(\mathbf{I} - \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} - \cdots\right)d\tau\right]\mathbf{B}\mathbf{k} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{\mathbf{A}^2t^3}{3!} - \cdots\right)\mathbf{B}\mathbf{k}\end{aligned}$$

  – Assume A is non-singular

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left[-(\mathbf{A}^{-1})(e^{-\mathbf{A}t} - \mathbf{I})\right]\mathbf{B}\mathbf{k} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}\mathbf{k}\end{aligned}$$

From SS to Time Response — Ramp Response

- Given: $\dot{x} = Ax + Bu$
- Start with $u(t) = t\boldsymbol{v}$
  Where **v** is a vector whose components are magnitudes of ramp functions applied at $t = 0$

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\tau\mathbf{v}\,d\tau \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\tau\,d\tau\,\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\frac{\mathbf{I}}{2}t^2 - \frac{2\mathbf{A}}{3!}t^3 + \frac{3\mathbf{A}^2}{4!}t^4 - \frac{4\mathbf{A}^3}{5!}t^5 + \cdots\right)\mathbf{B}\mathbf{v}\end{aligned}$$

  – Assume A is non-singular

$$\begin{aligned}\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + (\mathbf{A}^{-2})(e^{\mathbf{A}t} - \mathbf{I} - \mathbf{A}t)\mathbf{B}\mathbf{v} \\ &= e^{\mathbf{A}t}\mathbf{x}(0) + \left[\mathbf{A}^{-2}(e^{\mathbf{A}t} - \mathbf{I}) - \mathbf{A}^{-1}t\right]\mathbf{B}\mathbf{v}\end{aligned}$$

Example: Obtain the Step Response

- Given: $\begin{bmatrix}\dot{x}_1\\ \dot{x}_1\end{bmatrix} = \begin{bmatrix}-1 & -0.5\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0.5\\ 0\end{bmatrix}u, \quad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

$$y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad u(t) = 1(t)$$

- Solution:

$$\mathbf{A} = \begin{bmatrix}-1 & -0.5\\ 1 & 0\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}0.5\\ 0\end{bmatrix} \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix}s+1 & 0.5\\ -1 & s\end{bmatrix}^{-1} = \frac{1}{s^2 + s + 0.5}\begin{bmatrix}s & -0.5\\ 1 & s+1\end{bmatrix}$$

$$= \begin{bmatrix}\frac{s + 0.5 - 0.5}{(s+0.5)^2 + 0.5^2} & \frac{-0.5}{(s+0.5)^2 + 0.5^2}\\ \frac{1}{(s+0.5)^2 + 0.5^2} & \frac{s + 0.5 + 0.5}{(s+0.5)^2 + 0.5^2}\end{bmatrix}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}] = \begin{bmatrix}e^{-0.5t}(\cos 0.5t - \sin 0.5t) & -e^{-0.5t}\sin 0.5t\\ 2e^{-0.5t}\sin 0.5t & e^{-0.5t}(\cos 0.5t + \sin 0.5t)\end{bmatrix}$$

  - Set k=1, x(0)=0:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}k$$
$$= \mathbf{A}^{-1}(e^{\mathbf{A}t} - \mathbf{I})\mathbf{B}$$
$$= \begin{bmatrix}0 & 1\\ -2 & -2\end{bmatrix}\begin{bmatrix}0.5e^{-0.5t}(\cos 0.5t - \sin 0.5t) - 0.5\\ e^{-0.5t}\sin 0.5t\end{bmatrix}$$
$$= \begin{bmatrix}e^{-0.5t}\sin 0.5t\\ -e^{-0.5t}(\cos 0.5t + \sin 0.5t) + 1\end{bmatrix}$$

$$y(t) = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} = x_1 = e^{-0.5t}\sin 0.5t$$

Example II: Obtain the Step Response

- Given:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u \qquad u(t) = 1(t)$$

- Solution:

$$\mathbf{A} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix}0\\ 1\end{bmatrix} \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}[(s\mathbf{I} - \mathbf{A})^{-1}]$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix}2e^{-t} - e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \begin{bmatrix}2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)}\\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}[1]\,d\tau$$

