http://elec3004.com

Fourier Analysis of DT Signals and FFT

ELEC 3004: **Systems**: Signals & Controls

Dr. Surya Singh

Lecture 14
(with material from Lathi)

elec3004@itee.uq.edu.au

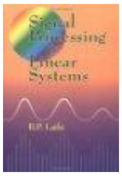
http://robotics.itee.uq.edu.au/~elec3004/

April 18, 2016

Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	24-Mar	Convolution Review
	28-Mar	Holiday
	31-Mar	
5	4-Apr	Frequency Response & Filter Analysis
	7-Apr	Filters
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Feedback
9	3-May	Introduction to Feedback Control
	5-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	12-May	Digitial Control
11	16-May	Digital Control Design
	19-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	2-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

Today

- Chapter 10
  (Discrete-Time System Analysis Using the *z*-Transform)
  – § 10.3 Properties of DTFT
  – § 10.5 Discrete-Time Linear System analysis by DTFT
  – § 10.7 Generalization of DTFT to the $\mathcal{Z}$–Transform

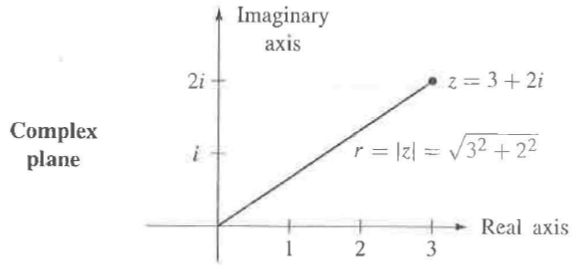
- Chapter 12
  (**Frequency Response and Digital Filters**)
- § 12.1 Frequency Response of Discrete-Time Systems
- § 12.3 Digital Filters
- § 12.4 Filter Design Criteria
- § 12.7 Nonrecursive Filters

Next Time

ELEC3004 is

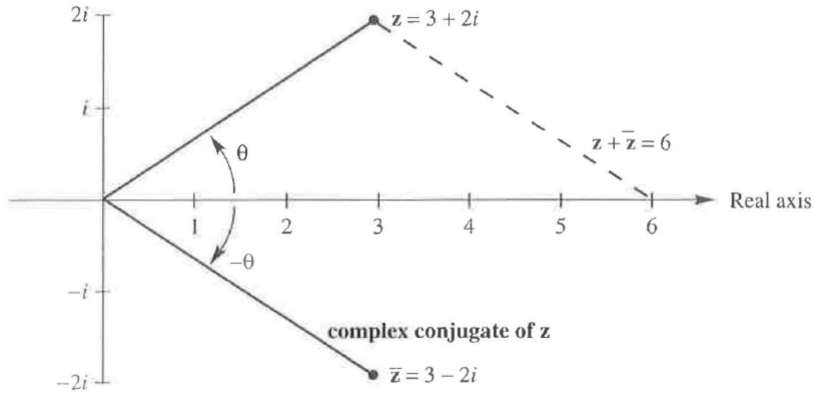
$$-e^{\pi i}$$

The Complex Plane Properties



- z=(a + bi)
- $z + \bar{z} = 2a$
- $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$

The Complex Plane Properties



- z=(a + bi) is also
- $z = r\cos\theta + ir\sin\theta$

The nth power of $z = r(\cos\theta + i\sin\theta)$ *is* $z^n = r^n(\cos n\theta + i\sin n\theta)$.

The Complex Plane Properties

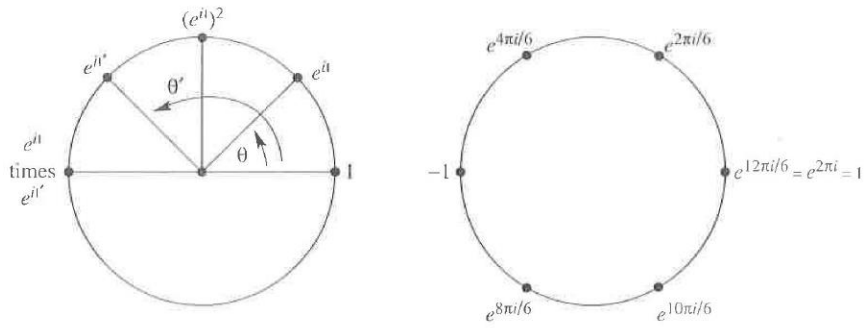


Figure 10.3 (a) Multiplying $e^{i\theta}$ times $e^{i\theta'}$. (b) The nth power of $e^{2\pi i/n}$ is $e^{2\pi i} = 1$.

DTFT

The Fourier Transform

- The continuous-time Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\exp(-j\omega t)dt$$

- What happens if we sample $x(t)|_{t=n\Delta t} = x_c(t)$?
- Represent $x_c(t)$ as sum of weighted impulses

$$x_c(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\delta(t-n\Delta t)$$

$$X_c(\omega) = \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} x(n\Delta t)\delta(t-n\Delta t)\right]\exp(-j\omega t)dt$$

Discrete-time Fourier Transform

- Changing order of integration & summation
  - and the simplifying (multiplication by impulse) gives

$$X_c(\omega) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\left[\int_{-\infty}^{\infty}\delta(t-n\Delta t)\exp(-j\omega t)dt\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n\Delta t)\exp(-j\omega n\Delta t)$$

- This is known as the DTFT
  - Requires an infinite number of samples $x(n\Delta t)$
  - discrete in time
  - continuous and periodic in frequency

DTFT of Finite Data Samples

- Assume only N samples of $x(n\Delta t)$
  - from $n = \{0, N-1\}$
- Therefore, can only approximate $X_c(w)$

$$\hat{X}_c(\omega) = \sum_{n=0}^{N-1} x(n\Delta t)\exp(-j\omega n\Delta t)$$

- How good an estimate is this?
  - Finite samples are same as infinite sequence multiplied by a rectangular time domain 'window'

$$\hat{x}(n\Delta t) = x(n\Delta t)\cdot\Pi\left(\frac{t}{T}\right), \qquad \text{where } T = N\Delta t$$

Where rect(t) = $\Pi(t) = u\left(t+\frac{T}{2}\right) - u\left(t-\frac{T}{2}\right)$

Window Effects

- Multiplication in time with rectangular window
- Equivalent to convolution in frequency
  - with 'sinc' function

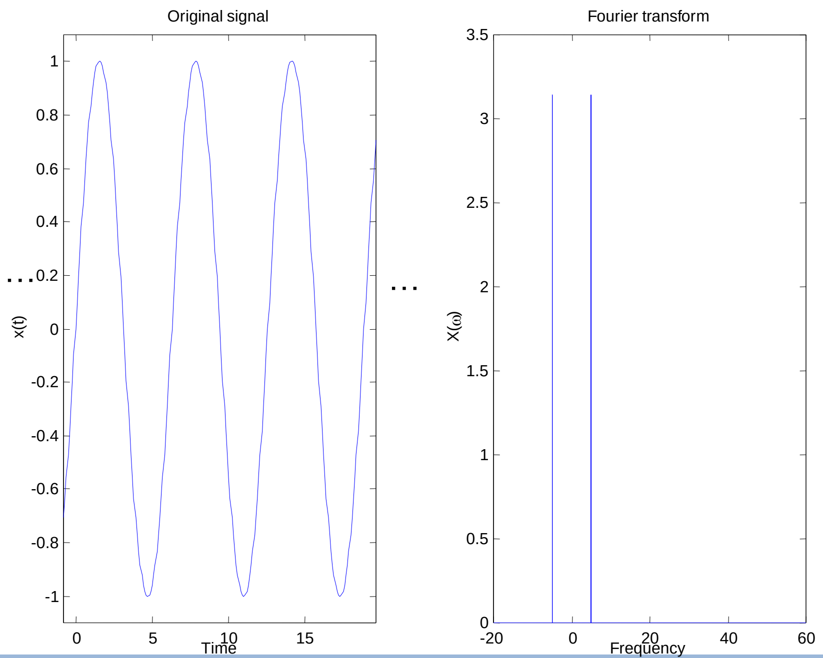
$$\hat{X}_c(\omega) = \frac{1}{2\pi} X_c(\omega) * T\,\mathrm{sinc}\left(\frac{T\omega}{2\pi}\right)$$

- In general, with arbitrary window function

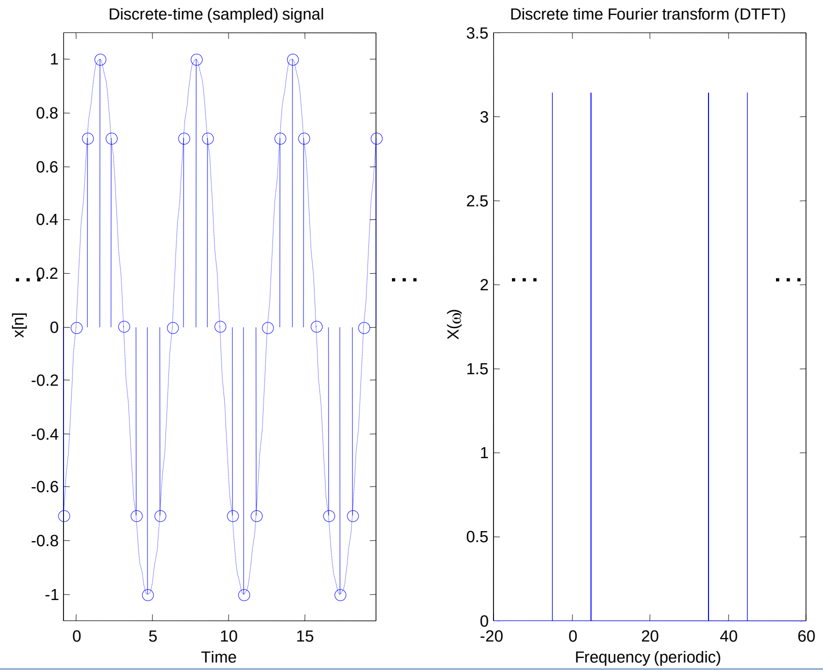
$$\hat{x}_c(t) = x_c(t)\cdot w_T(t)$$

$$\hat{X}_c(\omega) = \frac{1}{2\pi} X_c(\omega) * W_T(\omega)$$

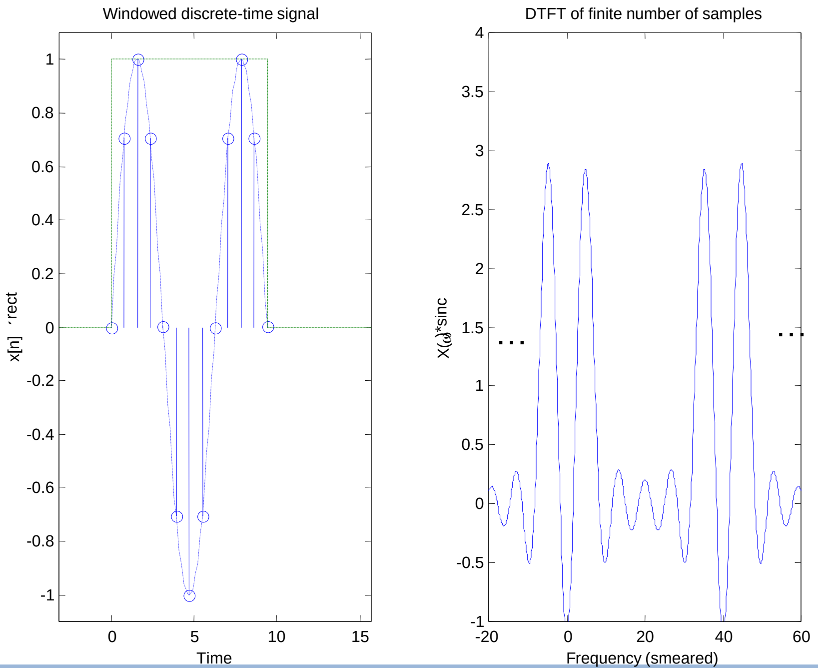
This is exactly same effect we saw in FIR filter design



Original signal
x(t)
Time
Fourier transform
X(ω)
Frequency

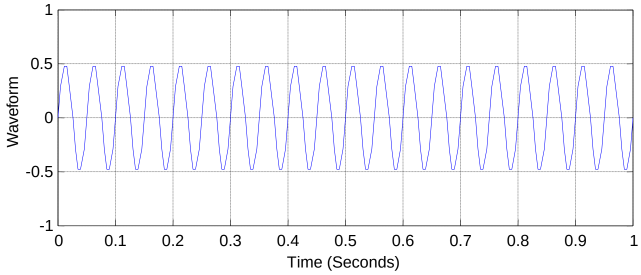


Discrete-time (sampled) signal
x[n]
Time
Discrete time Fourier transform (DTFT)
X(ω)
Frequency (periodic)

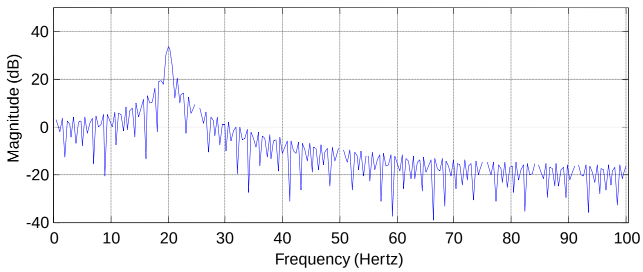


Reducing Window Effects

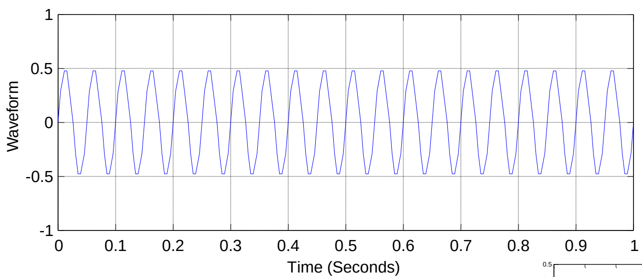
- We cannot avoid using a window function
  - as we must use a finite length of data
- Aim: to reduce window effect
  1. By choosing suitable window function
     - Hanning, Hamming, Blackman, Kaiser etc
  2. Increase number of samples (N)
     - reduces window effect (larger window)
     - increases resolution (No. samples)
     - Assumes signal is 'stationary' within sample window
       - Not true for most non-deterministic signals
       - e.g., speech, images etc



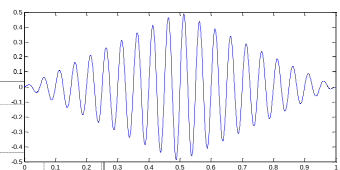
sinewave
frequency 20Hz
rectangular window
(1 second long)

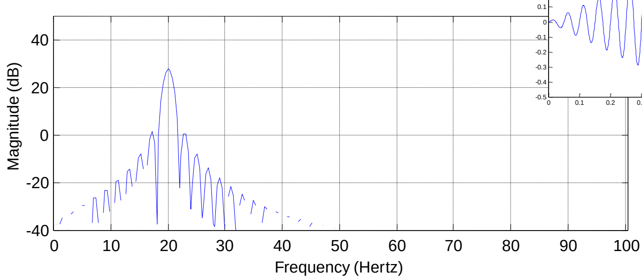


Note difficulty in detecting the single sinewave frequency present in time domain due to ‘smearing.’

