Digital Windows &
Frequency Response of Digital Filters

ELEC 3004: **Systems**: Signals & Controls
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Lecture 13
(with material from Lathi)

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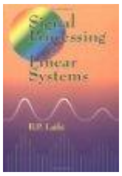
http://elec3004.com

Lecture Schedule:

Week	Date	Lecture Title
1	29-Feb	Introduction
	3-Mar	Systems Overview
2	7-Mar	Systems as Maps & Signals as Vectors
	10-Mar	Data Acquisition & Sampling
3	14-Mar	Sampling Theory
	17-Mar	Antialiasing Filters
4	21-Mar	Discrete System Analysis
	24-Mar	Convolution Review
	28-Mar	Holiday
	31-Mar	
5	4-Apr	Frequency Response & Filter Analysis
	7-Apr	Filters
6	11-Apr	Digital Filters
	14-Apr	Digital Filters
7	18-Apr	Digital Windows
	21-Apr	FFT
8	25-Apr	Holiday
	28-Apr	Feedback
9	3-May	Introduction to Feedback Control
	5-May	Servoregulation/PID
10	9-May	Introduction to (Digital) Control
	12-May	Digitial Control
11	16-May	Digital Control Design
	19-May	Stability
12	23-May	Digital Control Systems: Shaping the Dynamic Response & Estimation
	26-May	Applications in Industry
13	30-May	System Identification & Information Theory
	2-Jun	Summary and Course Review

Follow Along Reading:



B. P. Lathi
Signal processing and linear systems
1998
TK5102.9.L38 1998

Today

- Chapter 4
  - § 4.9 Data Truncation: Window Functions
- Chapter 12
  (**Frequency Response and Digital Filters**)
  - § 12.1 Frequency Response of Discrete-Time Systems
  - § 12.3 Digital Filters
  - § 12.4 Filter Design Criteria
  - § 12.7 Nonrecursive Filters

- Chapter 10
  (**Discrete-Time System Analysis Using the *z*-Transform**)
  - § 10.3 Properties of DTFT
  - § 10.5 Discrete-Time Linear System analysis by DTFT
  - § 10.7 Generalization of DTFT to the $\mathcal{Z}$–Transform
  - One of the days! ☺

Next Time

Announcement I

Google

Life at Google + Watch a Live Mock Interview

Learn about Google opportunities plus watch a Google engineer being interviewed live.

FREE pizza, Q&A and giveaways with Google Software Engineers!

RSVP: bit.ly/GoogleEvent2016

WHEN:
Tues 19th April 2016
5:30pm-8pm

WHERE:
Room 216, Building 42 (Prentice Building)
University of Queensland

google.com/students

BRING QUESTIONS BUILD ANSWERS

Announcement II

Question 5

(a) *deriving the equation of the circuit would give us*

$\omega^2 < \alpha^2$ *then by deriving it again and dividing by L, we have our 2nd order ODE*

$\omega^2 < \alpha^2$ *it is linear and causal because it's values will only change with respect to time and does not look into the future for other values.*

(b) *the oscillating frequency is a standard formula given as*

$\omega^2 < \alpha^2$

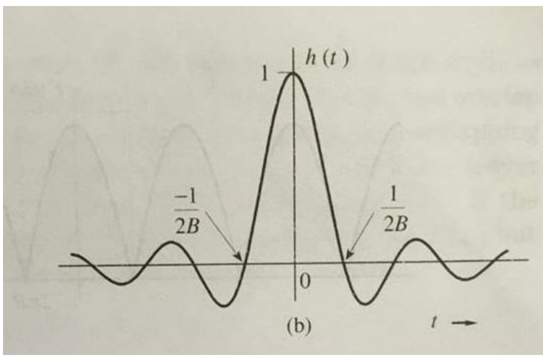
(c) *by simply deriving the first derivative of the equation and not dividing it by L we get:*