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}2e^{-t} - e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} + \begin{bmatrix}\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\\ e^{-t} - e^{-2t}\end{bmatrix}$$

  - Assume x(0)=0:

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} = \begin{bmatrix}2e^{-t} - e^{-2t} & e^{-t} - e^{-2t}\\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} + \begin{bmatrix}\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}\\ e^{-t} - e^{-2t}\end{bmatrix}$$

Digital PID Controls (Magic PID Made Easy Equations)

Implementation of Digital PID Controllers

We will consider the PID controller with an s-domain transfer function

$$\frac{U(s)}{X(s)} = G_c(s) = K_P + \frac{K_I}{s} + K_D s. \tag{13.54}$$

We can determine a digital implementation of this controller by using a discrete approximation for the derivative and integration. For the time derivative, we use the **backward difference rule**

$$u(kT) = \left.\frac{dx}{dt}\right|_{t=kT} = \frac{1}{T}(x(kT) - x[(k-1)T]). \tag{13.55}$$

The z-transform of Equation (13.55) is then

$$U(z) = \frac{1 - z^{-1}}{T} X(z) = \frac{z-1}{Tz} X(z).$$

The integration of $x(t)$ can be represented by the **forward-rectangular integration** at $t = kT$ as

$$u(kT) = u[(k-1)T] + Tx(kT), \tag{13.56}$$

Source: Dorf & Bishop, Modern Control Systems, §13.9, pp. 1030-1

Implementation of Digital PID Controllers (2)

where $u(kT)$ is the output of the integrator at $t = kT$. The z-transform of Equation (13.56) is

$$U(z) = z^{-1}U(z) + TX(z),$$

and the transfer function is then

$$\frac{U(z)}{X(z)} = \frac{Tz}{z-1}.$$

Hence, the z-domain transfer function of the **PID controller** is

$$\boxed{G_c(z) = K_P + \frac{K_I T z}{z-1} + K_D \frac{z-1}{Tz}.} \qquad (13.57)$$

The complete difference equation algorithm that provides the PID controller is obtained by adding the three terms to obtain [we use $x(kT) = x(k)$]

$$\begin{aligned} u(k) &= K_P x(k) + K_I[u(k-1) + Tx(k)] + (K_D/T)[x(k) - x(k-1)] \\ &= [K_P + K_I T + (K_D/T)]x(k) - K_D T x(k-1) + K_I u(k-1). \end{aligned} \qquad (13.58)$$

Equation (13.58) can be implemented using a digital computer or microprocessor. Of course, we can obtain a PI or PD controller by setting an appropriate gain equal to zero.

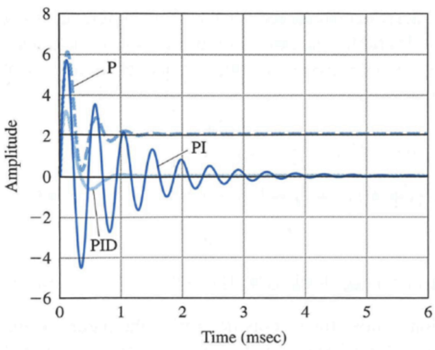
Source: Dorf & Bishop, Modern Control Systems, §13.9, pp. 1030-1

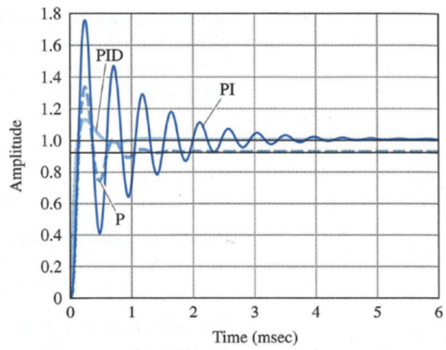
PID Intuition

Effects of increasing a parameter independently					
Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability
K_p	↓	⇑	Minimal change	↓	↓
K_I	↓	⇑	⇑	Eliminate	↓
K_D	Minor change	↓	↓	No effect / minimal change	Improve (if K_D small)

