http://elec3004.com

Systems Overview

ELEC 3004: **Systems**: Signals & Controls
Dr. Surya Singh

Lecture 2

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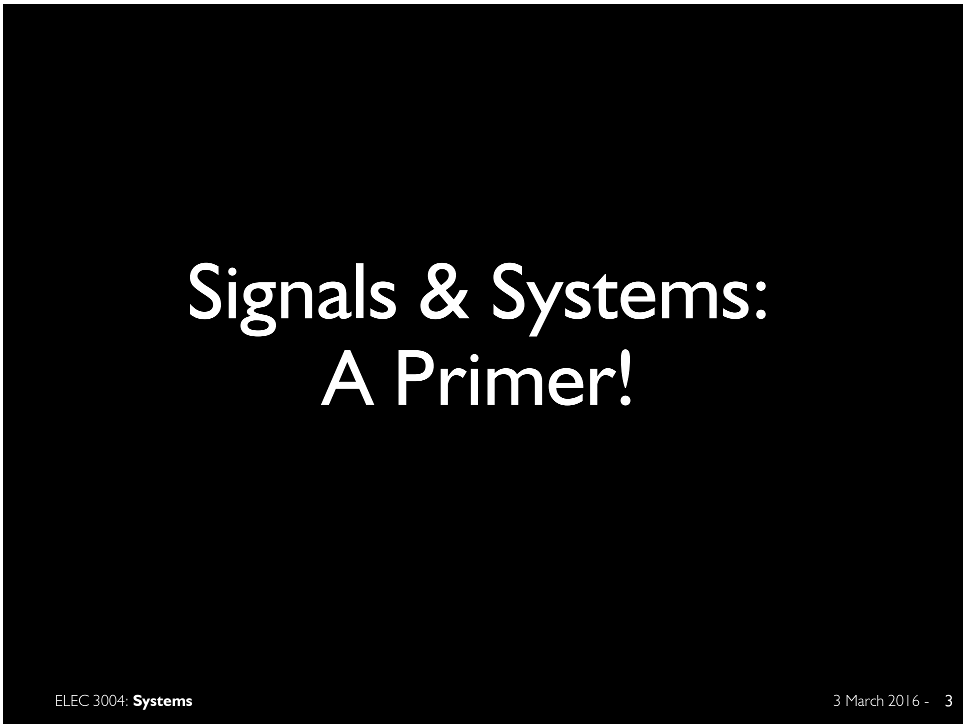
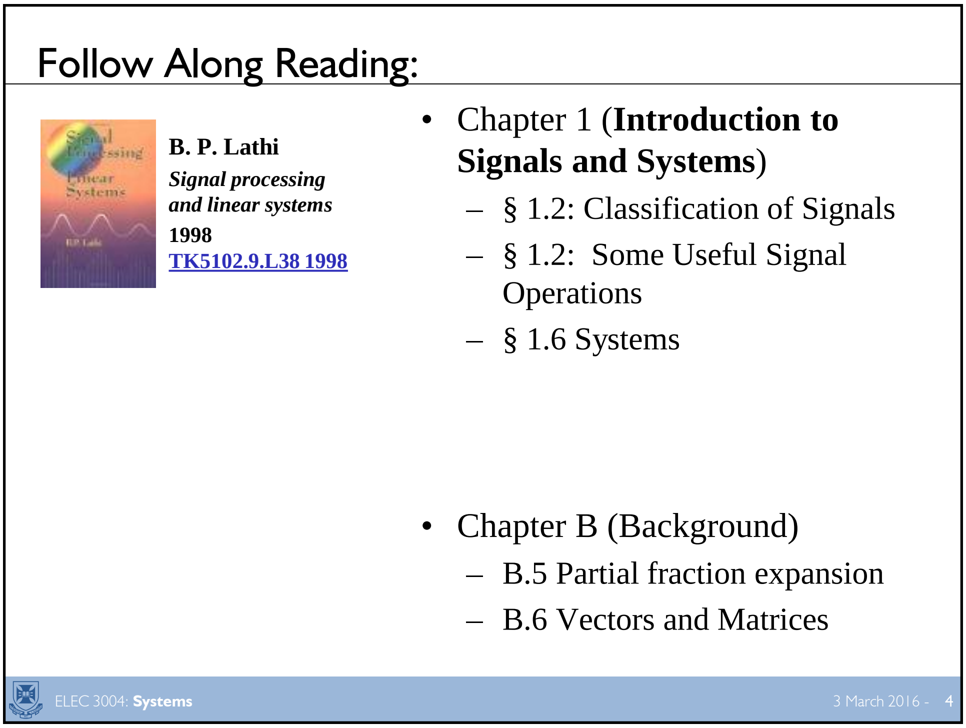
March 3, 2016

Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	9-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	16-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	23-Mar	Discrete System Analysis
	24-Mar	Convolution Review
0	30-Mar	Holiday & Holiday
	31-Mar	Holiday
6	13-Apr	Digital Filters
	14-Apr	Digital Filters
7	20-Apr	Digital Windows
	21-Apr	FFT
8	27-Apr	Discrete Systems Analysis
	28-Apr	Feedback
9	4-May	Introduction to Feedback Control
	5-May	Servoregulation/PID
10	11-May	Introduction to (Digital) Control
	12-May	Digitial Control
11	18-May	Digital Control Design
	19-May	Stability
12	25-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	1-Jun	System Identification & Information Theory
	2-Jun	Summary and Course Review



Signals & Systems: A Primer!