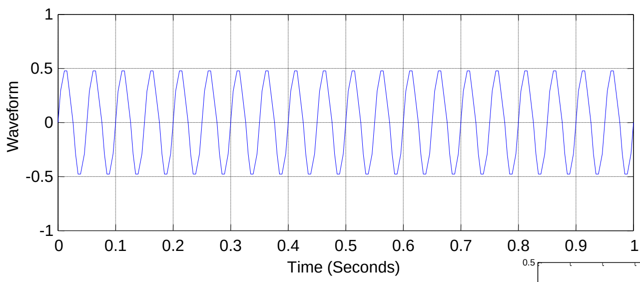


sinewave
frequency 20Hz
Triangular window
(AKA Bartlett)

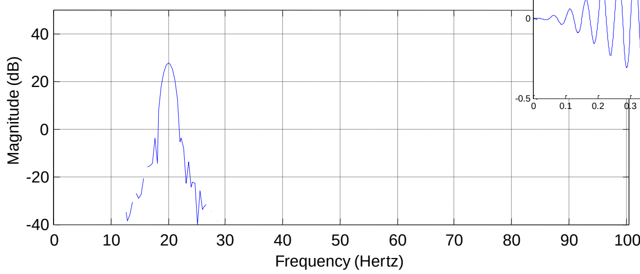


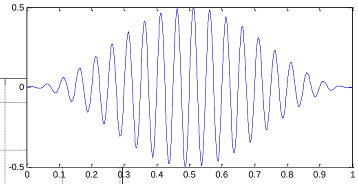


Wider main lobe
Increased side lobe attenuation



sinewave
frequency 20Hz
Hanning window





single sine wave
easier to detect

DTFT and the DFT

- Fourier transform, $\hat{X}_c(w)$, of sampled data is
  - continuous in frequency, range $\{0, w_s\}$
  - and periodic (w_s)
  - known as DTFT
- If calculating on digital computer
  - then only calculate $\hat{X}_c(w)$ at discrete frequencies
  - normally equally spaced over $\{0, w_s\}$
  - normally N samples, i.e., same as in time domain
  - i.e, samples Δw apart

Can reduce Δw by increasing N

$$\Delta\omega = \frac{2\pi}{N\Delta t}$$

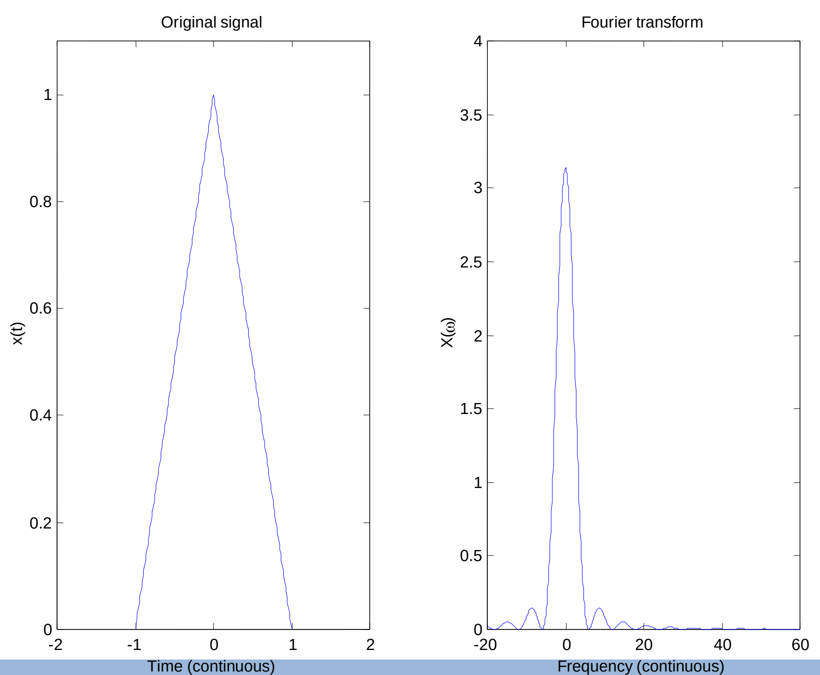
The DFT

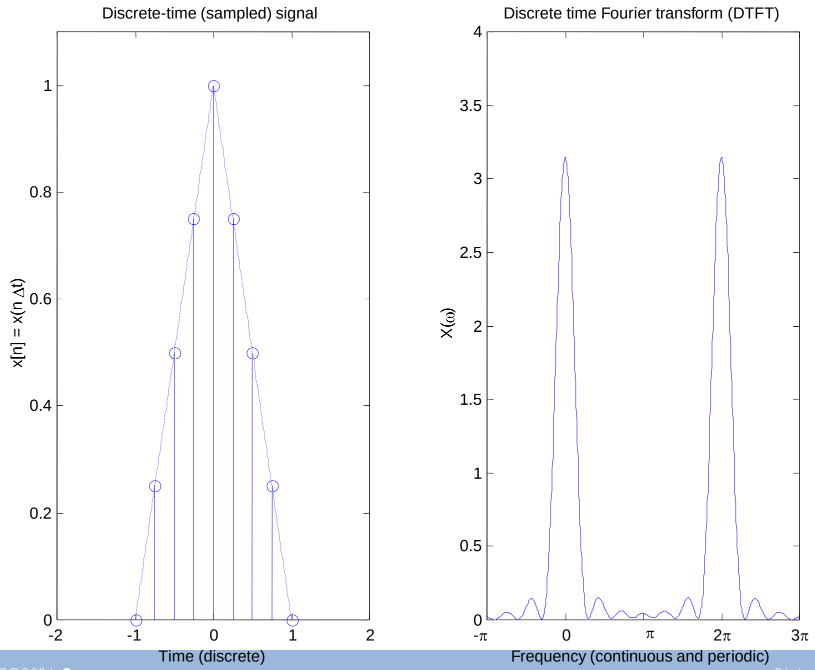
- Discrete Fourier Transform (DFT)
  - samples of DTFT, $X^{\wedge}_{c}(w)|_{w = k\,\Delta w}$

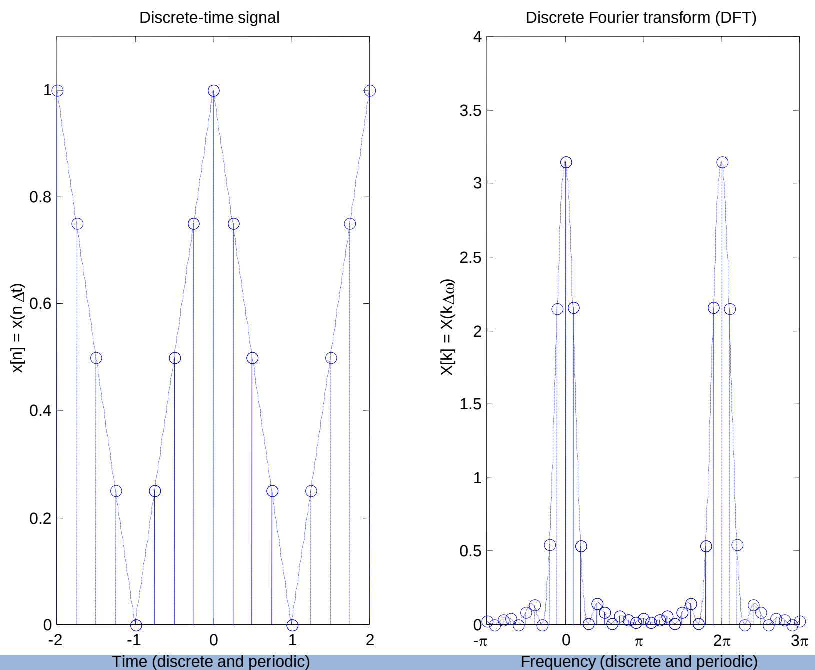
$$\hat{X}(k\Delta\omega) = X[k] = \sum_{n=0}^{N-1} x[n]\exp\left(\frac{-jnk2\pi}{N}\right)$$

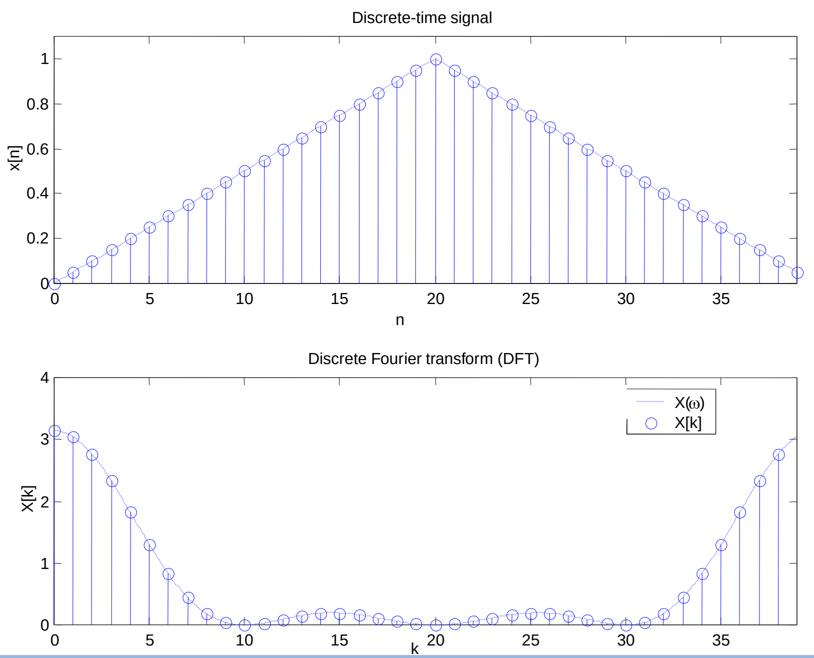
where $0 \le n, k \le N-1$

- Interpretation:
  - N equally spaced samples of $x(t)|_{t = n\,\Delta t}$
  - Calculates N equally spaced samples of $X(w)|_{w = k\Delta w}$
  - k often referred to a frequency 'bin': $X[k] = X(w_k)$




Discrete-time (sampled) signal
x[n] = x(n Δt)
Time (discrete)
Discrete time Fourier transform (DTFT)
X(ω)
Frequency (continuous and periodic)
-π
0
π
2π
3π


Discrete-time signal
x[n] = x(n Δt)
Time (discrete and periodic)
Discrete Fourier transform (DFT)
X[k] = X(kΔω)
Frequency (discrete and periodic)
-π
0
π
2π
3π



Inverse DFT

- Relates frequency domain samples to
  - time domain samples

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\exp\left(\frac{jnk2\pi}{N}\right)$$

- Note, differences to forward DFT
  - 1/N scaling and sign change on exponential
  - DFT & IDFT implemented with same algorithm
    - i.e., Fast Fourier Transform (FFT)
- Require both DFT and IDFT to implement (fast)
  - convolution as multiplication in frequency domain

Note, 1/N scaling can be on DFT only OR as 1/sqrt(N) on both DFT and IDFT

Fourier Transforms

Transform	Time Domain	Frequency Domain
Fourier Series (FS)	Continuous & Periodic	Discrete
Fourier Transform (FT)	Continuous	Continuous
Discrete-time Fourier Transform (DTFT)	Discrete	Continuous & Periodic
Discrete Fourier Transform (DFT)	Discrete & Periodic	Discrete & Periodic