PID Intuition: P and PI and PID

- Responses of P, PI, and PID control to





(a) step disturbance input (b) step reference input

Shaping the Dynamic Response: Pole Placement

Pole Placement (Following FPW – Chapter 6)

- FPW has a slightly different notation:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u,$$
$$y = \mathbf{H}\mathbf{x}.$$

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k),$$
$$y(k) = \mathbf{H}\mathbf{x}(k),$$

$$\mathbf{\Phi} = e^{\mathbf{F}T},$$

$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G},$$

Pole Placement

- Start with a simple feedback control law ("controller")

$$u = -\mathbf{K}\mathbf{x} = -[K_1 K_2 \dots] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

- It's actually a regulator
  ∵ it does not allow for a reference input to the system.
  (there is no "reference" **r** (r = 0))

- Substitute in the difference equation

$$x(k+1) = \Phi x(k) - \Gamma K x(k)$$

- $\mathcal{Z}$ Transform:

$$(zI - \Phi + \Gamma K)X(z) = 0$$

➔Characteristic Eqn:

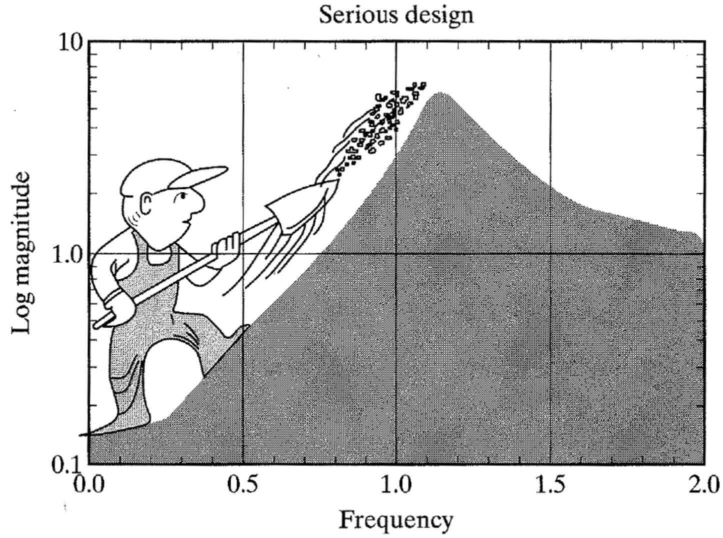
$$\det|zI - \Phi + \Gamma K| = 0$$

Pole Placement

Pole placement: Big idea:

- Arbitrarily select the desired root locations of the closed-loop system and see if the approach will work.
- AKA: full state feedback
  ∵ enough parameters to influence all the closed-loop poles
- Finding the elements of K so that the roots are in the desired locations. Unlike classical design, where we iterated on parameters in the compensator (hoping) to find acceptable root locations, the full state feedback, pole-placement approach guarantees success and allows us to arbitrarily pick any root locations, providing that *n* roots are specified for an n^{th}-order system.

Meaning…



Back to Pole Placement

- Given:

$$z_i = \beta_1, \beta_2, \beta_3, \dots$$

- This gives the desired control-characteristic equation as:

$$a_c(z) = (z - \beta_1)(z - \beta_2)(z - \beta_3) \dots =$$

- Now we "just solve" for **K** and "bingo"

Pole Placement Example (FPW p. 241)

Example 6.1: Suppose we want to design a control law for the satellite attitude-control system described by (2.45) with $\boldsymbol{x} = [x_1 \;\; x_2]$. Example 2.13 showed that the discrete model for this system is

$$\Phi = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \Gamma = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}.$$