$\omega^2 < \alpha^2$ *substituting q and getting the auxiliary equation*

$\omega^2 < \alpha^2$ *getting the roots by quadratic formula* $\omega^2 < \alpha^2$

- When using external tools, be sure to copy the LaTeX not the image (because it might change)
- In this case, the "image" is a web-link which has expired!
  - https://www.latex4technics.com/l4ttemp/ysio4z.png?1458878525541
- Only 3 students

Announcement III



- Please don't link external images/content please
  - It might expire and worse might disallow us from grading your solution ∵ it could be used to change the answer *a posteriori*
- Please don't link from Facebook as this reveals source ☺ (12527949_1066290980057720_1984531858_n.jpg)

Announcement IV: Student Management System

Request for Peer note-taker:

- Student Services requires students with good note-taking skills to provide notes for ELEC3004 Signals, Systems & Control. These notes are for exclusive use by students who have a disability. Students interested in being a peer note taker for the course ELEC3004 Signals, Systems & Control are requested to visit https://www.uq.edu.au/secure/dms/register/ to submit an expression of interest. This site also contains details on the role of a PNT and remuneration arrangements.

- Leesa Schwarz | Disability Administration Officer
  t: +61 7 3365 7394 | e: l.schwarz@uq.edu.au

Digital Windows!

Window Functions (Lathi 4.9)

- We often need to truncate data
  - Ex: Fourier transform of some signal, say $e^{-t}u(t)$
  - Truncate beyond a sufficiently large value of t (typically five time constants and above).
  - ∵ in numerical computations: we have data of finite duration.
  - Similarly, the impulse response *h(t)* of an ideal lowpass filter is noncausal, and approaches zero asymptotically as $|t| \to \infty$

- Data truncation can occur in both time and frequency domain
  - In signal sampling, to eliminate aliasing, we need to truncate the Signal spectrum beyond the half sampling frequency $\frac{\omega_s}{2}$, using an anti-aliasing filter

Window Functions

Truncation operation may be regarded as multiplying a signal of a large width by a window function of a smaller (finite) width. Simple truncation amounts to using a **rectangular window** $w_R(t)$ (Fig. 4.48a) in which we assign unit weight to all the data within the window width ($|t| < \frac{T}{2}$), and assign zero weight to all the data lying outside the window ($|t| > \frac{T}{2}$). It is also possible to use a window in which the weight assigned to the data within the window may not be constant. In a **triangular window** $w_T(t)$, for example, the weight assigned to data decreases linearly over the window width (Fig. 4.48b).

Consider a signal $f(t)$ and a window function $w(t)$. If $f(t) \Longleftrightarrow F(\omega)$ and $w(t) \Longleftrightarrow W(\omega)$, and if the windowed function $f_w(t) \Longleftrightarrow F_w(\omega)$, then

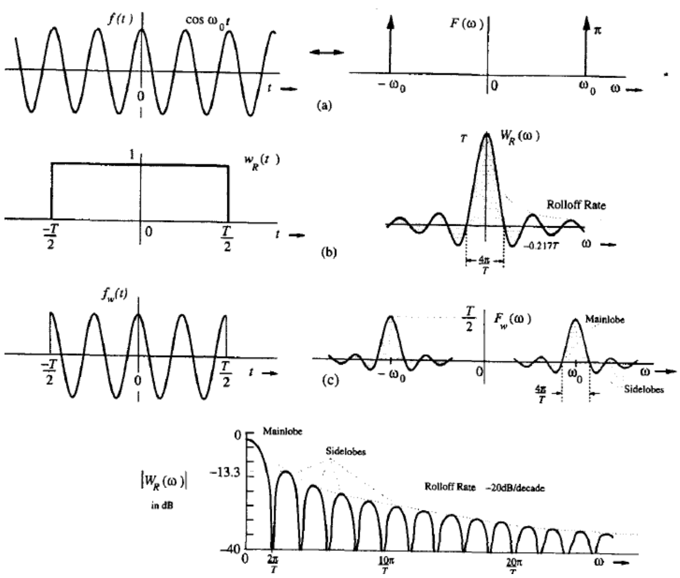
$$f_w(t) = f(t)w(t) \qquad \text{and} \qquad F_w(\omega) = \frac{1}{2\pi}F(\omega) * W(\omega)$$