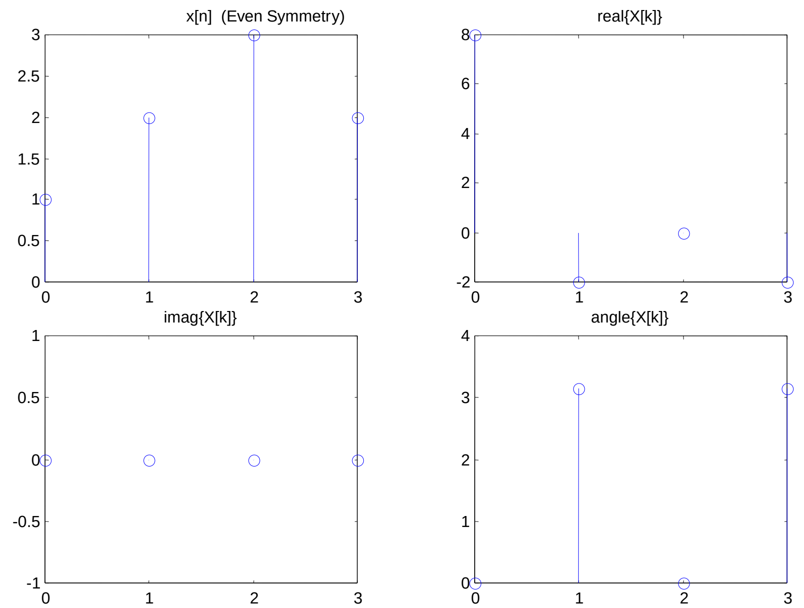
Properties of the DFT

if…

- $x[n]$ is real
- $x[n]$ is real and even
- $x[n]$ is real and odd

Then…

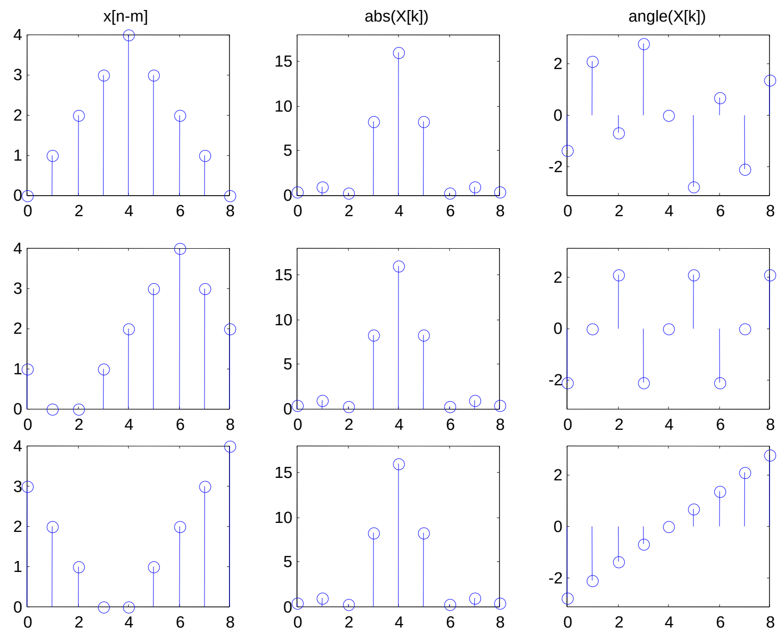
- $X[-k] = X[k]^*$
  - $\Re\{X[k]\}$ is even
  - $\Im\{X[k]\}$ is odd
  - $|X[k]|$ is even
  - $\angle X[k]$ is odd
- $X[k]$ is real and even
  - i.e., zero phase
- $X[k]$ is imaginary and odd



Note: DC at $k = 0$ & x[n] = x[n + N] & X[k] = X[k +N]

Properties of the DFT

- Periodic in frequency
  - period w_s i.e., the sampling frequency, or
  - period 2π (in normalised frequency)
- Repeats after N samples
  - $x[N + k] = X[k]$
- Mirror image (even) symmetry at $w_s/2$, i.e., π
  - $x[N - r] = X^*[r]$, where $r < N/2$
- Shift property
  - $x[n - m] = \exp(-jkm2\pi/N)\, X[k]$
  - i.e., $|X[k]|$ stays the same as input is shifted
  - only (phase) $\angle X[k]$ changes



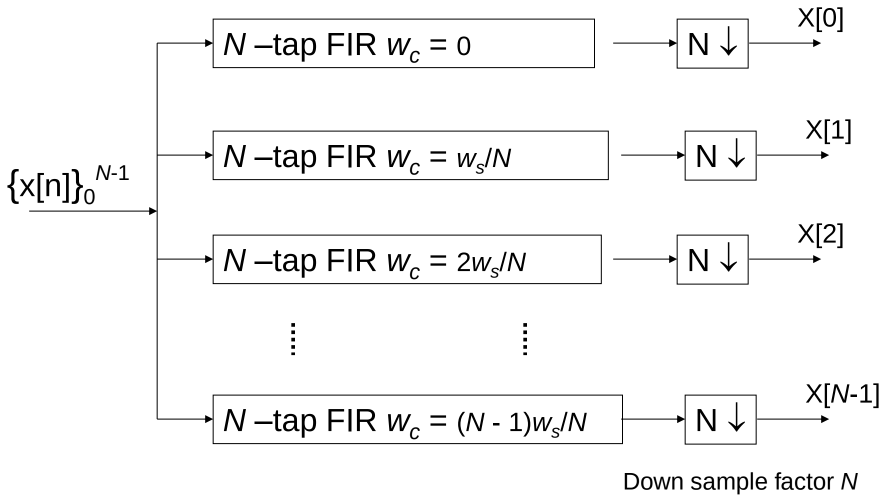
Note: all spectrums shifted to show $-w_s/2$ to $+w_s/2$, i.e., DC at $k = 4$

Analogies for the DFT

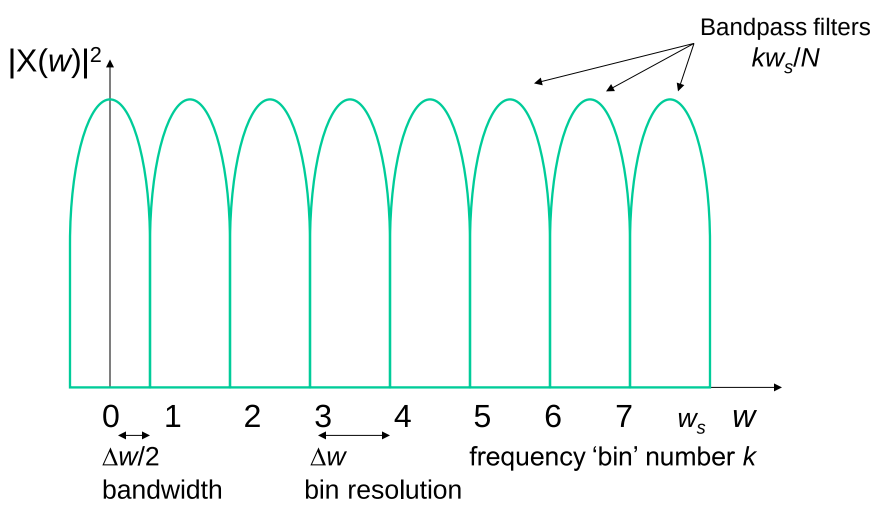
- Analogy for DFT is a **Filterbank**
  - Set of N FIR bandpass filters
  - with centre frequencies kw_s/N
    - k in range $\{0, N-1\}$
  - often called 'frequency bins'

- e.g., 8 point DFT
  - 8 bandpass filters (bins), spaced $\Delta w = w_s/8$ apart
  - Bandwidth of each filter $\Delta w/2$ therefore
    - output can be down-sampled by factor of 8
    - i.e., one sample, $x[k]$, per filter output (frequency bin)

Filterbank Analogy of DFT



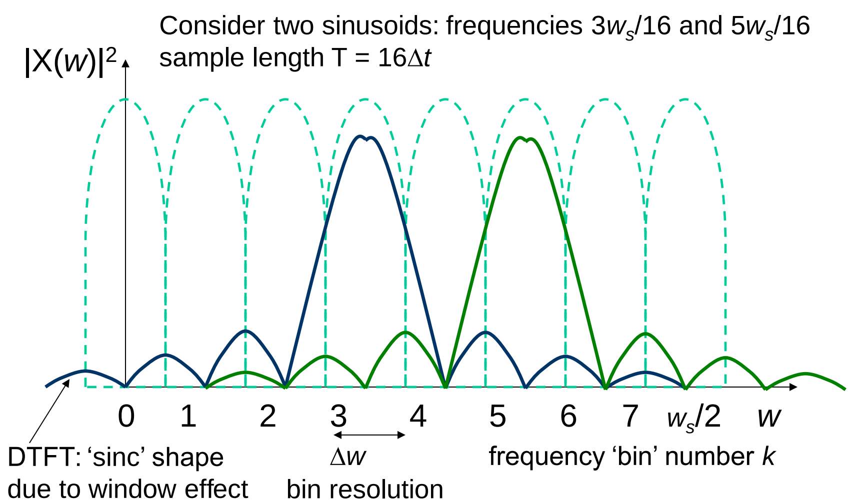
Filterbank Analogy of DFT



DFT Resolution

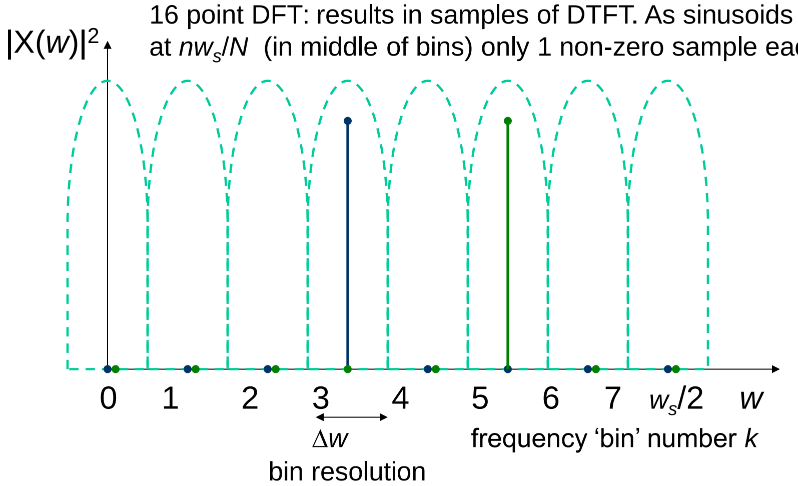
- Resolution is ability to distinguish
  - 2 (or more) closely spaced sinusoids
- <u>Minimum</u> resolution of DFT given by
  - $\Delta w = w_s/N = 2\pi/N\Delta t$
    - defined by sampling frequency, w_s
    - and number of samples, N
- Minimum resolution occurs when
  - integer number of complete cycles of input signal
  - in the N samples analysed
  - This is a 'best case' scenario
    - 'sinc' smearing always zero in adjacent frequency bins

DFT Resolution: Example



DFT Resolution: Example

16 point DFT: results in samples of DTFT. As sinusoids are at nw_s/N (in middle of bins) only 1 non-zero sample each.



Leakage Effects

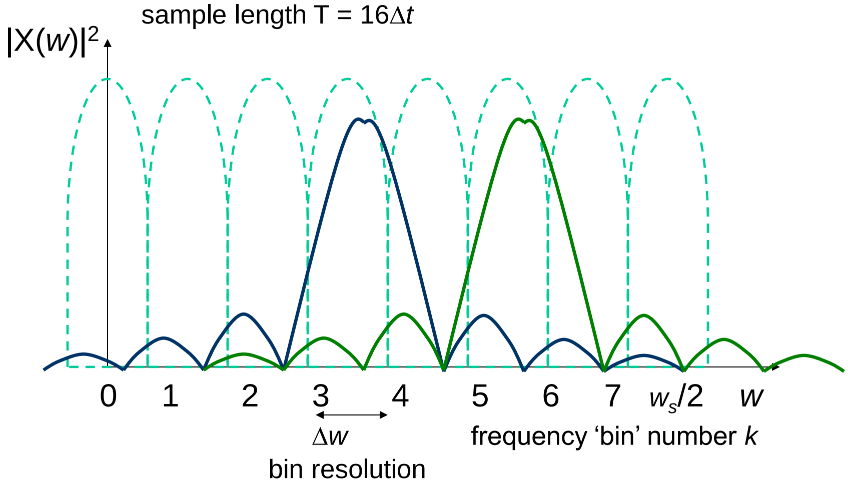
- In general, we can not capture
  - integer number of cycles of input
    - i.e., input will not be at bin frequencies nw_s/N
  - therefore, actual DFT resolution < Δw

- This is due to energy 'leakage'
  - between adjacent frequency bins

- Leakage due to finite data length
  - i.e., the 'window' effect
  - which 'smears' $X(w)$ -> $X[k]$
  - aim: to minimise window effect
    - using other than rectangular window

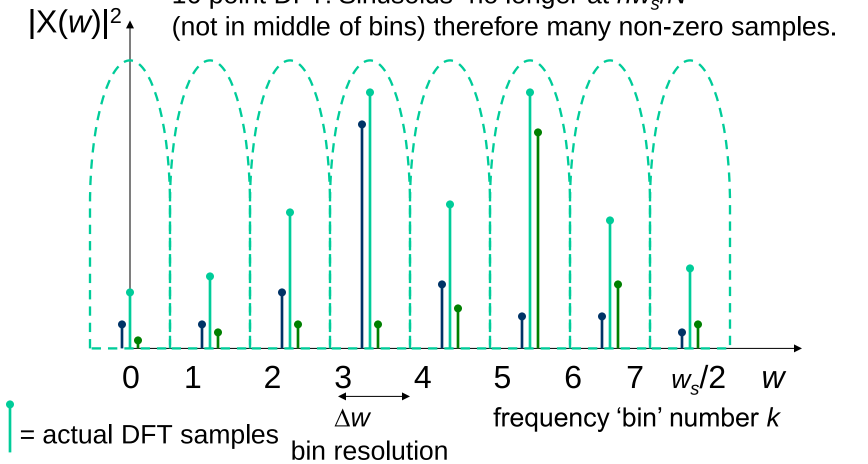
DFT Resolution: Example

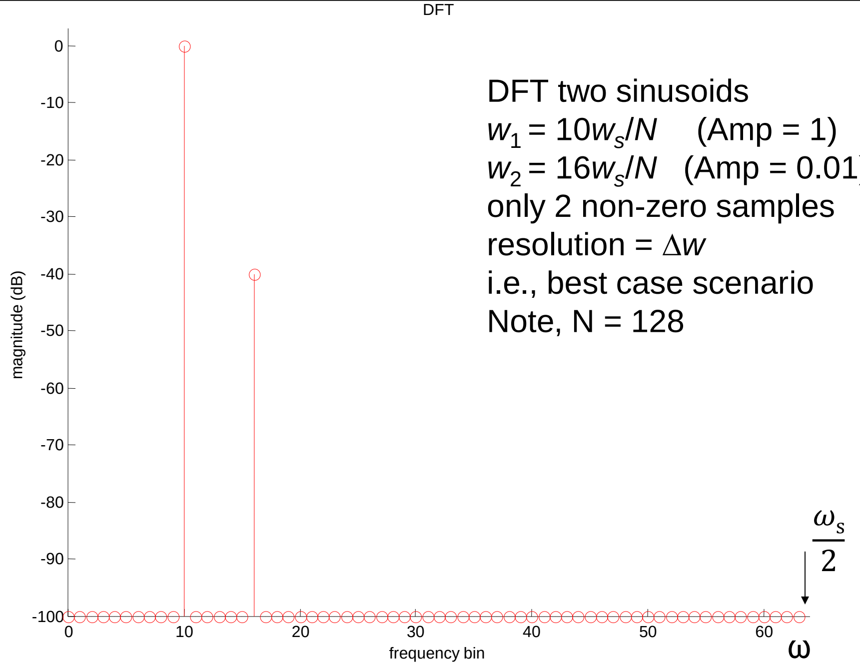
Consider two sinusoids: frequencies $3.1w_s/16$ and $5.1w_s/16$
sample length $T = 16\Delta t$



DFT Resolution: Example

16 point DFT: Sinusoids no longer at nw_s/N
(not in middle of bins) therefore many non-zero samples.