We want to pick z-plane roots of the closed-loop characteristic equation so that the equivalent s-plane roots have a damping ratio of $\zeta = 0.5$ and real part of $s = -1.8$ rad/sec (i.e., $s = -1.8 \pm j3.12$ rad/sec). Using $z = e^{sT}$ with a sample period of $T = 0.1$ sec, we find that $z = 0.8 \pm j0.25$, as shown in Fig. 6.1. The desired characteristic equation is then

$$z^2 - 1.6z + 0.70 = 0, \tag{6.9}$$

and the evaluation of (6.7) for any control law $\mathbf{K}$ leads to

$$\det \left| z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} [K_1 \quad K_2] \right| = 0$$

or

$$z^2 + (TK_2 + (T^2/2)K_1 - 2)z + (T^2/2)K_1 - TK_2 + 1 = 0. \tag{6.10}$$

Pole Placement Example (FPW p. 241)

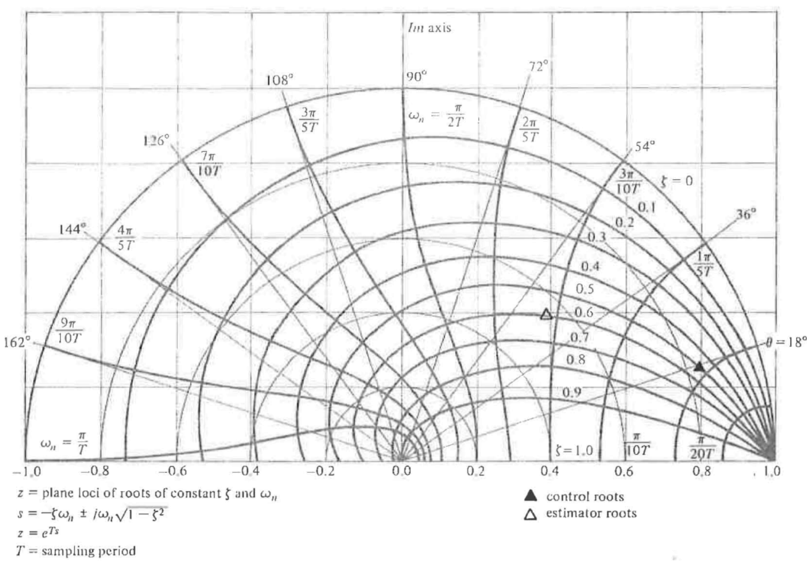
Equating coefficients in (6.9) and (6.10) with like powers of z, we obtain two simultaneous equations in the two unknown elements of $\mathbf{K}$:

$$TK_2 + (T^2/2)K_1 - 2 = -1.6,$$
$$(T^2/2)K_1 - TK_2 + 1 = 0.70,$$

which are easily solved for the coefficients and evaluated for $T = 0.1$ sec:

$$K_1 = \frac{0.10}{T^2} = 10, \qquad K_2 = \frac{0.35}{T} = 3.5.$$

Pole Placement Example (FPW p. 241)



Ackermann's Formula (FPW p. 245)

- Gains maybe approximated with:

$$\boxed{\mathbf{K} = [0 \ldots 0 \quad 1][\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \quad \mathbf{\Phi}^2\mathbf{\Gamma} \ldots \mathbf{\Phi}^{n-1}\mathbf{\Gamma}]^{-1}\alpha_c(\mathbf{\Phi}),}$$

- Where: C = controllability matrix, *n* is the order of the system (or number of state elements) and α_c:

$$\mathcal{C} = [\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \ldots]$$

$$\alpha_c(\mathbf{\Phi}) = \mathbf{\Phi}^n + \alpha_1\mathbf{\Phi}^{n-1} + \alpha_2\mathbf{\Phi}^{n-2} + \cdots + \alpha_n\mathbf{I},$$

  - α_i: coefficients of the desired characteristic equation

$$\alpha_c(z) = |\, z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma K}\,| = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n.$$

Ackermann's Formula Example (FPW p.246)

Example 6.2: Applying Ackermann's formula to the satellite attitude-control system of Example 6.1, we find from (6.9) that

$$\alpha_1 = -1.6, \qquad \alpha_2 = +0.70,$$

and therefore

$$\alpha_c(\mathbf{\Phi}) = \begin{bmatrix} 1 & 2T \\ 0 & 1 \end{bmatrix} - 1.6\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + 0.70\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix}.$$