Window Functions

$$f_w(t) = f(t)w(t) \qquad \text{and} \qquad F_w(\omega) = \frac{1}{2\pi} F(\omega) * W(\omega)$$

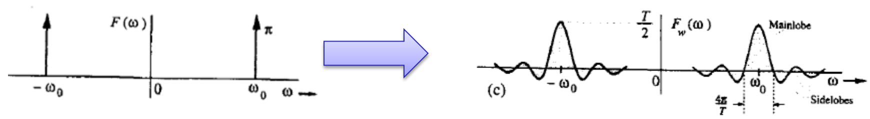
According to the width property of convolution, it follows that the width of $F_w(\omega)$ equals the sum of the widths of $F(\omega)$ and $W(\omega)$. Thus, truncation of a signal increases its bandwidth by the amount of bandwidth of $w(t)$. Clearly, the truncation of a signal causes its spectrum to spread (or smear) by the amount of the bandwidth of $w(t)$. Recall that the signal bandwidth is inversely proportional to the signal duration (width). Hence, the wider the window, the smaller is its bandwidth, and the smaller is the **spectral spreading**. This result is predictable because a wider window means we are accepting more data (closer approximation), which should cause smaller distortion (smaller spectral spreading). Smaller window width (poorer approximation) causes more spectral spreading (more distortion). There are also other effects produced by the fact that $W(\omega)$ is really not strictly bandlimited, and its spectrum $\rightarrow 0$ only asymptotically. This causes the spectrum of $F_w(\omega) \rightarrow 0$ asymptotically also at the same rate as that of $W(\omega)$, even though the $F(\omega)$ may be strictly bandlimited. Thus, windowing causes the spectrum of $F(\omega)$ to leak in the band where it is supposed to be zero. This effect is called **leakage**. These twin effects, the spectral spreading and the leakage, will now be clarified by an example.

Window Functions



Window Functions

For an example, let us take $f(t) = \cos \omega_0 t$ and a rectangular window $w_R(t) = \text{rect}(\frac{t}{T})$, illustrated in Fig. 4.46b. The reason for selecting a sinusoid for $f(t)$ is that its spectrum consists of spectral lines of zero width (Fig. 4.46a). This choice will make the effect of spectral spreading and leakage clearly visible. The spectrum of the truncated signal $f_w(t)$ is the convolution of the two impulses of $F(\omega)$ with the sinc spectrum of the window function. Because the convolution of any function with an impulse is the function itself (shifted at the location of the impulse), the resulting spectrum of the truncated signal is ($1/2\pi$ times) the two sinc pulses at $\pm\omega_0$, as depicted in Fig. 4.46c. Comparison of spectra $F(\omega)$ and $F_w(\omega)$ reveals the effects of truncation. These are:



Window Functions

1 The spectral lines of $F(\omega)$ have zero width. But the truncated signal is spread out by $4\pi/T$ about each spectral line. The amount of spread is equal to the width of the mainlobe of the window spectrum. One effect of this **spectral spreading** (or smearing) is that if $f(t)$ has two spectral components of frequencies differing by less than $4\pi/T$ rad/s ($2/T$ Hz), they will be indistinguishable in the truncated signal. The result is loss of spectral resolution. We would like the spectral spreading (mainlobe width) to be as small as possible.