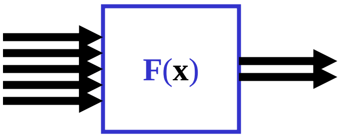


Follow Along Reading:

B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

- Chapter 1 (**Introduction to Signals and Systems**)
  - § 1.2: Classification of Signals
  - § 1.2: Some Useful Signal Operations
  - § 1.6 Systems
- Chapter B (Background)
  - B.5 Partial fraction expansion
  - B.6 Vectors and Matrices

An Overview of Systems

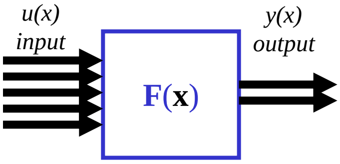


- Today we are going to look at **F**(x)!
- F(x): System Model
  - The rules of operation that describe it's behaviour of a "system"
  - Predictive power of the responses
  - Analytic forms > Empirical ones
    - Analytic formula offer various levels of detail
    - Not everything can be experimented on *ad infinitum*
    - Also offer Design Intuition (let us devise new "systems")
    - Let's us do **analysis**! (determine the outputs for an input)
  - Various Analytic Forms
    - Constant, Polynomial, **Linear**, Nonlinear, Integral, **ODE**, PDE, Bayesian…

System Classifications/Attributes

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Linear Systems



- Model describes the relationship between the input **u**(x) and the output **y**(x)
- If it is a Linear System (wk 3):

$$y(t) = \int_0^t F(t-\tau)\, u(\tau)\, d\tau$$

- If it is also a (Linear and) **lumped**, it can be expressed **algebraically** as:

$$\dot{x}(t) = A(t)\, x(t) + B(t)\, u(t)$$
$$y(t) = C(t)\, x(t) + D(t)\, u(t)$$

- If it is also (Linear and) **time invariant** the matrices can be reduced to:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

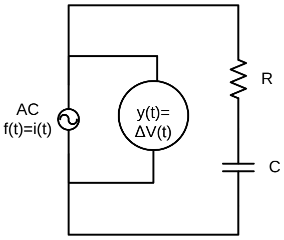
$\mathscr{L}$aplacian: $$y(s) = F(s)\, u(s)$$

Modelling Ties Back with ELEC 2004

- Linear Circuit Theorems, Operational Amplifiers
- Operational Amplifiers
- Capacitors and Inductors, RL and RC Circuits
- AC Steady State Analysis
- AC Power, Frequency Response
- Laplace Transform
- Reduction of Multiple Sub-Systems
- Fourier Series and Transform
- Filter Circuits

➔ Modelling Tools!

Example 1: RC Circuits



$$y\left(t\right) = Rf\left(t\right) + \tfrac{1}{C}\int_{-\infty}^{t} f\left(\tau\right) d\tau$$

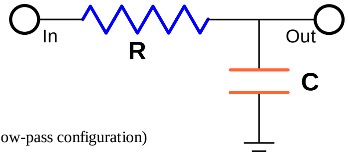
$$y\left(t\right) = Rf\left(t\right) + \tfrac{1}{C}\int_{-\infty}^{0} f\left(\tau\right) d\tau + \tfrac{1}{C}\int_{0}^{t} f\left(\tau\right) d\tau$$

$$y\left(t\right) = v_C\left(0\right) + Rf\left(t\right) + \tfrac{1}{C}\int_{0}^{t} f\left(\tau\right) d\tau$$

$$y\left(t\right) = v_C\left(t_0\right) + Rf\left(t\right) + \tfrac{1}{C}\int_{t_0}^{t} f\left(\tau\right) d\tau$$

First Order RC Filter

- Passive, First-Order **R**esistor-**C**apacitor Design:



(Low-pass configuration)

$$T\left(s\right) = \frac{a_1 s + a_0}{s + \omega_0}$$

- 3dB (½ Signal Power):

$$\omega = 2\pi f$$

$$f_c = \frac{1}{2\pi \mathbf{RC}}$$

- Magnitude:

$$\left|V_{\text{out}}\right| = \sqrt{\frac{1}{\left(\omega RC\right)^2}}\left|V_{\text{in}}\right|$$