Furthermore, we find that

$$[\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma}] = \begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix}$$

and

$$[\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma}]^{-1} = 1/T^2\begin{bmatrix} -1 & +3T/2 \\ 1 & -T/2 \end{bmatrix},$$

and finally

$$\mathbf{K} = [K_1 \quad K_2] = (1/T^2)[0 \quad 1]\begin{bmatrix} -1 & 3T/2 \\ 1 & -T/2 \end{bmatrix}\begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix};$$

therefore

$$[K_1 \quad K_2] = \frac{1}{T^2}[0.1 \quad 0.35T]$$
$$= [10 \quad 3.5],$$

which is the same result as that obtained earlier.

Shaping the Dynamic Response: SISO

Design of regulators for single-input, single-output systems

6.2 DESIGN OF REGULATORS FOR SINGLE-INPUT, SINGLE-OUTPUT SYSTEMS

The present section is concerned with the design of a gain matrix

$$G = g' = [g_1, g_2, \ldots, g_k] \tag{6.6}$$

for the single-input, single-output system

$$\dot{x} = Ax + Bu \tag{6.7}$$

where

$$B = b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix} \tag{6.8}$$

With the control law $u = -Gx = -g'x$ (6.7) becomes

$$\dot{x} = (A - bg')x$$

Our objective is to find the matrix $G = g'$ which places the poles of the closed-loop dynamics matrix

$$A_c = A - bg' \tag{6.9}$$

Design of regulators for single-input, single-output systems

at the locations desired. We note that there are k gains $g_1, g_2, \ldots, g_k$ and k poles for a kth order system, so there are precisely as many gains as needed to specify each of the closed-loop poles.

One way of determining the gains would be to set up the characteristic polynomial for A_c:

$$|sI - A_c| = |sI - A + bg'| = s^k + \bar{a}_1 s^{k-1} + \cdots + \bar{a}_k \tag{6.10}$$

The coefficients $\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k$ of the powers of s in the characteristic polynomial will be functions of the k unknown gains. Equating these functions to the numerical values desired for $\bar{a}_1, \ldots, \bar{a}_k$ will result in k simultaneous equations the solution of which will yield the desired gains $g_1, \ldots, g_k$.

This is a perfectly valid method of determining the gain matrix g', but it entails a substantial amount of calculation when the order k of the system is higher than 3 or 4. For this reason, we would like to develop a direct formula for g in terms of the coefficients of the open-loop and closed-loop characteristic equations.

If the original system is in the companion form given in (3.90), the task is particularly easy, because

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{k-1} & -a_k \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{6.11}$$

Design of regulators for single-input, single-output systems

$$bg' = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} [g_1, g_2, \ldots, g_k] = \begin{bmatrix} g_1 & g_2 & \cdots & g_k \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Hence

$$A_c = A - bg' = \begin{bmatrix} -a_1 - g_1 & -a_2 - g_2 & \cdots & -a_k - g_k \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

The gains $g_1, \ldots, g_k$ are simply added to the coefficients of the open-loop A matrix to give the closed-loop matrix A_c. This is also evident from the block-diagram representation of the closed-loop system as shown in Fig. 6.1. Thus for a system in the companion form of Fig. 6.1, the gain matrix elements are given by

$$a_i + g_i = \hat{a}_i \qquad i = 1, 2, \ldots, k$$

or

$$g = \hat{a} - a \tag{6.12}$$

where

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_k \end{bmatrix} \qquad \hat{a} = \begin{bmatrix} \hat{a}_1 \\ \vdots \\ \hat{a}_k \end{bmatrix} \tag{6.13}$$

Design of regulators for single-input, single-output systems

are vectors formed from the coefficients of the open-loop and closed-loop characteristic equations, respectively.