2 In addition to the mainlobe spreading, the truncated signal also has sidelobes, which decay slowly with frequency. The spectrum of $f(t)$ is zero everywhere except at $\pm\omega_0$. On the other hand, the truncated signal spectrum $F_w(\omega)$ is zero nowhere because of sidelobes. These sidelobes decay asymptotically as $1/\omega$. Thus, the truncation causes spectral **leakage** in the band where the spectrum of the signal $f(t)$ is zero. The peak **sidelobe** magnitude is 0.217 times the mainlobe magnitude (13.3 dB below the peak mainlobe magnitude). Also, the sidelobes decay at a rate $1/\omega$, which is -6 dB/octave (or -20 dB/decade). This is the **rolloff rate** of sidelobes. We want smaller sidelobes with a faster rate of decay (high rolloff rate). Figure 4.46d shows $|W_R(\omega)|$ (in dB) as a function of ω. This plot clearly shows the mainlobe and sidelobe features, with the first sidelobe amplitude -13.3 dB below the mainlobe amplitude, and the sidelobes decaying at a rate of -6 dB/octave (or -20 dB per decade).

Remedies for Side Effects of Truncation

For better results, we must try to minimize the truncation's twin side effects, the spectral spreading (mainlobe width) and leakage (sidelobe). Let us consider each of these ills.

1 The spectral spread (mainlobe width) of the truncated signal is equal to the bandwidth of the window function $w(t)$. We know that the signal bandwidth is inversely proportional to the signal width (duration). Hence, to reduce the spectral spread (mainlobe width), we need to increase the window width.

2 To improve the leakage behavior, we must search for the cause of the slow decay of sidelobes. In Chapter 3, we saw that the Fourier spectrum decays as $1/\omega$ for a signal with jump discontinuity, and decays as $1/\omega^2$ for a continuous signal whose first derivative is discontinuous, and so on.† Smoothness of a signal is measured by the number of continuous derivatives it possesses. The smoother the signal, the faster the decay of its spectrum. Thus, we can achieve a given leakage behavior by selecting a suitably smooth window.

3 For a given window width, the remedies for the two effects are incompatible. If we try to improve one, the other deteriorates. For instance, among all the windows of a given width, the rectangular window has the smallest spectral spread (mainlobe width), but has high level sidelobes, which decay slowly. A tapered (smooth) window of the same width has smaller and faster decaying sidelobes, but it has a wider mainlobe.‡ But we can compensate for the increased mainlobe width by widening the window. Thus, we can remedy both the side effects of truncation by selecting a suitably smooth window of sufficient width.

Remedies for Side Effects of Truncation

There are several well-known tapered-window functions, such as Bartlett (triangular), Hanning (von Hann), Hamming, Blackman, and Kaiser, which truncate the data gradually. These windows offer different tradeoffs with respect to spectral spread (mainlobe width), the peak sidelobe magnitude, and the leakage rolloff rate as indicated in Table 4.3.[8, 9] Observe that all windows are symmetrical about the origin (even functions of t). Because of this feature, $W(\omega)$ is a real function of ω; that is, $\angle W(\omega)$ is either 0 or π. Hence, the phase function of the truncated signal has a minimal amount of distortion.

Figure 4.47 shows two well-known tapered-window functions, the von Hann (or Hanning) window $w_{\text{HAN}}(x)$ and the Hamming window $w_{\text{HAM}}(x)$. We have intentionally used the independent variable x because windowing can be performed in time domain as well as in frequency domain; so x could be t or ω, depending on the application.

Remedies for Side Effects of Truncation

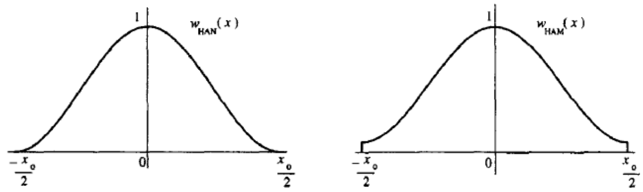


Fig. 4.47 Hanning and Hamming windows.