- Phase:

$$\phi = \tan^{-1}\left(-\omega RC\right)$$

Expanding on this: Types of Linear Systems

From Last Week:

- LDS:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- LTI – LDS:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Types of Linear Systems

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Types of Linear Systems

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To Review:

- Continuous-time linear dynamical system (CT LDS):

$$\frac{dx}{dt} = A(t)x(t) + B(t)u(t), \quad y(t) = C(t)x(t) + D(t)u(t)$$

- $\boldsymbol{t} \in \mathbb{R}$ denotes time
- $x(t) \in \mathbb{R}^n$ is the state (vector)
- $u(t) \in \mathbb{R}^m$ is the input or control
- $y(t) \in \mathbb{R}^p$ is the output

Types of Linear Systems

- LDS:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- $A(t) \in \mathbb{R}^{n\times n}$ is the dynamics matrix
- $B(t) \in \mathbb{R}^{n\times m}$ is the input matrix
- $C(t) \in \mathbb{R}^{p\times n}$ is the output or sensor matrix
- $D(t) \in \mathbb{R}^{p\times m}$ is the feedthrough matrix

➔ state equations, or “*m*-input, *n*-state, *p*-output’ LDS

Types of Linear Systems

- LDS:

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➔ state equations, or "*m*-input, *n*-state, *p*-output' LDS

Types of Linear Systems

- LDS:

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)$$
$$y(t) = C(t)\,x(t) + D(t)\,u(t)$$

- **Time-invariant:** where A(t), B(t), C(t) and D(t) are **constant**
- **Autonomous:** there is no input *u* (B,D are irrelevant)
- **No Feedthrough**: D = 0

- SISO: *u(t)* and *y(t)* are scalars
- MIMO: *u(t)* and *y(t)*: They're vectors: Big Deal ?

Discrete-time Linear Dynamical System

- Discrete-time Linear Dynamical System (DT LDS) has the form:

$$x(t+1) = A(t)x(t) + B(t)u(t), \qquad y(t) = C(t)x(t) + D(t)u(t)$$

- ***t*** ∈ ℤ denotes time index : ℤ={0, ±1, …, ± **n**}
- *x(t), u(t), y(t)* ∈ are sequences
- Differentiation handled as difference equation:
  ➔ first-order vector recursion

Discrete Variations & Stability

$$y\,(s) = F\,(s)\,u\,(s)$$

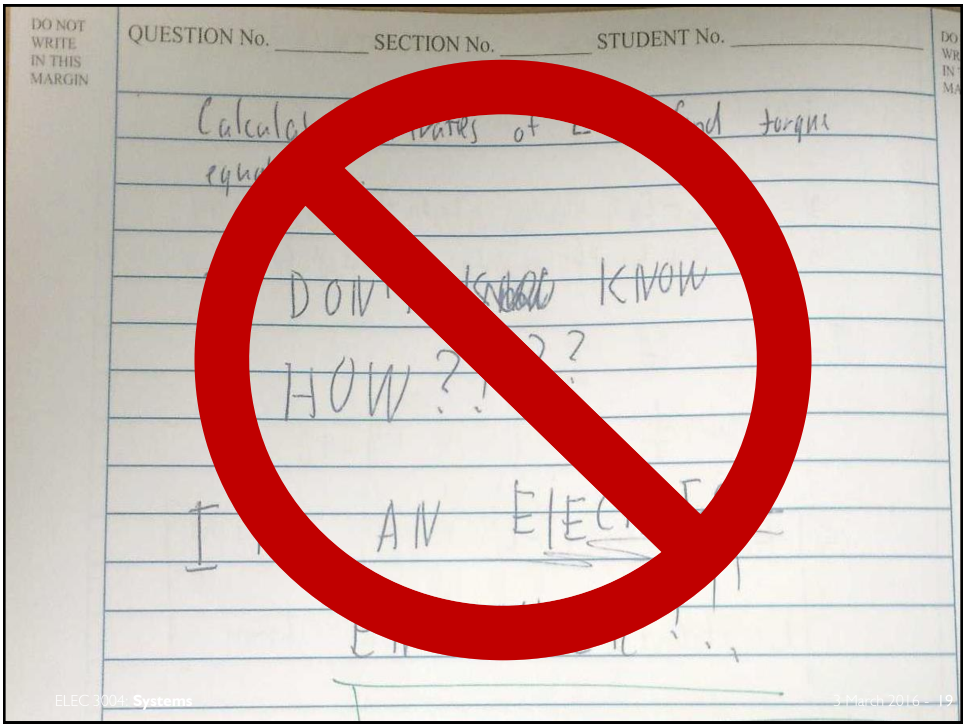
- Is in continuous time …
- To move to discrete time it is more than just "sampling" at: 2 × (biggest Frequency)
- Discrete-Time Exponential

$$F(t) \to F\,[kT]$$

$$e^{\frac{k}{T}} = \gamma^k$$

$$\frac{1}{T} = \ln \gamma$$

- SISO to MIMO
  - Single Input, Single Output
  - Multiple Input, Multiple Output
- BIBO:
  - Bounded Input, Bounded Output
- Lyapunov:
  - Conditions for Stability
  ➔ Are the results of the system asymptotic or exponential



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QUESTION No. ______ SECTION No. ______ STUDENT No. ______

Calcula[illegible] [illegible]vatives of [illegible] [illegible]nd torque
equa[illegible]

DON'T ~~know~~ KNOW
HOW ?! ??