The dynamics of a typical system are usually not in companion form. It is necessary to transform such a system into companion form before (6.12) can be used. Suppose that the state of the transformed system is $\bar{x}$, achieved through the transformation

$$\bar{x} = Tx \tag{6.14}$$

Then, as shown in Chap. 3,

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{b}u \tag{6.15}$$

where

$$\bar{A} = TAT^{-1} \qquad \text{and} \qquad \bar{b} = Tb$$

For the transformed system the gain matrix is

$$\bar{g} = \hat{a} - \bar{a} = \hat{a} - a \tag{6.16}$$

since $\bar{a} = a$ (the characteristic equation being invariant under a change of state variables). The desired control law in the original system is

$$u = -g'x = -g'T^{-1}\bar{x} = -\bar{g}'\bar{x} \tag{6.17}$$

From (6.17) we see that

$$\bar{g}' = g'T^{-1}$$

Thus the gain in the original system is

$$g = T'\bar{g} = T'(\hat{a} - a) \tag{6.18}$$

Design of regulators for single-input, single-output systems

In words, the desired gain matrix for a general system is the difference between the coefficient vectors of the desired and actual characteristic equation, premultiplied by the inverse of the transpose of the matrix T that transforms the general system into the companion form of (3.90), the A matrix of which has the form (6.11).

The desired matrix T is obtained as the product of two matrices U and V:

$$T = VU \tag{6.19}$$

The first of these matrices transforms the original system into an intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} \tag{6.20}$$

in the second companion form (3.107) and the second transformation U transforms the intermediate system into the first companion form.

Consider the intermediate system

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{b}u \tag{6.21}$$

with $\tilde{A}$ and $\tilde{b}$ in the form of (3.107). Then we must have

$$\tilde{A} = UAU^{-1} \qquad \text{and} \qquad \tilde{b} = Ub \tag{6.22}$$

Design of regulators for single-input, single-output systems

The desired matrix U is precisely the inverse of the controllability test matrix Q of Sec. 5.4. To prove this fact, we must show that

$$U^{-1}\tilde{A} = AU^{-1} \tag{6.23}$$

or

$$Q\tilde{A} = AQ \tag{6.24}$$

Now, for a single-input system

$$Q = [b, Ab, \ldots, A^{k-1}b]$$

Thus, with $\tilde{A}$ given by (3.107), the left-hand side of (6.23) is

$$Q\tilde{A} = [b, Ab, \ldots, A^{k-1}b]\begin{bmatrix} 0 & 0 & \cdots & -a_k \\ 1 & 0 & \cdots & -a_{k-1} \\ 0 & 1 & \cdots & -a_{k-2} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a_1 \end{bmatrix}$$

$$= [Ab, A^2b, \ldots, A^{k-1}b, -a_kb - a_{k-1}Ab - \cdots - a_kA^{k-1}b] \tag{6.25}$$

The last term in (6.25) is

$$(-a_kI - a_{k-1}A - \cdots - a_kA^{k-1})b \tag{6.26}$$

Now, by the Cayley-Hamilton theorem, (see Appendix):

$$A^k = -a_1A^{k-1} - a_2A^{k-2} - \cdots - a_kI$$

so (6.26) is A^kb. Thus the left-hand side of (6.24) as given by (6.25) is

$$Q\tilde{A} = [Ab, A^2b, \ldots, A^kb] = A[b, Ab, \ldots, A^{k-1}b] = AQ$$

which is the desired result.

</assessable>

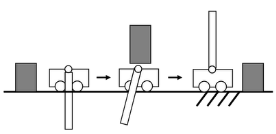
WARNING: NOT ASSESSABLE

- Nothing beyond this point is on the exam.
  (except for the exam review ☺)
- Do not pay attention.
- Do not attempt to learn.