There are hundreds of windows, each with differing characteristics. But the choice depends on a particular application. The rectangular window has the narrowest mainlobe. The Bartlett (triangle) window (also called the Fejer or Cesaro) is inferior in all respects to the Hanning window. For this reason it is rarely used in practice. Hanning is preferred over Hamming in spectral analysis because it has faster sidelobe decay. For filtering applications, on the other hand, the Hamming window is the choice because it has the smallest sidelobe magnitude for a given mainlobe width. The Hamming window is the most widely used, general purpose window. The Kaiser window, which uses $I_0(\alpha)$, the Bessel function of the order 0, is more versatile and adjustable. Selecting a proper value of α $(0 \leq \alpha \leq 10)$ allows the designer to tailor the window to suit a particular application. The parameter α controls the mainlobe and sidelobe trade-off. When $\alpha = 0$, the Kaiser window is the rectangular window. For $\alpha = 5.4414$, it is the Hamming window, and when $\alpha = 8.885$, it is the Blackman window. As α increases, the mainlobe width increases and the sidelobe level decreases.

Frequency Response of Discrete-Time Systems

Filter Design Using Windows

We shall design an ideal lowpass filter of bandwidth W rad/s. For this filter, the impulse response $h(t) = \frac{W}{\pi}\text{sinc}(Wt)$ (Fig. 4.48c) is noncausal and, therefore, unrealizable. Truncation of $h(t)$ by a suitable window (Fig. 4.48a) makes it realizable, although the resulting filter is now an approximation to the desired ideal filter.† We shall use a rectangular window $w_R(t)$ and a triangular (Bartlett) window $w_T(t)$ to truncate $h(t)$, and then examine the resulting filters. The truncated impulse responses $h_R(t)$ and $h_T(t)$ for the two cases are depicted in Fig. (4.48d).

$$h_R(t) = h(t)w_R(t) \qquad \text{and} \qquad h_T(t) = h(t)w_T(t)$$

Hence, the windowed filter transfer function is the convolution of $H(\omega)$ with the Fourier transform of the window, as illustrated in Fig. 4.48e and f. We make the following observations.

1. The windowed filter spectra show **spectral spreading** at the edges, and instead of a sudden switch there is a gradual transition from the passband to the stopband of the filter. The transition band is smaller ($2\pi/T$ rad/s) for the rectangular case compared to the triangular case ($4\pi/T$ rad/s).
2. Although $H(\omega)$ is bandlimited, the windowed filters are not. But the stopband behavior of the triangular case is superior to that of the rectangular case. For the rectangular window, the leakage in the stopband decreases slowly (as $1/\omega$) compared to that of the triangular window (as $1/\omega^2$). Moreover, the rectangular case has a higher peak sidelobe amplitude compared to that of the triangular window.

Filter Design Using Windows

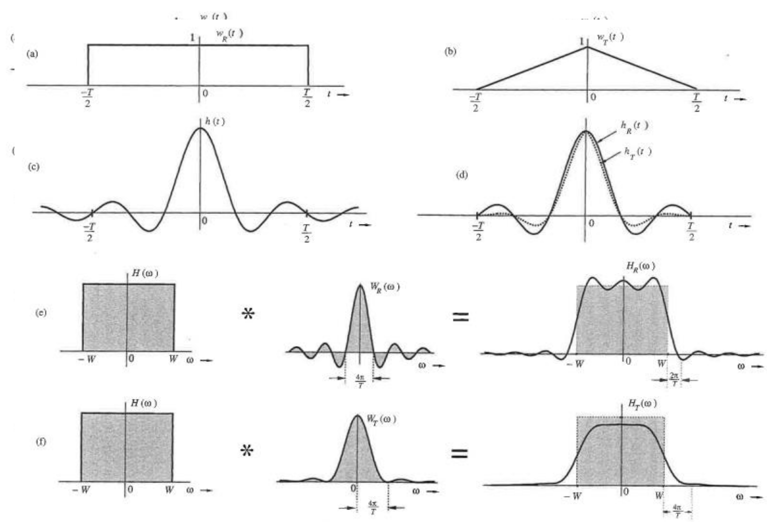


Fig. 4.48 Filter design using windows.