I [illegible] AN ELEC[illegible]
EN[illegible]!!

Equivalence Across Domains

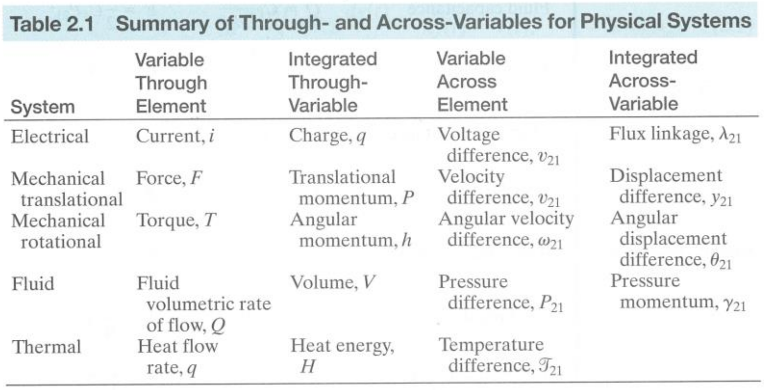


Table 2.1 Summary of Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Source: Dorf & Bishop, *Modern Control Systems*, 12th Ed., p. 73

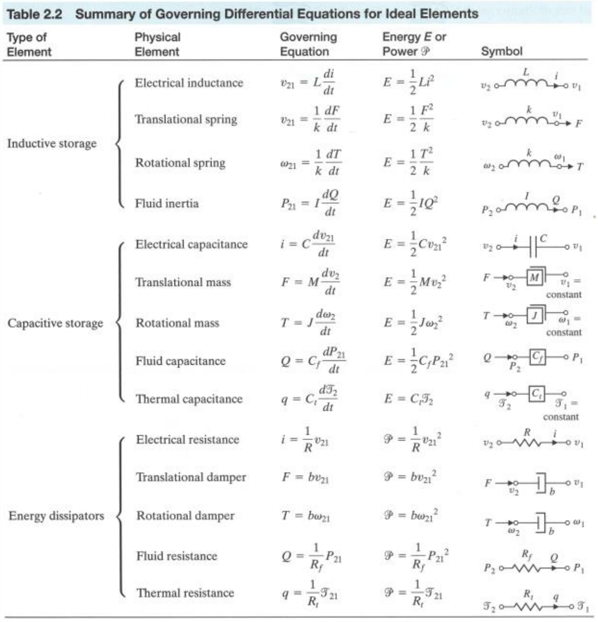
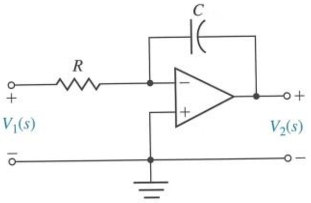