Example 1:
Inverted Pendulum

Digital Control

Wikipedia,
Cart and pole



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$
$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

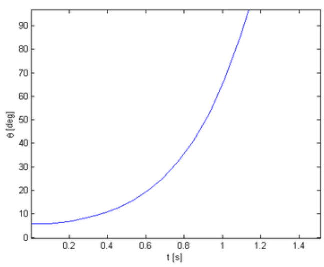
$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

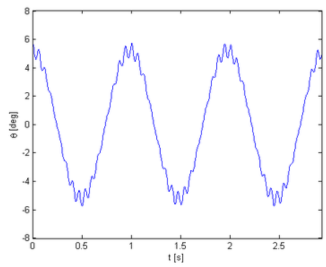
and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

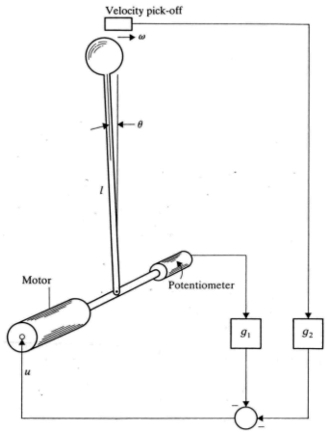
substituting L in these equations and simplifying leads to the equations that describe the motion o

$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$
$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$





Inverted Pendulum



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$

$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

substituting L in these equations and simplifying leads to the equations that describe the motion o

$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$

$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$

Inverted Pendulum – Equations of Motion

- The equations of motion of an inverted pendulum (under a small angle approximation) may be linearized as:

$$\dot{\theta} = \omega$$

$$\dot{\omega} = \ddot{\theta} = Q^2\theta + Pu$$

Where:

$$Q^2 = \left(\frac{M+m}{Ml}\right)g$$

$$P = \frac{1}{Ml}.$$

If we further assume unity ***Ml*** ($Ml \approx 1$), then $P \approx 1$

Inverted Pendulum –State Space

- We then select a state-vector as:

$$\boldsymbol{x} = \begin{bmatrix}\theta\\ \omega\end{bmatrix}, \text{ hence } \dot{\boldsymbol{x}} = \begin{bmatrix}\dot{\theta}\\ \dot{\omega}\end{bmatrix} = \begin{bmatrix}\omega\\ \dot{\omega}\end{bmatrix}$$

- Hence giving a state-space model as:

$$A = \begin{bmatrix}0 & 1\\ Q^2 & 0\end{bmatrix}, B = \begin{bmatrix}0\\ 1\end{bmatrix}$$

- The resolvent of which is:

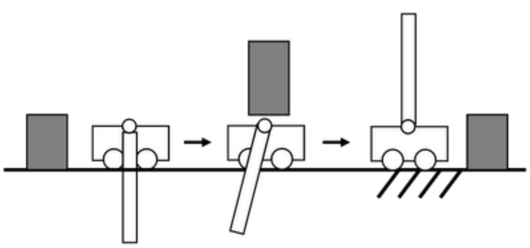
$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix}s & -1\\ -Q^2 & s\end{bmatrix}^{-1} = \frac{1}{s^2 - Q^2}\begin{bmatrix}s & 1\\ Q^2 & s\end{bmatrix}$$

- And a state-transition matrix as:

$$\Phi(t) = \begin{bmatrix}\cosh Qt & \dfrac{\sinh Qt}{Q}\\ Q\ sinh\ Qt & \cosh Qt\end{bmatrix}$$

Cart & Pole in State-Space With Obstacles?

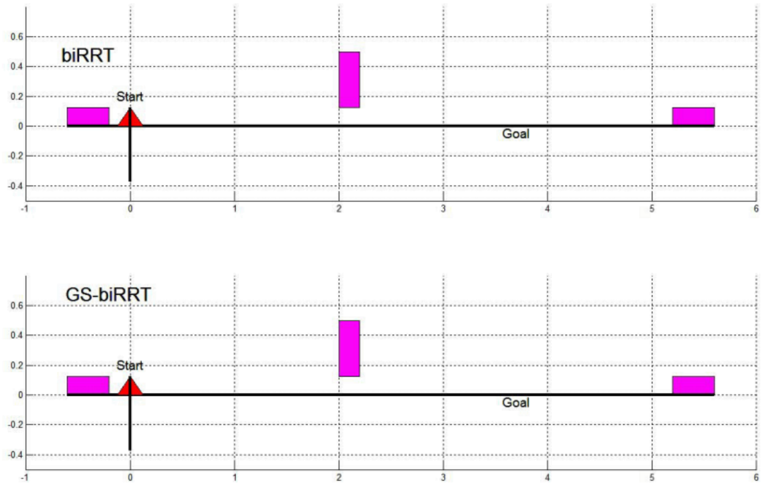
Swing-up is a little more than stabilization…