Frequency Response of Discrete-Time Systems

For (asymptotically stable) continuous-time systems we showed that the system response to an input $e^{j\omega t}$ is $H(j\omega)e^{j\omega t}$, and that the response to an input $\cos \omega t$ is $|H(j\omega)| \cos[\omega t + \angle H(j\omega)]$. Similar results hold for discrete-time systems. We now show that for an (asymptotically stable) LTID system, the system response to an input $e^{j\Omega k}$ is $H[e^{j\Omega}]e^{j\Omega k}$ and the response to an input $\cos \Omega k$ is $|H[e^{j\Omega}]| \cos(\Omega k + \angle H[e^{j\Omega}])$.

The proof is similar to the one used in continuous-time systems. In Sec. 9.4-2 we showed that an LTID system response to an (everlasting) exponential z^k is also an (everlasting) exponential $H[z]z^k$. It is helpful to represent this relationship by a directed arrow notation as

$$z^k \Longrightarrow H[z]z^k \qquad (12.1)$$

Setting $z = e^{\pm j\Omega}$ in this relationship yields

$$e^{j\Omega k} \Longrightarrow H[e^{j\Omega}]e^{j\Omega k} \qquad (12.2a)$$

$$e^{-j\Omega k} \Longrightarrow H[e^{-j\Omega}]e^{-j\Omega k} \qquad (12.2b)$$

Addition of these two equations yields

$$2\cos \Omega k \Longrightarrow H[e^{j\Omega}]e^{j\Omega k} + H[e^{-j\Omega}]e^{-j\Omega k} = 2\text{Re}\left(H[e^{j\Omega}]e^{j\Omega k}\right) \qquad (12.3)$$

Expressing $H[e^{j\Omega}]$ in the polar form

$$H[e^{j\Omega}] = |H[e^{j\Omega}]|e^{j\angle H[e^{j\Omega}]} \qquad (12.4)$$

Eq. (12.3) can be expressed as

Frequency Response of Discrete-Time Systems

$$\cos \Omega k \Longrightarrow |H[e^{j\Omega}]| \cos\left(\Omega k + \angle H[e^{j\Omega}]\right) \qquad (12.5)$$

In other words, the system response $y[k]$ to a sinusoidal input $\cos \Omega k$ is given by

$$y[k] = |H[e^{j\Omega}]| \cos\left(\Omega k + \angle H[e^{j\Omega}]\right) \qquad (12.6a)$$

Following the same argument, the system response to a sinusoid $\cos(\Omega k + \theta)$ is

$$y[k] = |H[e^{j\Omega}]| \cos\left(\Omega k + \theta + \angle H[e^{j\Omega}]\right) \qquad (12.6b)$$

This result applies only to asymptotically stable systems because Eq. (12.1) is valid only for values of z lying in the region of convergence of $H[z]$. For $z = e^{j\Omega}$, z lies on the unit circle ($|z| = 1$). The region of convergence for unstable and marginally stable systems does not include the unit circle.

This important result shows that the response of an asymptotically stable LTID system to a discrete-time sinusoidal input of frequency Ω is also a discrete-time sinusoid of the same frequency. *The amplitude of the output sinusoid is $|H[e^{j\Omega}]|$ times the input amplitude, and the phase of the output sinusoid is shifted by $\angle H[e^{j\Omega}]$ with respect to the input phase.* Clearly $|H[e^{j\Omega}]|$ is the amplitude gain, and a plot of $|H[e^{j\Omega}]|$ versus Ω is the amplitude response of the discrete-time system. Similarly, $\angle H[e^{j\Omega}]$ is the phase response of the system, and a plot of $\angle H[e^{j\Omega}]$ vs Ω shows how the system modifies or shifts the phase of the input sinusoid. Note that $H[e^{j\Omega}]$ incorporates the information of both amplitude and phase response and therefore is called the **frequency response** of the system.