Table 2.2 Summary of Governing Differential Equations for Ideal Elements

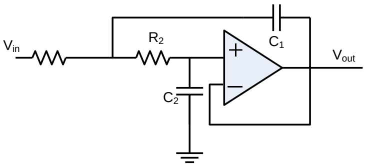
Type of Element	Physical Element	Governing Equation	Energy E or Power $\mathcal{P}$	Symbol
Inductive storage	Electrical inductance	$v_{21} = L\frac{di}{dt}$	$E = \frac{1}{2}Li^2$	v_2 L i v_1
	Translational spring	$v_{21} = \frac{1}{k}\frac{dF}{dt}$	$E = \frac{1}{2}\frac{F^2}{k}$	v_2 k v_1 F
	Rotational spring	$\omega_{21} = \frac{1}{k}\frac{dT}{dt}$	$E = \frac{1}{2}\frac{T^2}{k}$	ω_2 k ω_1 T
	Fluid inertia	$P_{21} = I\frac{dQ}{dt}$	$E = \frac{1}{2}IQ^2$	P_2 I Q P_1
Capacitive storage	Electrical capacitance	$i = C\frac{dv_{21}}{dt}$	$E = \frac{1}{2}Cv_{21}{}^2$	v_2 i C v_1
	Translational mass	$F = M\frac{dv_2}{dt}$	$E = \frac{1}{2}Mv_2{}^2$	F v_2 M $v_1 =$ constant
	Rotational mass	$T = J\frac{d\omega_2}{dt}$	$E = \frac{1}{2}J\omega_2{}^2$	T ω_2 J $\omega_1 =$ constant
	Fluid capacitance	$Q = C_f\frac{dP_{21}}{dt}$	$E = \frac{1}{2}C_fP_{21}{}^2$	Q P_2 C_f P_1
	Thermal capacitance	$q = C_t\frac{d\mathcal{T}_2}{dt}$	$E = C_t\mathcal{T}_2$	q $\mathcal{T}_2$ C_t $\mathcal{T}_1 =$ constant
Energy dissipators	Electrical resistance	$i = \frac{1}{R}v_{21}$	$\mathcal{P} = \frac{1}{R}v_{21}{}^2$	v_2 R i v_1
	Translational damper	$F = bv_{21}$	$\mathcal{P} = bv_{21}{}^2$	F v_2 b v_1
	Rotational damper	$T = b\omega_{21}$	$\mathcal{P} = b\omega_{21}{}^2$	T ω_2 b ω_1
	Fluid resistance	$Q = \frac{1}{R_f}P_{21}$	$\mathcal{P} = \frac{1}{R_f}P_{21}{}^2$	P_2 R_f Q P_1
	Thermal resistance	$q = \frac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{P} = \frac{1}{R_t}\mathcal{T}_{21}$	$\mathcal{T}_2$ R_t q $\mathcal{T}_1$

Source: Dorf & Bishop, *Modern Control Systems*, 12th Ed., p. 74

Circuits



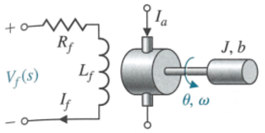
$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$



$$\frac{v_{out}}{v_{in}} = \frac{1}{C_1C_2R_1R_2s^2+C_2(R_1+R_2)s+1}$$

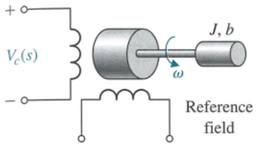
Motors

5. DC motor, field-controlled, rotational actuator



$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$$

7. AC motor, two-phase control field, rotational actuator



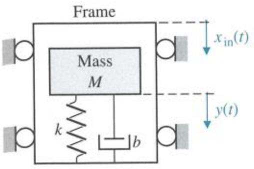
$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

Mechanical Systems

15. Accelerometer, acceleration sensor



$$x_o(t) = y(t) - x_{\text{in}}(t),$$

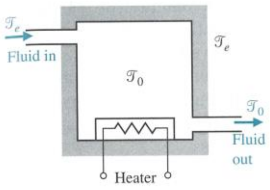
$$\frac{X_o(s)}{X_{\text{in}}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$

For low-frequency oscillations, where $\omega < \omega_n$,

$$\frac{X_o(j\omega)}{X_{\text{in}}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

Thermal Systems

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$ = temperature difference due to thermal process

C_t = thermal capacitance

Q = fluid flow rate = constant

S = specific heat of water

R_t = thermal resistance of insulation

$q(s)$ = transform of rate of heat flow of heating element

First Order Systems

First order systems

$$ay' + by = 0 \quad (\text{with } a \neq 0)$$

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

$$Ty' + y = 0 \quad \text{or} \quad y' + ry = 0$$

where

- $T = a/b$ is a *time* (units: seconds)
- $r = b/a = 1/T$ is a *rate* (units: $1/$sec)

First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT + 1} = \frac{y(0)}{s + 1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, y *decays* exponentially

- T gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half $(0.693 = \log 2)$
- $4.6T$ gives time to decay by 0.01 $(4.6 = \log 100)$

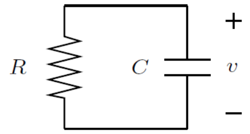
if $T < 0$, y *grows* exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100

First Order Systems

Examples

simple RC circuit:



circuit equation: $RCv' + v = 0$

solution: $v(t) = v(0)e^{-t/(RC)}$

population dynamics:

- $y(t)$ is population of some bacteria at time t
- growth (or decay if negative) rate is $y' = by - dy$ where b is birth rate, d is death rate
- $y(t) = y(0)e^{(b-d)t}$ (grows if $b > d$; decays if $b < d$)

Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a(\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}(y'')}) + b(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

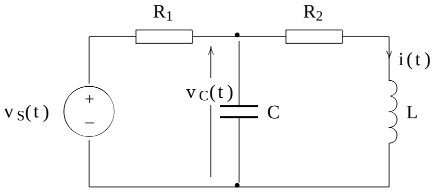
Second Order Systems

so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

Example of 2nd Order: RLC Circuits



- KCL:

$$\frac{V_s(t) - V_c(t)}{R_1} = C\frac{d}{dt}V_c(t) + i(t)$$

- KVL:

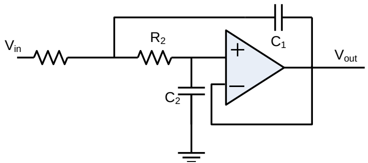
$$V_c(t) = L\frac{d}{dt}i(t) + R_2 i(t)$$

- Combining:

$$V_s(t) = R_1 LC\frac{d^2}{dt^2}i(t) + (L + R_1 R_2 C)\frac{d}{dt}i(t) + (R_1 + R_2)\,i(t)$$

2^{nd} Order Active RC Filter (Sallen–Key)

- 2^{nd} Order System Sallen–Key Low-Pass Topology:



Build this for
Real in
ELEC 4403

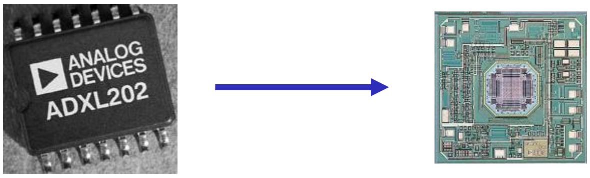
- KCL: $\frac{v_{\text{in}}-v_x}{R_1} = C_1 s\left(v_x - v_{\text{out}}\right) + \frac{v_x - v_{\text{out}}}{R_2}$
- Combined with Op-Amp Law:

$$\frac{v_{\text{in}}-v_{\text{out}}(C_2 s R_2+1)}{R_1} = C_1 s v_{\text{out}}\left(C_2 s R_2 + 1\right) - v_{\text{out}} + \frac{v_{\text{out}}(C_2 s R_2+1)-v_{\text{out}}}{R_2}$$

- Solving for Gives a 2^{nd} order System:

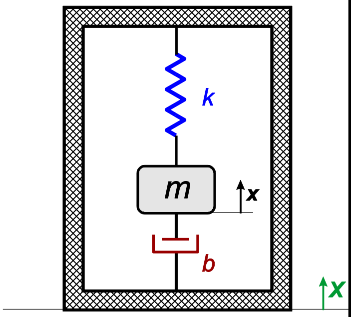
$$\frac{v_{out}}{v_{in}} = \frac{1}{C_1C_2R_1R_2s^2+C_2(R_1+R_2)s+1}$$

Another 2^{nd} Order System:
Accelerometer or Mass Spring Damper (MSD)



- General accelerometer:
  - Linear spring (*k*) (0^{th} order w/r/t o)
  - Viscous damper (*b*) (1^{st} order)
  - Proof mass (*m*) (2^{nd} order)

➔ Electrical system analogy:
  - resistor (R) : damper (b)
  - inductance (L) : spring (k)
  - capacitance (C) : mass (m)



Measuring Acceleration:
Sense *a* by measuring spring motion *Z*

- Start with Newton’s 2nd Law:

$$ma = F$$

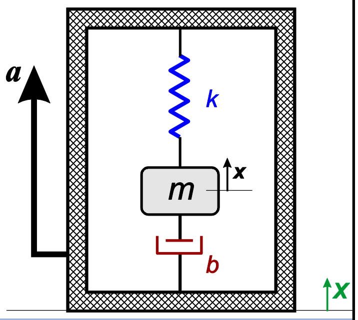
- Substitute:

$$m\frac{d^2x}{dt^2} = k\left(X - x\right) + b\frac{d\left(X - x\right)}{dt}$$

$$\mathbf{Z} \equiv \left(X - x\right) \rightarrow x = X - \mathbf{Z}$$

$$\Rightarrow m\frac{d^2X}{dt^2} = m\frac{d^2\mathbf{Z}}{dt^2} + k\mathbf{Z} + b\frac{d\mathbf{Z}}{dt}$$

The “displacement” measured by the unit (the motion of *m* relative the accelerometer frame)



- Solve ODE:

$$X\left(t\right) = X_0 e^{i\omega t} \qquad \mathbf{Z}\left(t\right) = Z_0 e^{i\omega t}$$

Measuring Acceleration [2]

- Substitute candidate solutions:

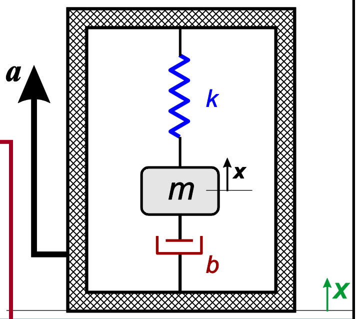
$$m\frac{d^2\left(X_0 e^{i\omega t}\right)}{dt^2} = m\frac{d^2\left(Z_0 e^{i\omega t}\right)}{dt^2} + k\left(Z_0 e^{i\omega t}\right) + b\frac{d\left(Z_0 e^{i\omega t}\right)}{dt}$$

$$-m\omega^2 X_0 e^{i\omega t} = -m\omega^2 Z_0 e^{i\omega t} + kZ_0 e^{i\omega t} + \left(i\omega\right) bZ_0 e^{i\omega t}$$