DFT two sinusoids

$w_1 = 10w_s/N$ (Amp = 1)

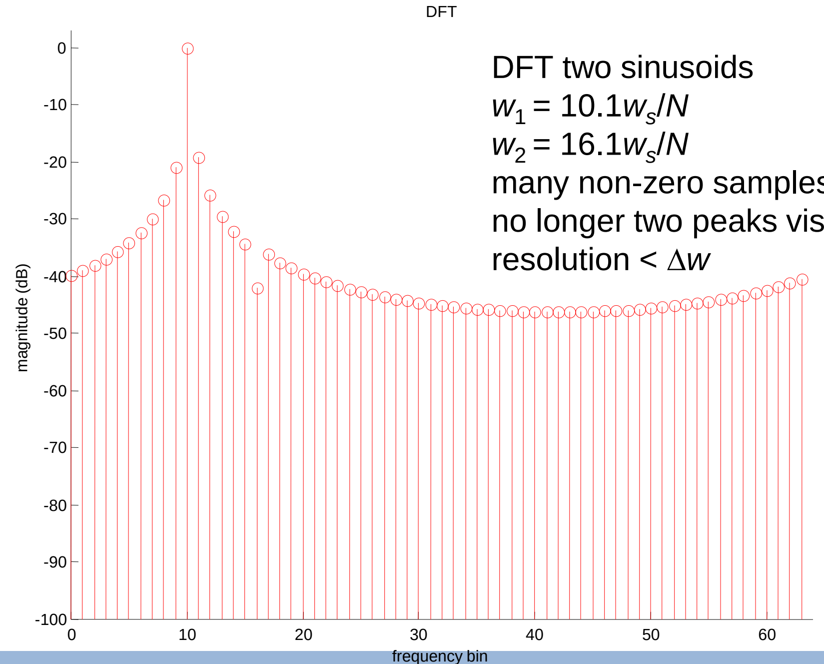
$w_2 = 16w_s/N$ (Amp = 0.01)

only 2 non-zero samples

resolution = Δw

i.e., best case scenario

Note, N = 128



DFT two sinusoids

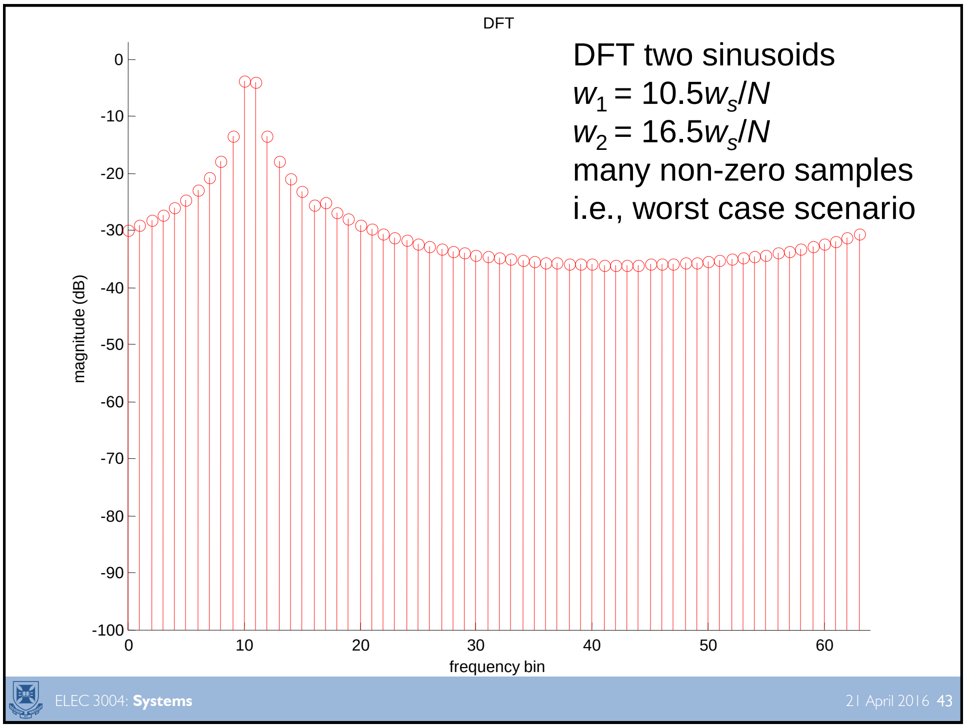
$w_1 = 10.1w_s/N$

$w_2 = 16.1w_s/N$

many non-zero samples

no longer two peaks visible

resolution < Δw


DFT
DFT two sinusoids
$w_1 = 10.5w_s/N$
$w_2 = 16.5w_s/N$
many non-zero samples
i.e., worst case scenario
magnitude (dB)
frequency bin
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
0
10
20
30
40
50
60
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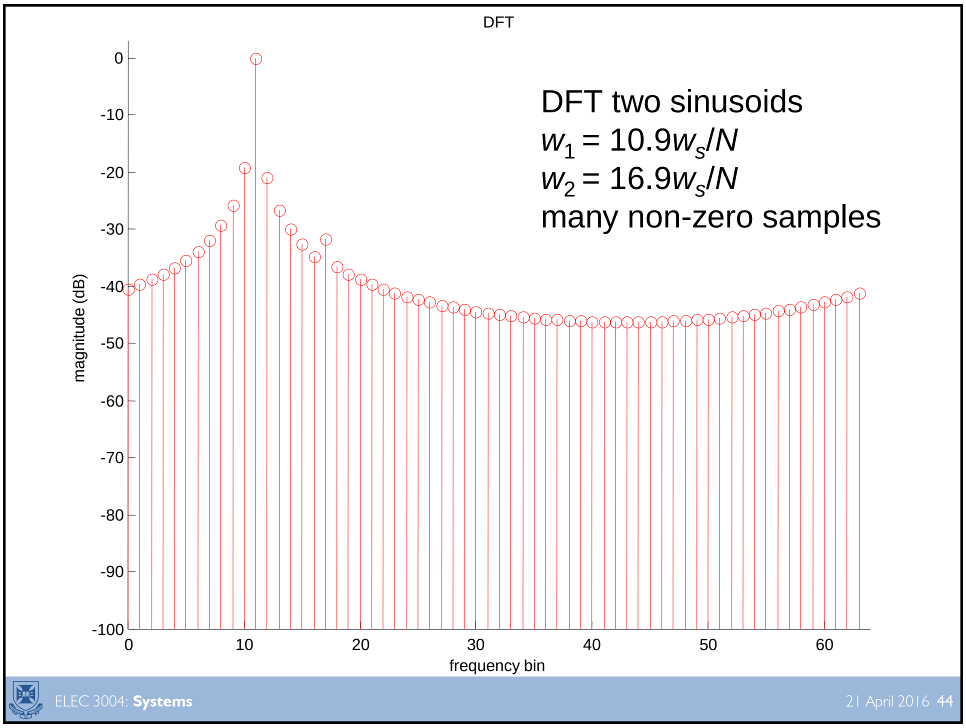
DFT
DFT two sinusoids
$w_1 = 10.9w_s/N$
$w_2 = 16.9w_s/N$
many non-zero samples
magnitude (dB)
frequency bin
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
0
10
20
30
40
50
60
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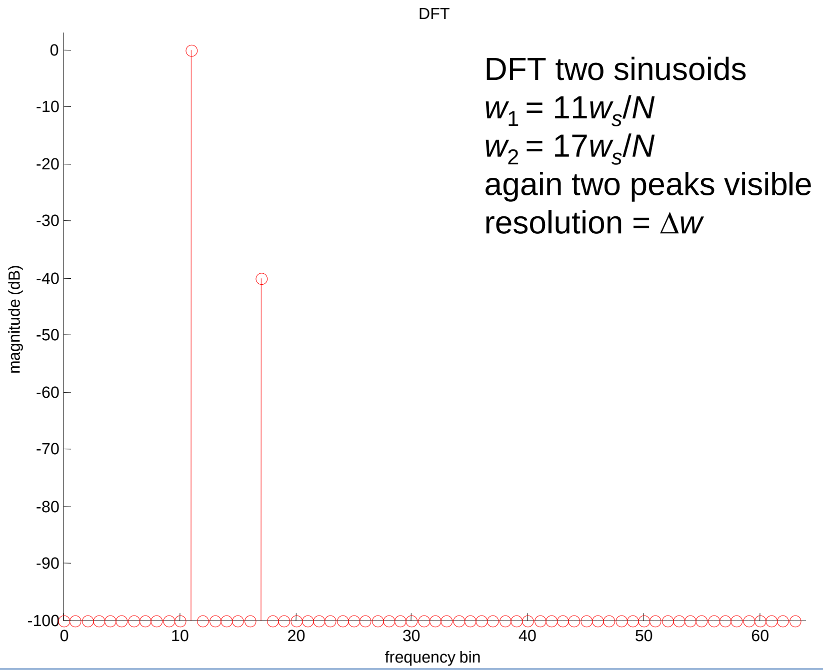
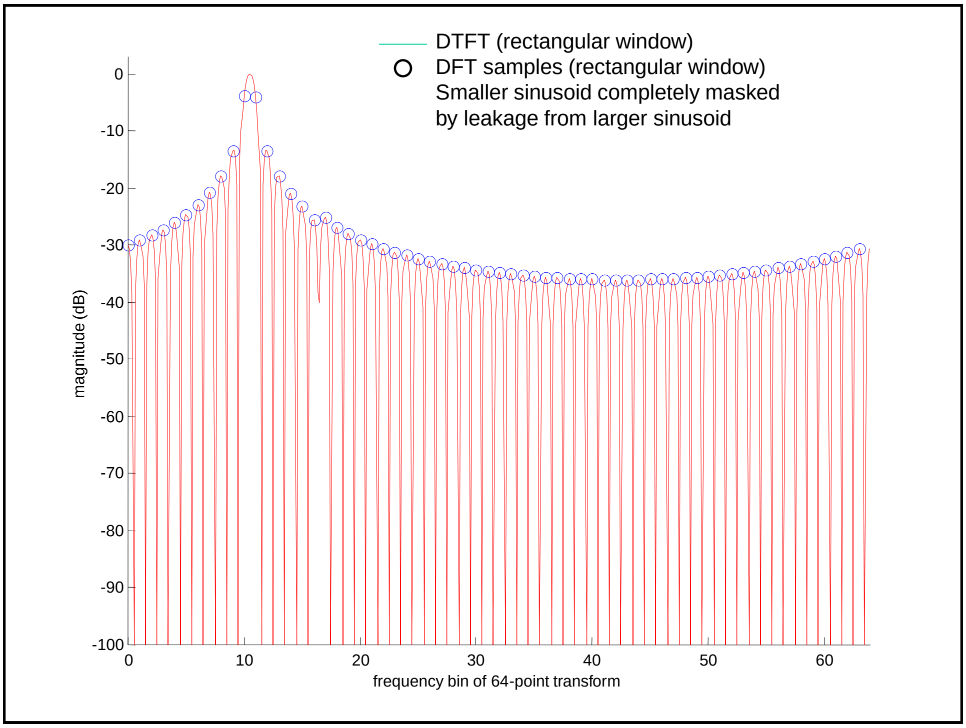


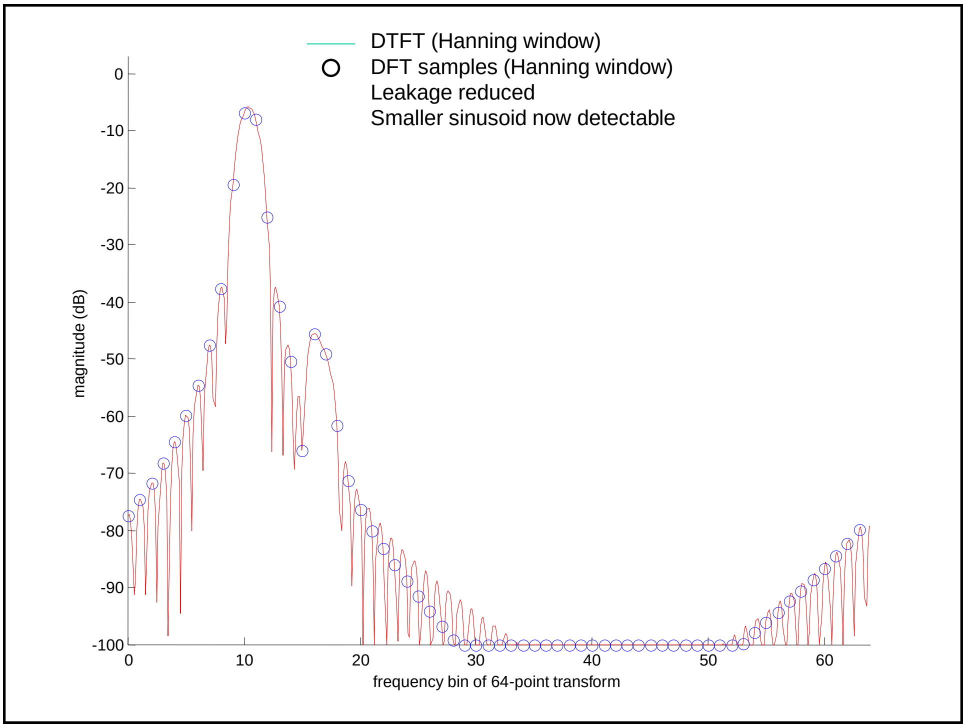
Reducing Leakage with Window Functions: Example