See also: METR4202 – Tutorial 11:
http://robotics.itee.uq.edu.au/~metr4202/tpl/t11-Week11-pendulum.pdf

Cart & Pole in State-Space

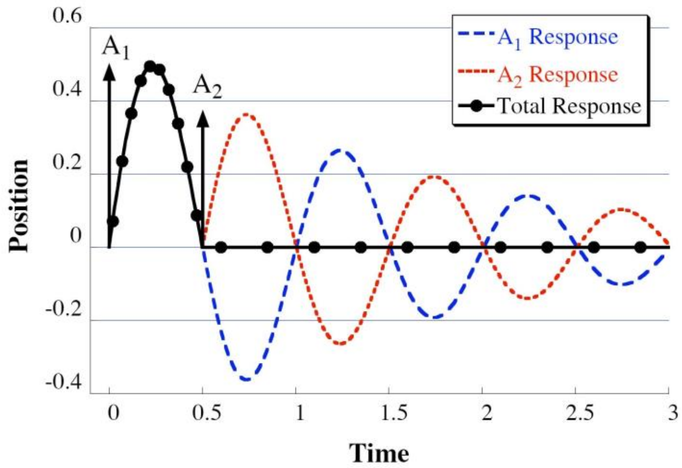
Swing-up is a little more than stabilization…



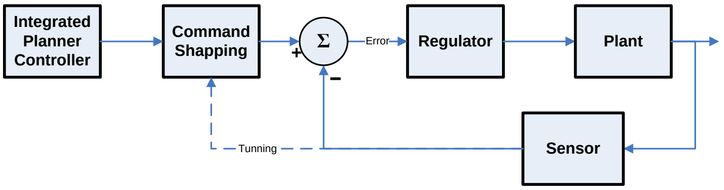
Example 2: Command Shaping

Experiments: Scanning Over Obstacle

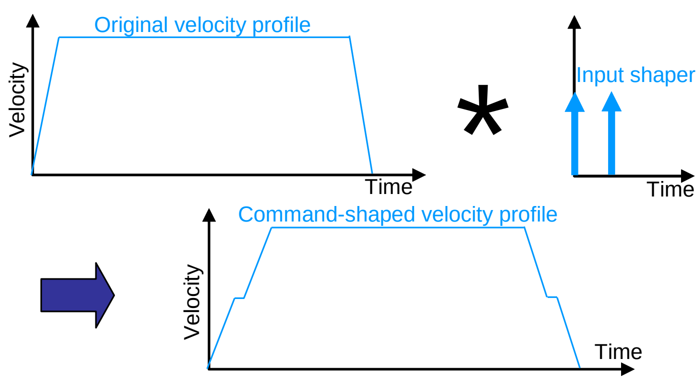
Command Shaping



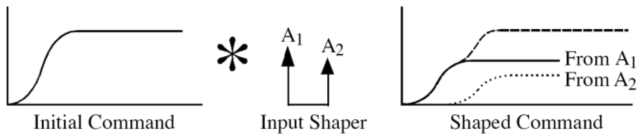
Robust Control:
Command Shaping for Vibration Reduction



Command Shaping



Command Shaping



- Zero Vibration (ZV)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{1+K} & \frac{K}{1+K} \\ 0 & \frac{T_d}{2} \end{bmatrix} \qquad K = e^{\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

- Zero Vibration and Derivative (ZVD)

$$\begin{bmatrix} A_i \\ t_i \end{bmatrix} = \begin{bmatrix} \frac{1}{(1+K)^2} & \frac{2K}{(1+K)^2} & \frac{K^2}{(1+K)^2} \\ 0 & \frac{T_d}{2} & T_d \end{bmatrix}$$

Experiments: Command Shaping

Next Time…

- **Digital Control via Emulation!**

- Review:
  - Chapter 5 of FPW

- Deeper Pondering??