- Define Natural Frequency (ω_0)
  & Simplify for Z_0
  (the spring displacement “magnitude”):

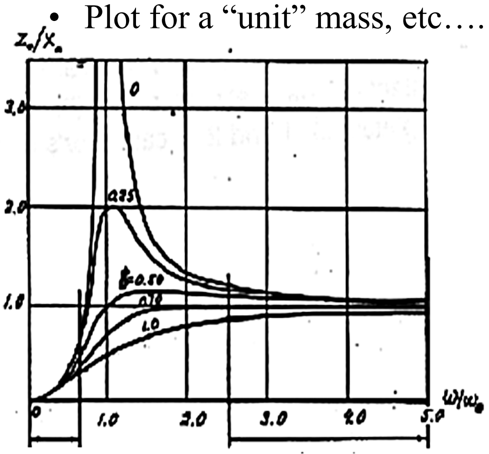
$$\omega_0 \equiv \sqrt{\frac{k}{m}}$$

$$Z_0 = \frac{m\omega^2 X_0}{m\omega^2 - k - i\omega b} = \boxed{\frac{X_0}{\sqrt{1 - \frac{\omega_0^2}{\omega^2} - \frac{b^2}{m^2\omega^2}}}}$$



Acceleration: 2nd Order System

- Plot for a "unit" mass, etc….



Accelerometer Seismometer

- For $\omega \ll \omega_0$:

$$Z_0 \approx \frac{\omega^2 X_0}{\omega_0^2} = \frac{a}{\omega_0^2}$$

$$\to a = Z_0 \omega_0^2$$

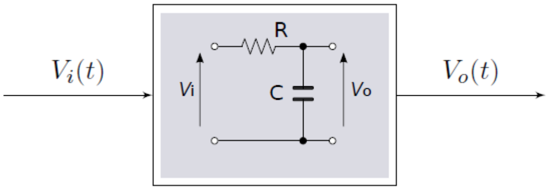
➔ **it's an Accelerometer**

- For $\omega \sim \omega_0$
  - As: b → 0, Z → ∞
  - Sensitivity ↑
- For $\omega \gg \omega_0$:

$$Z_0 \approx X_0$$

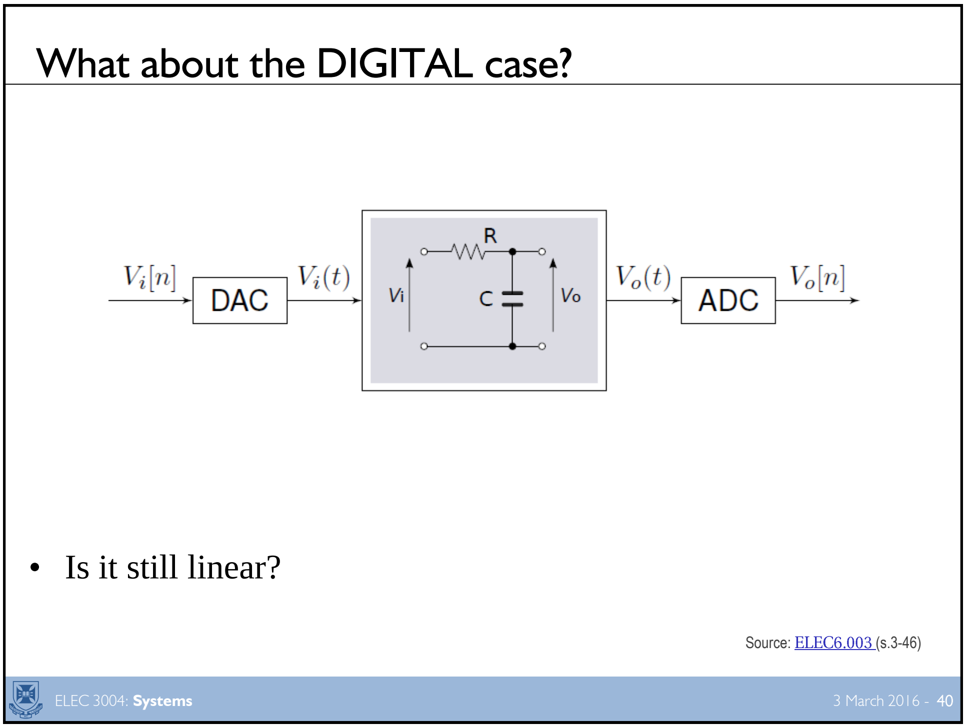
➔ **it's a Seismometer**

Example: Speaking of Circuits



$$C\frac{dV_o(t)}{dt} = \frac{V_i(t) - V_o(t)}{R}.$$

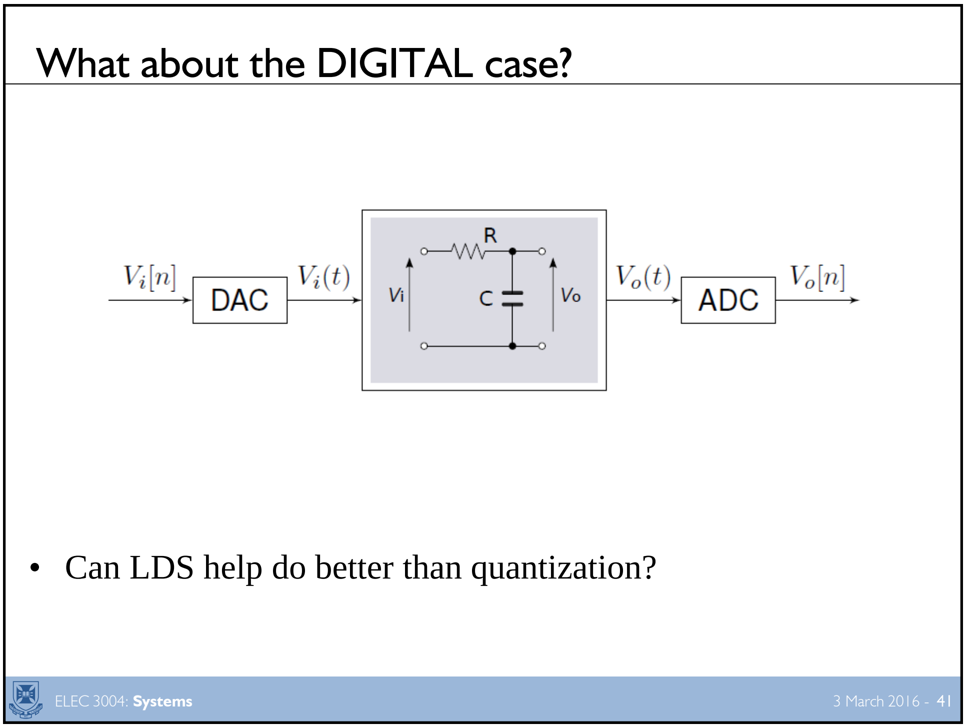
Source: ELEC6.003 (s.3-42)



What about the DIGITAL case?

- Is it still linear?

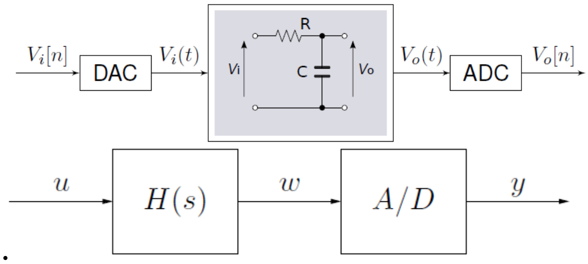
Source: ELEC6.003 (s.3-46)



What about the DIGITAL case?

- Can LDS help do better than quantization?

What about the DIGITAL case?



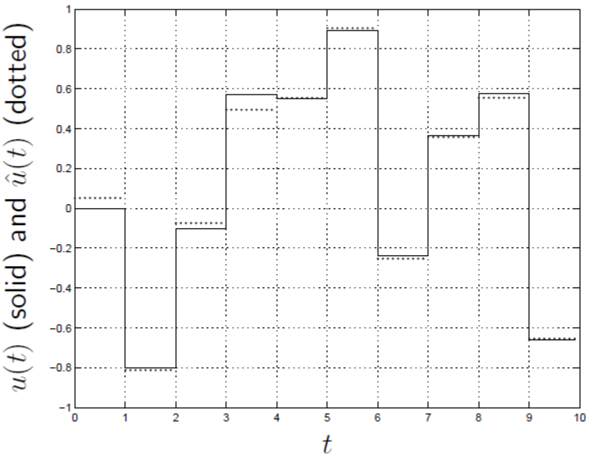
➔

- Problem:
  Estimate signal ***u***, given quantized, filtered signal ***y***
- Some solutions:
  - ignore quantization
  - design equalizer *G(s)* for *H(s)* (i.e., *GH* ≅ 1)
  - approximate *u* as *G(s)y*

  ➔ Pose as an estimation problem

Source: EE263 (s.1-124)

What about the DIGITAL case?



- RMS error 0.03, well below quantization error (!)

Source: EE263 (s.1-124)

Next Time…

- We'll talk about Other System Properties ☺

- We will introduce this via the lens of:

  "Systems as Maps. Signals as Vectors"
- Review:
  - Phasers, complex numbers, polar to rectangular, and general functional forms.
  - Chapter B and Chapter 1 of Lathi
    (particularly the first sections on signals & classification thereof)

- Register on Platypus

- Try the practise assignment