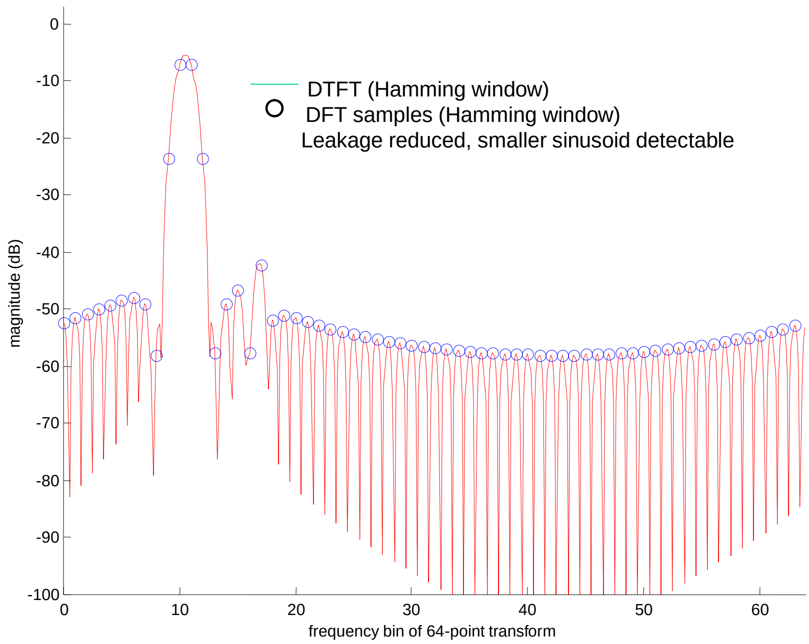
- Consider, two sinusoids,
  1. $\sin(10.5w_s/N)$: amplitude 1
  2. $0.01 \sin(16.5w_s/N)$: amplitude 0.01
     - i.e., significantly smaller (-40dB)

- This produces worst case leakage as
  - both sinusoids fall at edge of frequency bins
  - leakage due to large sinusoid > amplitude of smaller sinusoid (will be ‘masked’)

- Leakage can be reduced by using
  - non-rectangular window (Hanning/Hamming)
  - as used in FIR filter design


DTFT (rectangular window)
DFT samples (rectangular window)
Smaller sinusoid completely masked
by leakage from larger sinusoid
magnitude (dB)
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
0
10
20
30
40
50
60
frequency bin of 64-point transform


DTFT (Hanning window)
DFT samples (Hanning window)
Leakage reduced
Smaller sinusoid now detectable
magnitude (dB)
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
0
10
20
30
40
50
60
frequency bin of 64-point transform



Window Functions

Window	-3dB bandwidth	Loss (dB)	Peak sidelobe (dB)	Sidelobe roll off (dB/octave)
Rectangular	$0.89/N\Delta t$	0	-13	-6
Hanning	$1.4/N\Delta t$	4	-32	-18
Hamming	$1.3/N\Delta t$	2.7	-43	-6
Dolph-Chebyshev	$1.44/N\Delta t$	3.2	-60	0

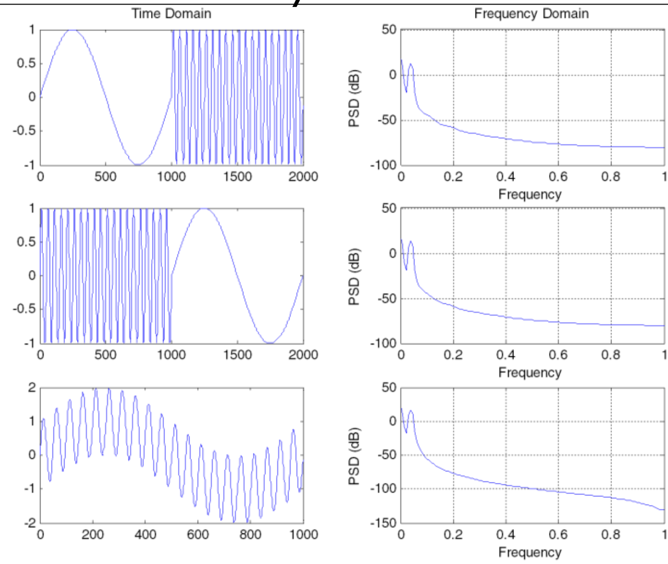
Note, trade-off between increased sidelobe attenuation
And increased 3dB (peak) bandwidth

Limitations of Fourier Analysis

Low – high

High – low

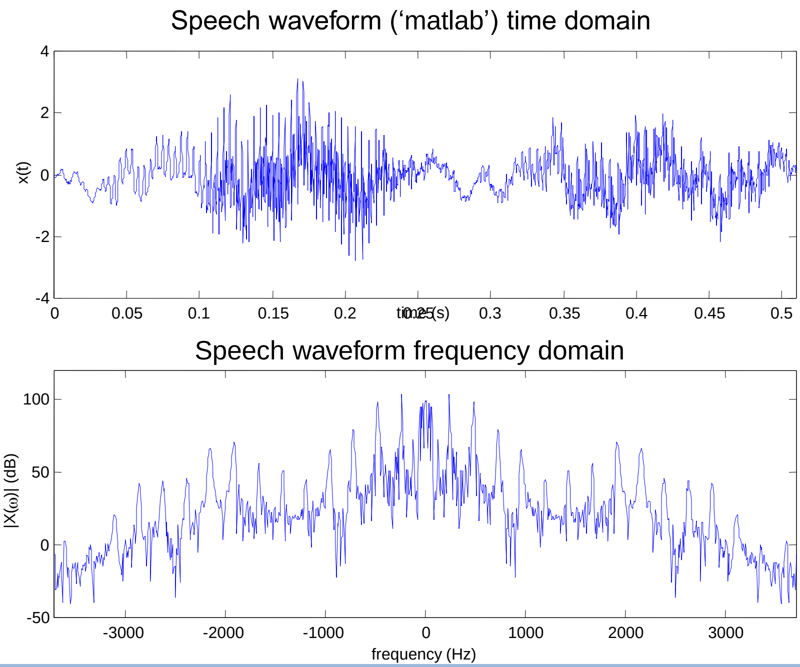
Mixed

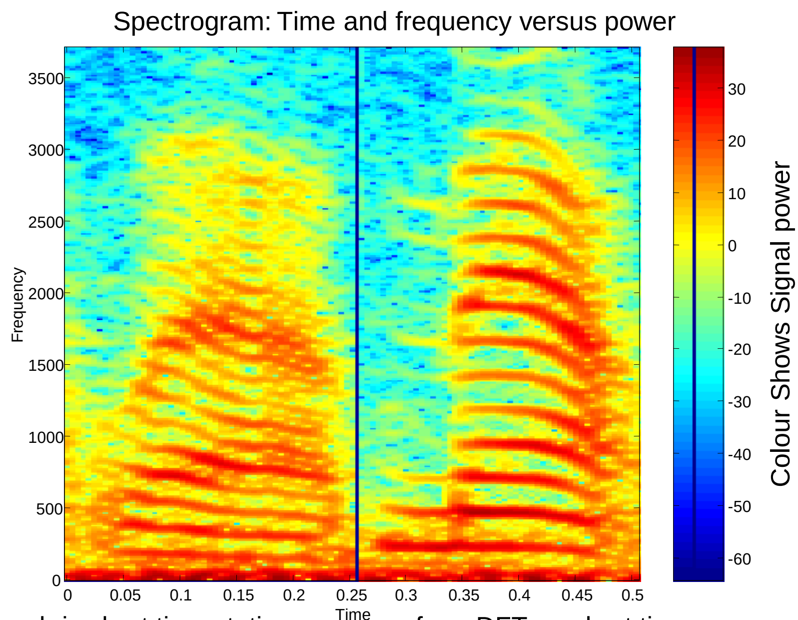


Note: These signals differ in Phase. PSD is zero phase as $F\{\phi_{xx}(k)\}$ real & even

Spectrum Analysis of Non-Stationary Signals

- Spectrum of non-deterministic Signal $X(w)$
  - is only valid if $x(t)$ is stationary
  - i.e., statistics of $x(t)$ do not change over time
- Real-world signals often only stationary over a short time period of time
  - e.g., speech: assumed stationary over t < 60ms
- Therefore, take 'short-time' DFT of signal
  - i.e., take multiple DFT's over stationary periods
  - plot how frequency components change over time
  - for speech the plot of time V frequency V power
    - is called a **Spectrogram**





Speech is short time stationary, so perform DFT on short time sequences

Summary

- FT of sampled data is known as
  - discrete-time Fourier transform (DTFT)
  - discrete in time
  - continuous & periodic in frequency
- DFT is sampled version of DTFT
  - discrete in both time and frequency
  - periodic in both time and frequency
    - due to sampling in both time and frequency
- DFT is implemented using the FFT
- Leakage reduced (dynamic range increased)
  - with non-rectangular window functions

DTFT Meets Linear Algebra

❤

2D DFT

$$\mathcal{F}(u,v) = \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux+vy)/N}$$

$$f(x,y) = \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \mathcal{F}(u,v)e^{j2\pi(ux+vy)/N}$$

2D DFT

- Each DFT coefficient is a complex value
  - There is a single DFT coefficient for each spatial sample
  - A complex value is expressed by two real values in either Cartesian or polar coordinate space.
    - Cartesian: R(u,v) is the *real* and I(u, v) the *imaginary* component
    - Polar: |F(u,v)| is the *magnitude* and phi(u,v) the *phase*

$$\mathcal{F}(u,v) = R(u,v) + jI(u,v)$$

$$\mathcal{F}(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

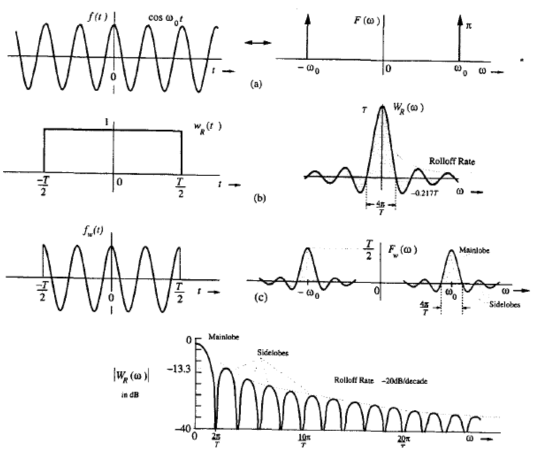
2D DFT

- Representing the DFT coefficients as magnitude and phase is a more useful for processing and reasoning.
  - The magnitude is a measure of strength or length
  - The phase is a direction and lies in [-pi, +pi]
- The magnitude and phase are easily obtained from the real and imaginary values

$$
\begin{aligned}
|\mathcal{F}(u,v)| &= \sqrt{R^2(u,v) + I^2(u,v)} \\
\phi(u,v) &= \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right].
\end{aligned}
$$

Windowing for the DFT

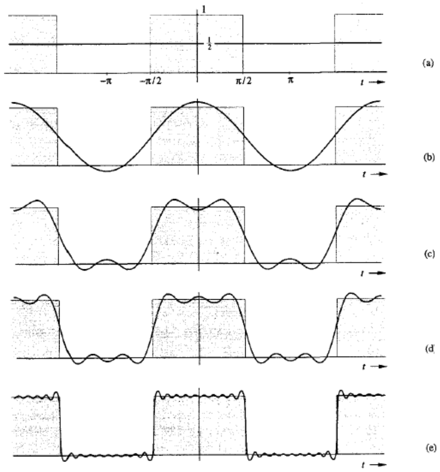
$$f_w(t) = f(t)w(t) \quad \text{and} \quad F_w(\omega) = \frac{1}{2\pi} F(\omega) * W(\omega)$$

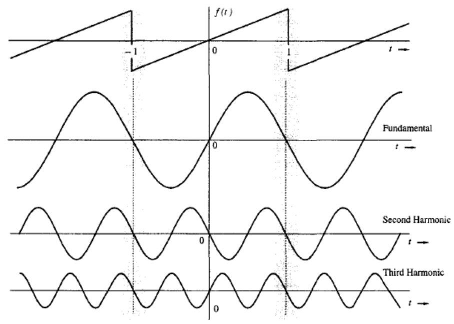


Source: Lathi, p.303

Harmonics

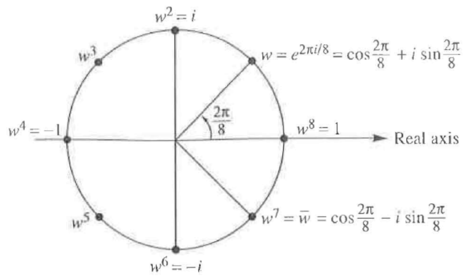
- Synthesis of a square pulse: periodic signal by successive addition of its harmonics (Lathi, p. 202-3)





FFT

Fourier Matrix



$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}.$$

$$Fc = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & w & w^2 & w^3 \\ 1 & w^2 & w^4 & w^6 \\ 1 & w^3 & w^6 & w^9 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_0 + c_1 + c_2 + c_3 \\ c_0 + c_1 w + c_2 w^2 + c_3 w^3 \\ c_0 + c_1 w^2 + c_2 w^4 + c_3 w^6 \\ c_0 + c_1 w^3 + c_2 w^6 + c_3 w^9 \end{bmatrix}$$

The DFT

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(\frac{-j2\pi nk}{N} \right)$$

- Sample number *n* where $0 \le n < N-1$
  - time 0 to $N\Delta t$
- Frequency sample (bin) number *k* where $0 \le k < N-1$
  - frequency 0 to ω_s ($\omega_s = 2\pi/\Delta t$)
- Discrete in both time x[*n*] and frequency X[*k*]
- Periodic in both time and frequency (due to sampling)
- Remember: $H(w) = H(z)|_{z = \exp(jw\Delta t)}$
  - i.e., DFT samples around unit circle in the z-plane

DFT in Matlab

```
function X = MyDFT(x)
% function X = MyDFT(x)
% Niave/direct implementation of the Discrete Fourier Transform (DFT)

% Calculate N samples of the DTFT, i.e., same a No. samples
N = length(x);
% Initialize (complex) X to zero
X = complex(zeros(size(x)),zeros(size(x)));
for n = 0:N-1
    for k = 0:N-1
        % Calculate each sample of DFT using each sample of input.
        % Note: Matlab indexes vectors from 1 to N,
        % whilst DFT is defined from from 0 to (N-1)
        X(k+1) = X(k+1) + x(n+1)*exp(-j*n*k*2*pi/N);
    end
end
```


Computational Complexity

- Each frequency sample X[k]
  - Requires N complex multiply accumulate (MAC) operations
- ∴ for N frequency samples
  - There are N 2 complex MAC
- e.g.,
  - 8-point DFT requires 64 MAC
  - 64-point DFT requires 4,096 MAC
  - 256-point DFT requires 65,536 MAC
  - 1024-point DFT requires 1,048,576 MAC
  - i.e., number of MACs gets very large, very quickly!

DFT Notation

$$X[k] = \sum_{n=0}^{N-1} x[n] \cdot W_N^{nk}$$

$$\text{where } W_N^{nk} = \exp\left(\frac{-j2\pi nk}{N}\right)$$

W_N^{nk} are called "N^{th} roots of unity"

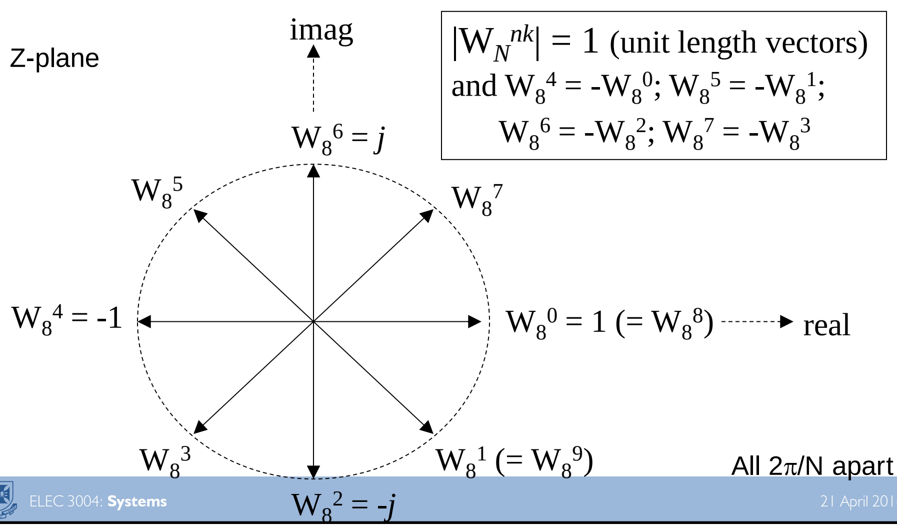
e.g., $N = 8$:

$W_8^0 = \exp(0) = 1$;

$W_8^1 = \exp(-j\pi/4) = \cos(\pi/4) - j\sin(\pi/4) = 0.7 - j0.7$;

$W_8^2 = -j$; $W_8^3 = -0.7 - j0.7$; $W_8^4 = -1$; etc

Nth Roots of Unity



DFT Expansion

$$X[k]=\sum_{n=0}^{N-1} x[n]\cdot W_N^{nk}$$

$$\begin{array}{lclclclcl}
X(k=0) & = & x(0) & + & x(1) & + & \dots & + & x(N-1) \\
X(k=1) & = & x(0) & + & x(1)W_N^{1} & + & \cdots & + & x(N-1)W_N^{N-1} \\
X(k=2) & = & x(0) & + & x(1)W_N^{2} & + & \cdots & + & x(N-1)W_N^{N-2} \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
X(k=N-1) & = & x(0) & + & x(1)W_N^{N-1} & + & \cdots & + & x(N-1)W_N^{1}
\end{array}$$

Remember $W_N^{0} = 1$

DFT Matrix Formulation

DFT expansion can also be written as a matrix operation:

$$\underbrace{\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ . \\ X(N-1) \end{bmatrix}}_{X[k]} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & . & 1 \\ 1 & W_N^{1} & W_N^{2} & . & W_N^{N-1} \\ 1 & W_N^{2} & W_N^{4} & . & W_N^{N-2} \\ . & . & . & . & . \\ 1 & W_N^{N-1} & W_N^{N-2} & . & W_N^{1} \end{bmatrix}}_{\text{DFT Matrix}} \cdot \underbrace{\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ x(N-1) \end{bmatrix}}_{x[n]}$$

Example: 8-point DFT Matrix

no rotation DC row 1 rotation 2 rotations etc

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\ W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$

Increasing rotational frequency down the rows of the DFT matrix

Example: 8-point DFT Matrix

x(0) x(2) x(4) x(6) Even samples

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^0 & W_8^2 & W_8^4 & W_8^6 \\ W_8^0 & W_8^3 & W_8^6 & W_8^1 & W_8^4 & W_8^7 & W_8^2 & W_8^5 \\ W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 & W_8^0 & W_8^4 \\ W_8^0 & W_8^5 & W_8^2 & W_8^7 & W_8^4 & W_8^1 & W_8^6 & W_8^3 \\ W_8^0 & W_8^6 & W_8^4 & W_8^2 & W_8^0 & W_8^6 & W_8^4 & W_8^2 \\ W_8^0 & W_8^7 & W_8^6 & W_8^5 & W_8^4 & W_8^3 & W_8^2 & W_8^1 \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix}$$

Repeated complex multiplications in EVEN rows

Re-ordered DFT Matrix

Separate even and odd row operations (and re-order input vector)

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ X(4)\\ X(5)\\ X(6)\\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^4 & W_8^2 & W_8^6 & W_8^1 & W_8^5 & W_8^3 & W_8^7 \\ W_8^0 & W_8^0 & W_8^4 & W_8^4 & W_8^2 & W_8^2 & W_8^6 & W_8^6 \\ W_8^0 & W_8^4 & W_8^6 & W_8^2 & W_8^3 & W_8^7 & W_8^1 & W_8^5 \\ W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^4 & W_8^4 & W_8^4 & W_8^4 \\ W_8^0 & W_8^4 & W_8^2 & W_8^6 & W_8^5 & W_8^1 & W_8^7 & W_8^3 \\ W_8^0 & W_8^0 & W_8^4 & W_8^4 & W_8^6 & W_8^6 & W_8^2 & W_8^2 \\ W_8^0 & W_8^4 & W_8^6 & W_8^2 & W_8^7 & W_8^3 & W_8^5 & W_8^1 \end{bmatrix} \cdot \begin{bmatrix} x(0)\\ x(4)\\ x(2)\\ x(6)\\ x(1)\\ x(5)\\ x(3)\\ x(7) \end{bmatrix}$$

Even samples (first four columns) Odd samples (last four columns)

Phasor Rotational Symmetry

To highlight repeated computations on odd samples
as $W_8^4 = -W_8^0$, $W_8^5 = -W_8^1$, $W_8^6 = -W_8^2$, $W_8^7 = -W_8^3$

$$\begin{bmatrix} X(0)\\ X(1)\\ X(2)\\ X(3)\\ X(4)\\ X(5)\\ X(6)\\ X(7) \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & -W_8^0 & W_8^2 & -W_8^2 & W_8^1 & -W_8^1 & W_8^3 & -W_8^3 \\ W_8^0 & W_8^0 & -W_8^0 & -W_8^0 & W_8^2 & W_8^2 & -W_8^2 & -W_8^2 \\ W_8^0 & -W_8^0 & -W_8^2 & W_8^2 & W_8^3 & -W_8^3 & W_8^1 & -W_8^1 \\ W_8^0 & W_8^0 & W_8^0 & W_8^0 & -W_8^0 & -W_8^0 & -W_8^0 & -W_8^0 \\ W_8^0 & -W_8^0 & W_8^2 & -W_8^2 & -W_8^1 & W_8^1 & -W_8^3 & W_8^3 \\ W_8^0 & W_8^0 & -W_8^0 & -W_8^0 & -W_8^2 & -W_8^2 & W_8^2 & W_8^2 \\ W_8^0 & -W_8^0 & -W_8^2 & W_8^2 & -W_8^3 & W_8^3 & -W_8^1 & W_8^1 \end{bmatrix} \cdot \begin{bmatrix} x(0)\\ x(4)\\ x(2)\\ x(6)\\ x(1)\\ x(5)\\ x(3)\\ x(7) \end{bmatrix}$$

Upper & lower left-hand quarters are identical
Right hand quarters identical except sign difference!

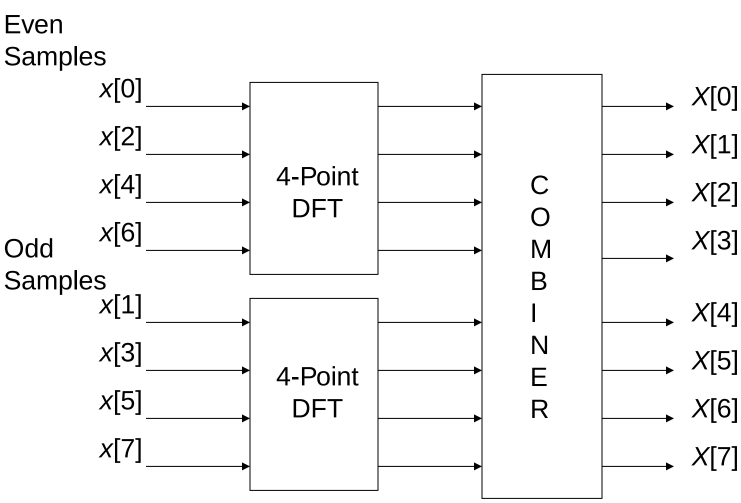
Adding "Twiddle Factors"

$$\begin{bmatrix}
W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \times W_8^0 & W_8^0 \times W_8^0 & W_8^0 \times W_8^0 & W_8^0 \times W_8^0 \\
W_8^0 & -W_8^0 & W_8^2 & -W_8^2 & W_8^1 \times W_8^0 & W_8^1 \times -W_8^0 & W_8^1 \times W_8^2 & W_8^1 \times -W_8^2 \\
W_8^0 & W_8^0 & -W_8^0 & -W_8^0 & W_8^2 \times W_8^0 & W_8^2 \times W_8^0 & W_8^2 \times -W_8^0 & W_8^2 \times -W_8^0 \\
W_8^0 & -W_8^0 & -W_8^2 & W_8^2 & W_8^3 \times W_8^0 & W_8^3 \times -W_8^0 & W_8^3 \times -W_8^2 & W_8^3 \times W_8^2 \\
W_8^0 & W_8^0 & W_8^0 & W_8^0 & -W_8^0 \times W_8^0 & -W_8^0 \times W_8^0 & -W_8^0 \times W_8^0 & -W_8^0 \times W_8^0 \\
W_8^0 & -W_8^0 & W_8^2 & -W_8^2 & -W_8^1 \times W_8^0 & -W_8^1 \times -W_8^0 & -W_8^1 \times W_8^2 & -W_8^1 \times -W_8^2 \\
W_8^0 & W_8^0 & -W_8^0 & -W_8^0 & -W_8^2 \times W_8^0 & -W_8^2 \times W_8^0 & -W_8^2 \times -W_8^0 & -W_8^2 \times -W_8^0 \\
W_8^0 & -W_8^0 & -W_8^2 & W_8^2 & -W_8^3 \times W_8^0 & -W_8^3 \times -W_8^0 & -W_8^3 \times -W_8^2 & -W_8^3 \times W_8^2
\end{bmatrix}$$

i.e., 8-point DFT reduced to two 4-point DFT's only need calculate upper left and right quarters

Twiddle Factors make the left and right hand quarters identical

8-Point DFT as Two 4-Point DFTs



Combiner adds twiddle factors to data

Radix-2 FFT

Each 4-point DFT can be reduced to two 2-point DFT's

$$\begin{bmatrix} W^0 & W^0 & W^0 & W^0 \\ W^0 & -W^0 & W^2 & -W^2 \\ W^0 & W^0 & -W^0 & -W^0 \\ W^0 & -W^0 & -W^2 & W^2 \end{bmatrix} = \begin{bmatrix} W^0 & W^0 & W^0 \times W^0 & W^0 \times W^0 \\ W^0 & -W^0 & W^2 \times W^0 & W^2 \times -W^0 \\ W^0 & W^0 & -W^0 \times W^0 & -W^0 \times W^0 \\ W^0 & -W^0 & -W^2 \times W^0 & -W^2 \times -W^0 \end{bmatrix}$$

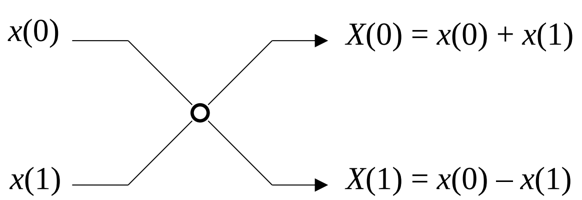
2x2 Quadrants are identical (with twiddle factors)

Two-point "Butterfly" operation

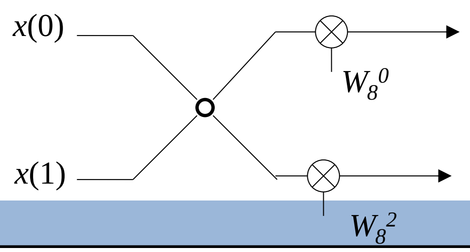
$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \begin{bmatrix} W^0 & W^0 \\ W^0 & -W^0 \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

$$\begin{bmatrix} X(0) \\ X(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} x(0) \\ x(1) \end{bmatrix}$$

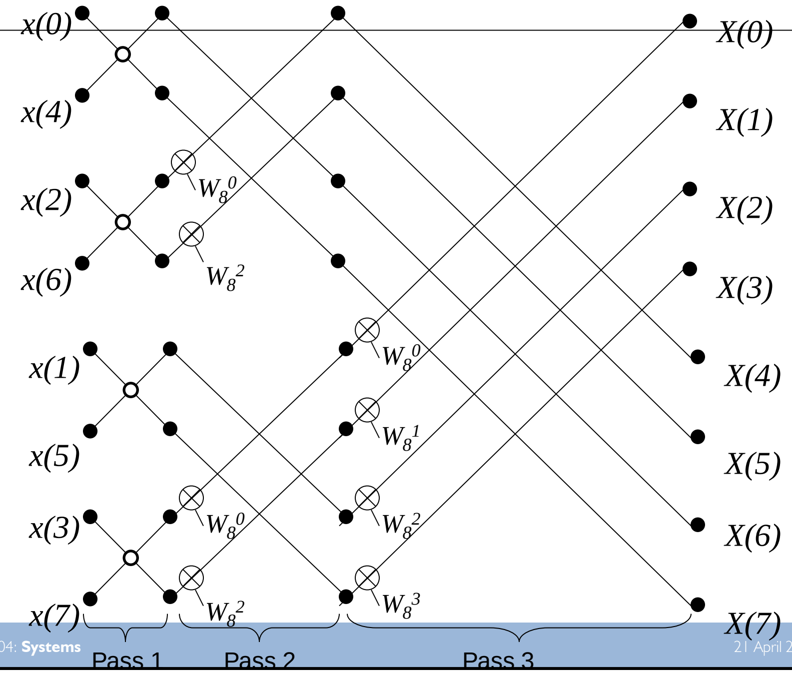
Two Point Butterfly



With twiddle factors:



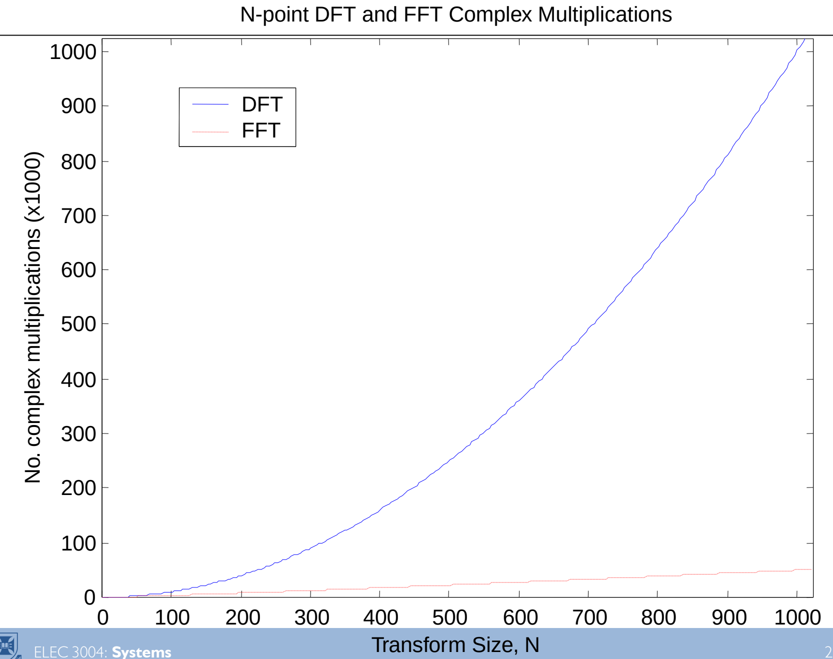
8-point radix-2 DIT FFT flowgraph:



Features of the FFT

- Reduce complex multiplications from N2 to:
  - (N/2)log2(N)
  - As there are log2(N) passes
  - Each pass requires N/2 complex multiplications
- Disadvantages
  - More complex memory addressing
    - To get appropriate samples pairs for each butterfly
  - FFT can be slower (than DFT) for small N (< 16)

Remember: $\log_2(N) = x$, where $N = 2^x$ & integer x



Alternative FFT Algorithms

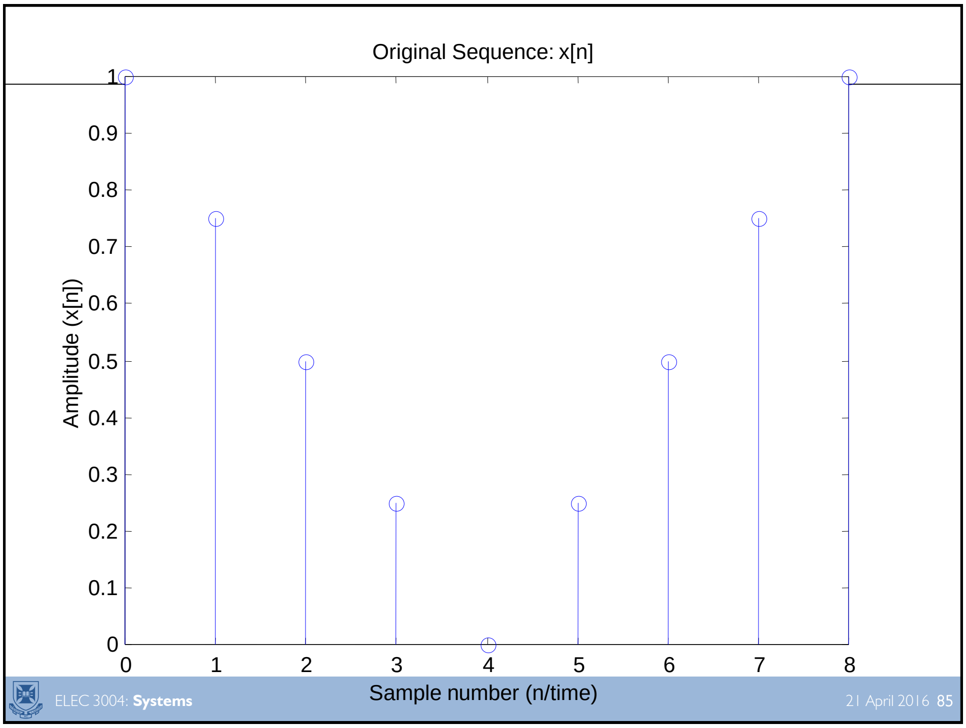
- Only case covered so far is
  - (one case of) radix-2 decimation in time (DIT) FFT
  - requires sequence length, N, to be a power of 2
  - achieved by ‘zero padding’ sequence to desired, N
- Decimation in Frequency
  - similar to DIT, twiddle factors on outputs
- Alternatives to radix-2 decomposition
  - Radix 3: for sequence length, N = power of 3
  - Radix 4: twice as fast as radix 2 FFT
    - half number of passes, log4(N)
  - Split radix: mixtures of the above

Inverse FFT

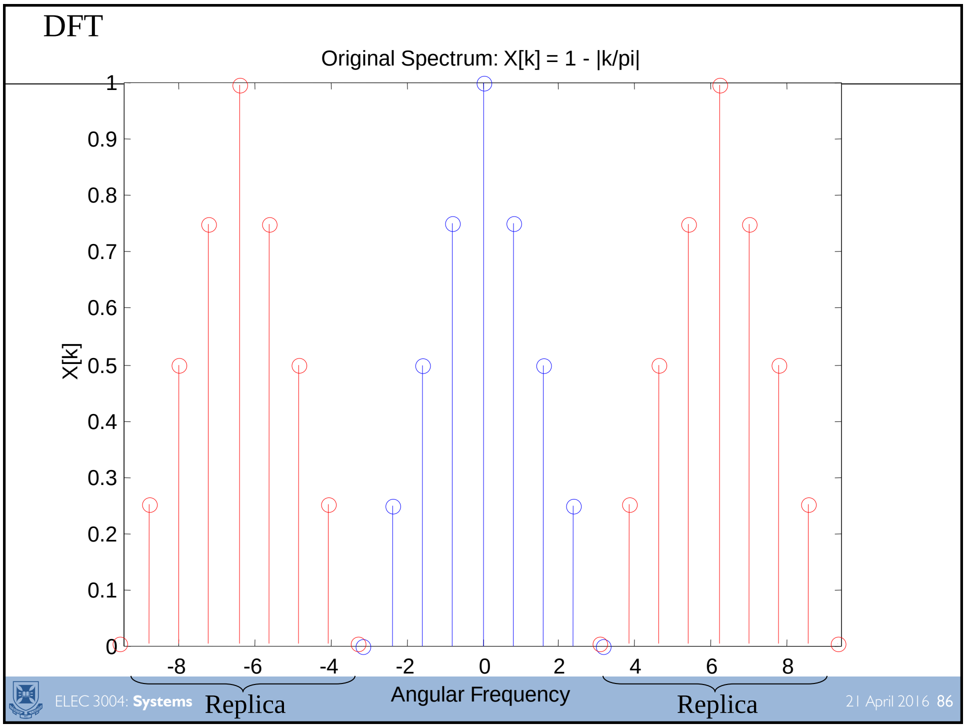
- IDFT obtained by
  - changing sign of WNnk
  - scaling by 1/N
- Therefore, we can use same FFT algorithm
  - change sign of twiddle factors
  - and scale output to get x[n]

Interpolation using the DFT

- DFT samples the DTFT
  - Normally N samples in both time & Frequency
  - But we can increase the (DFT) sample density!
  - By zero padding
- Zero Pad in time domain
  - Calculates additional samples of DTFT
- Zero Pad in frequency domain
  - Adds additional high frequency components (zero)
  - DFT zero padding ≡ sinc interpolation
    - Windowed by length, N, of DFT (not ideal sinc)

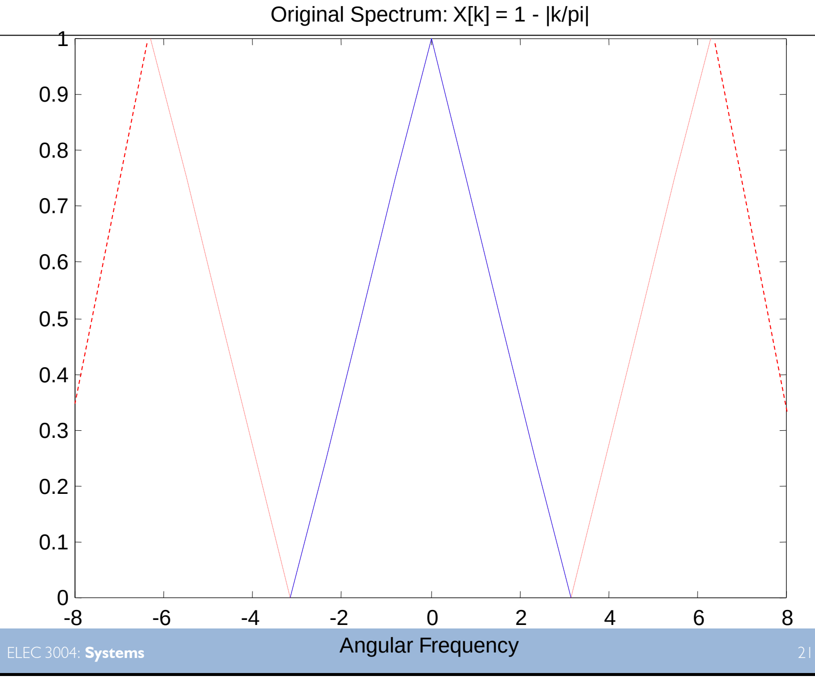


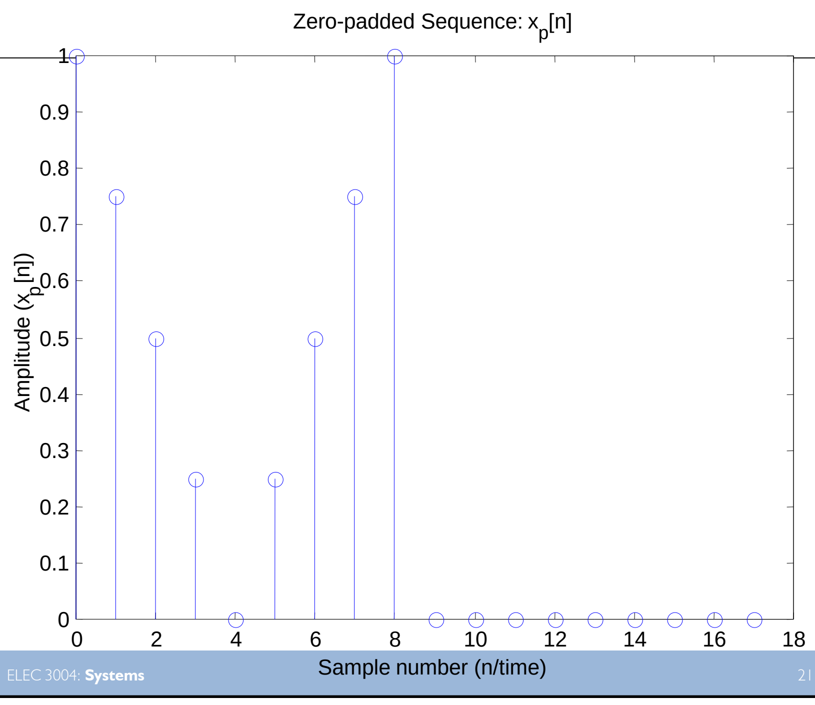
Original Sequence: x[n]
Amplitude (x[n])
Sample number (n/time)
0
1
2
3
4
5
6
7
8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
ELEC 3004: Systems
21 April 2016 85



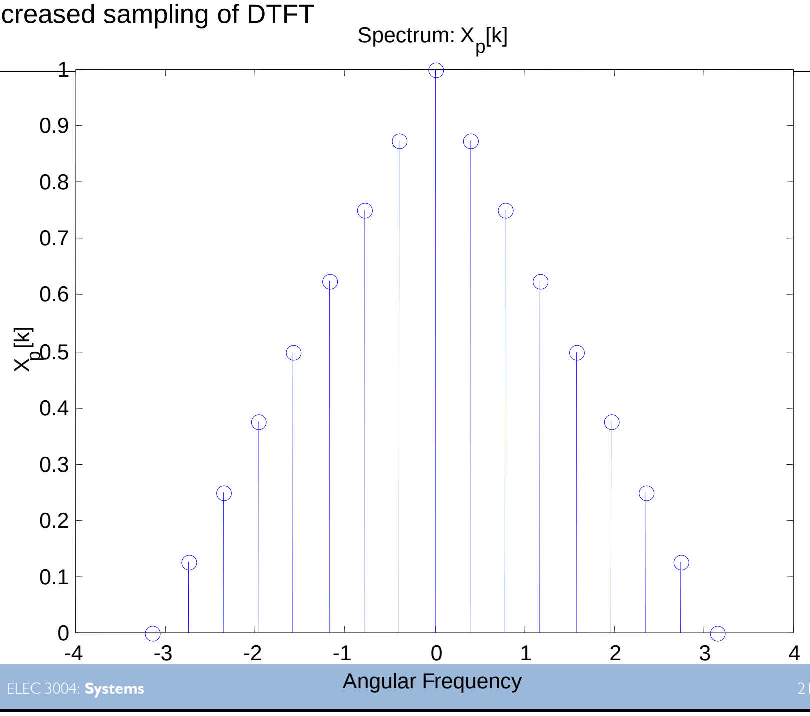
DFT
Original Spectrum: X[k] = 1 - |k/pi|
X[k]
Angular Frequency
-8
-6
-4
-2
0
2
4
6
8
Replica
Replica
ELEC 3004: Systems
21 April 2016 86

DTFT

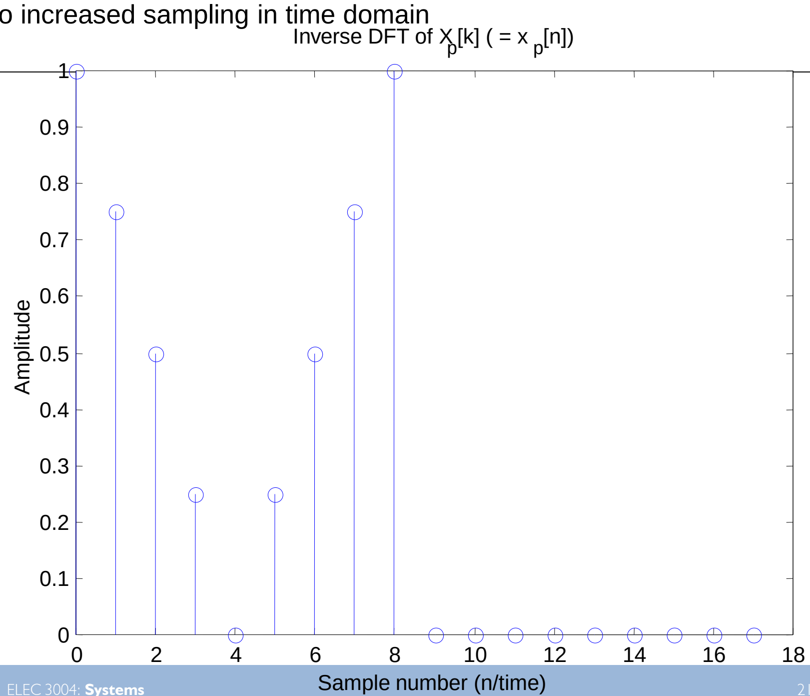

Original Spectrum: X[k] = 1 - |k/pi|
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-8
-6
-4
-2
0
2
4
6
8
Angular Frequency


Zero-padded Sequence: x_p[n]
Amplitude (x_p[n])
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14
16
18
Sample number (n/time)

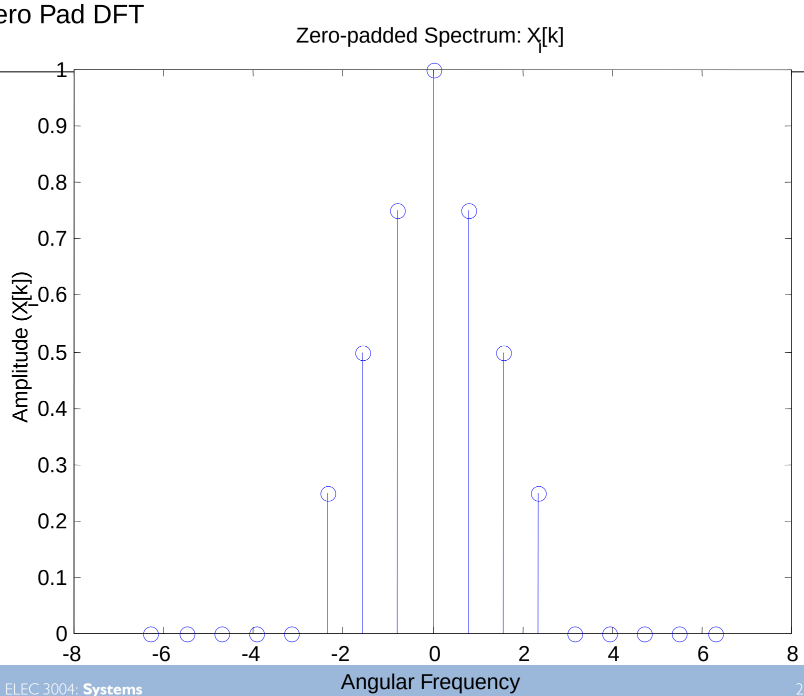
Increased sampling of DTFT



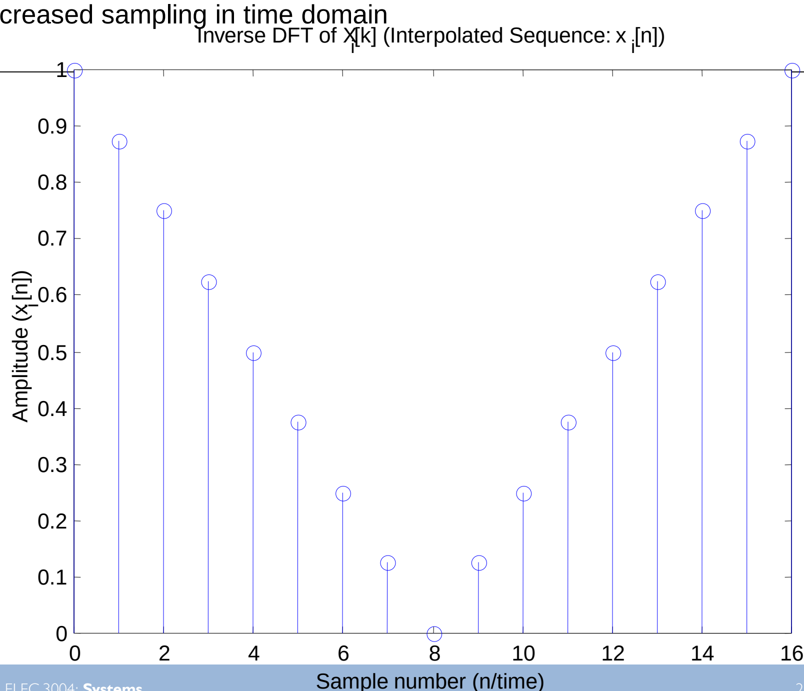
No increased sampling in time domain



Zero Pad DFT



Increased sampling in time domain



Interpolation via DFT (FFT)

- Interpolation of X[k]
  - zero pad sequence x[n]
    - either start or end of x[n] (or both)
  - increased sampling of DTFT spectrum, X(w)
- Interpolation of x[n]
  - zero pad discrete spectrum X[k]
    - evenly, both at start or end of the sequence
    - to ensure xu[n] remains real
    - i.e., pad to preserve symmetry of X[k]

Applications of the FFT

- Spectral Analysis
  - Estimate (power) spectrum with less computations
  - i.e., what frequencies in our signal are carrying power (i.e., carrying information) ?
- Fast (circular) Convolution
  - Convolution requires N 2 MAC operations ☹
  - more efficient alternative using the FFT ☺
    - Take FFT of both sequences
    - Multiply them together (point-wise)
    - Take IFFT to get the result
- Fast Cross-correlation
  - E.g., correlation detector in digital comm's

Spectral Analysis

- Power Spectral Density (PSD) defined as
  - Fourier Transform of Autocorrelation function

$$S_{xx}(w) = \sum_{m=-\infty}^{\infty} \phi_{xx}(m) \exp(-jwm\Delta t)$$

- In practice, we estimate $S_{xx}(w)$ from $\{x[n]\}_0^{N-1}$
  - i.e., a finite length of sampled data
- This can be done using N - point DFT
  - and implemented using the FFT algorithm

Spectral Analysis

- Estimate of PSD is given by

$$\hat{S}_{xx}[k] = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n] \exp\left(\frac{-jnk2\pi}{N}\right)\right|^2$$

- This is known as a **periodogram**
  - DFT effectively implements narrow-band filter bank
  - calculate power (i.e., square) at each frequency k
- Again, window functions often required
  - to improve PSD estimate
  - e.g., Hanning, Hamming, Bartlet etc

Spectral Analysis

- Alternatively, we can estimate PSD as
  - DFT (FFT) of the estimate of the autocorrelation

$$\hat{S}_{xx}[k] = \sum_{m=-M}^{M} \hat{\phi}_{xx}[m] \exp\left(\frac{-jmk2\pi}{2M+1} \right)$$

where $\hat{\phi}_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-1} x[n]x[n+m]$

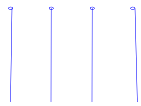
- Assuming $x[n]$ is ergodic (at least stationary)
- Normally restricted range of PSD
  - e.g., $0 < M < N/10$

Spectral Analysis

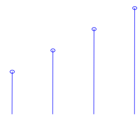
- Note,
  - when finding PSD as DFT of ϕ^xx[m]
  - ϕ^xx[m] has an odd length! (2M + 1)
- Therefore, to use the radix-2 FFT we need to
  - zero pad ϕ^xx[m] to length = power of 2
- e.g., for M = 2, ϕ^xx[m] is of length 5
  - we need to zero pad to length 8, i.e.,
  - {ϕ^xx[-2] ϕ^xx[-1] ϕ^xx[0] ϕ^xx[1] ϕ^xx[2] 0 0 0}
  - Note, sequence made causal (no change to PSD)
- This estimate of PSD is known as correlogram
  - Note, periodogram is most common estimate of PSD

(Linear) Convolution

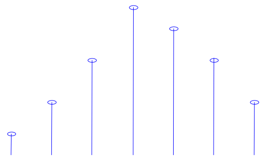
$h[n] = \{1\ 1\ 1\ 1\}$



$x[n] = \{0.5\ 0.75\ 1.0\ 1.25\}$



$y[n] = x[n]*h[n] = \{0.5\ 1.25\ \ 2.25\ \ 3.5\ \ 3.0\ \ 2.25\ \ 1.25\}$



In general: length($y[n]$) = length($x[n]$) + length($h[n]$) - 1

Circular Convolution

Given $X[k] = \text{DFT}\{x[n]\}$ and $H[k] = \text{DFT}\{h[n]\}$

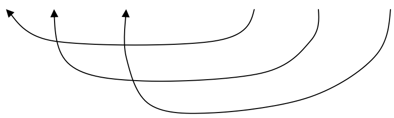
from convolution theorem we know
$\text{IDFT}\{X[k]\cdot H[k]\} \equiv x[n]*h[n]$

$\text{IDFT}\{X[k]\cdot H[k]\} = \{3.5\ 3.5\ 3.5\ 3.5\}$ ← Wrong Length!

Solution: zero pad both sequences to required length

$h_p[n] = \{1\ 1\ 1\ 1\ 0\ 0\ 0\ \}$ $x_p[n] = \{0.5\ 0.75\ 1.0\ 1.25\ 0\ 0\ 0\}$

$\text{IDFT}\{X_p[k]\cdot H_p[k]\} = [0.5\ 1.25\ \ 2.25\ \ 3.5\ \ 3.0\ \ 2.25\ \ 1.25]$

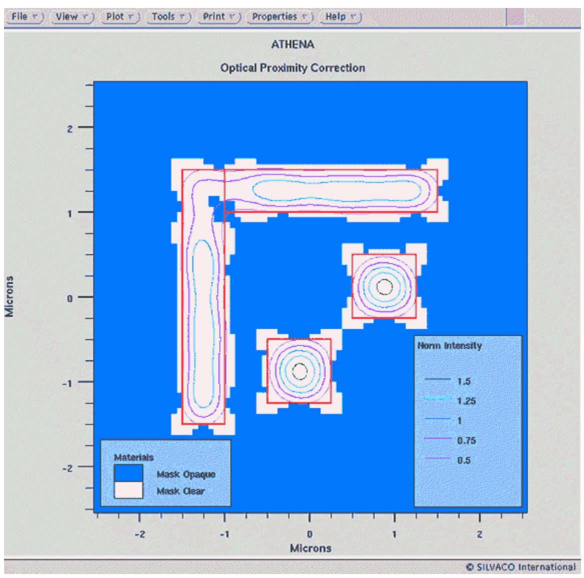


i.e., $x[n]$ and $h[n]$
are periodic in time

Summary

- FFT exploits symmetries in the DFT
  - Successively splits DFT in half
    - odd and even samples
  - Reduction to elementary butterfly operation
    - with ‘twiddle factors’
  - Reduce computations from N 2 to (N/2)log2(N) ☺
- FFT can be used to implement DFT for
  - PSD estimates (periodogram and correlogram)
  - Circular (fast) convolution (and correlation)
    - Requires zero padding to obtain “correct” answer

Fun Application: Optical Proximity Correction



Next Time…

- **Digital Filters**

- Review:
  - Chapter 12 of Lathi

- Ponder?

$$y[k] = f[k] * h[k] \qquad Y(\Omega) = F(\Omega)H(\Omega)$$

where $F(\Omega)$, $Y(\Omega)$, and $H(\Omega)$ are DTFTs of $f[k]$, $y[k]$, and $h[k]$, respectively; that is,

$$f[k] \Longleftrightarrow F(\Omega), \quad y[k] \Longleftrightarrow Y(\Omega), \quad \text{and} \quad h[k] \Longleftrightarrow H(\Omega)$$