These results, although parallel to those for continuous-time systems, differ from them in one significant aspect. In the continuous-time case, the frequency response is $H(j\omega)$. A parallel result for the discrete-time case would lead to frequency response $H[j\Omega]$. Instead, we found the frequency response to be $H[e^{j\Omega}]$. This deviation causes some interesting differences between the behavior of continuous-time and discrete-time systems.

Frequency Response of Discrete-Time Systems

■ **Example 12.1**

For a system specified by the equation

$$y[k+1] - 0.8y[k] = f[k+1]$$

find the system response to the input **(a)** $1^k = 1$ **(b)** $\cos\left[\frac{\pi}{6}k - 0.2\right]$ **(c)** a sampled sinusoid $\cos 1500t$ with sampling interval $T = 0.001$.

The system equation can be expressed as

$$(E - 0.8)y[k] = Ef[k]$$

Therefore, the transfer function of the system is

$$H[z] = \frac{z}{z - 0.8} = \frac{1}{1 - 0.8z^{-1}}$$

The frequency response is

$$H[e^{j\Omega}] = \frac{1}{1 - 0.8e^{-j\Omega}} \tag{12.7}$$

$$= \frac{1}{1 - 0.8(\cos\Omega - j\sin\Omega)}$$

$$= \frac{1}{(1 - 0.8\cos\Omega) + j0.8\sin\Omega}$$

Therefore

$$\left|H[e^{j\Omega}]\right| = \frac{1}{\sqrt{(1 - 0.8\cos\Omega)^2 + (0.8\sin\Omega)^2}}$$

$$= \frac{1}{\sqrt{1.64 - 1.6\cos\Omega}} \tag{12.8a}$$

and

$$\angle H[e^{j\Omega}] = -\tan^{-1}\left[\frac{0.8\sin\Omega}{1 - 0.8\cos\Omega}\right] \tag{12.8b}$$

The amplitude response $|H[e^{j\Omega}]|$ can also be obtained by observing that $|H|^2 = HH^*$. Therefore

$$\left|H[e^{j\Omega}]\right|^2 = H[e^{j\Omega}]H^*[e^{j\Omega}]$$

$$= H[e^{j\Omega}]H[e^{-j\Omega}] \tag{12.9}$$

From Eq. (12.7) it follows that

$$\left|H[e^{j\Omega}]\right|^2 = \left(\frac{1}{1 - 0.8e^{-j\Omega}}\right)\left(\frac{1}{1 - 0.8e^{j\Omega}}\right)$$

$$= \frac{1}{1.64 - 1.6\cos\Omega}$$

which yields the result found earlier in Eq. (12.8a).

Frequency Response of Discrete-Time Systems

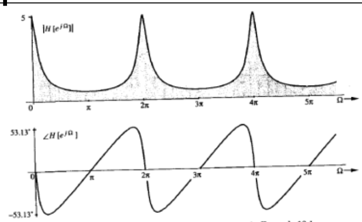


Fig. 12.1 Frequency response of an LTID system in Example 12.1.

Figure 12.1 shows plots of amplitude and phase response as functions of Ω. We now compute the amplitude and the phase response for the various inputs:

(a) $f[k] = 1^k = 1$

Since $1^k = (e^{j\Omega})^k$ with $\Omega = 0$, the amplitude response is $H[e^{j0}]$. From Eq. (12.8a) we obtain

$$H[e^{j0}] = \frac{1}{\sqrt{1.64 - 1.6\cos(0)}} = \frac{1}{\sqrt{0.04}} = 5 = 5\angle 0$$

Therefore

$$|H[e^{j0}]| = 5 \quad \text{and} \quad \angle H[e^{j0}] = 0$$

These values also can be read directly from Figs. 12.1a and 12.1b, respectively, corresponding to $\Omega = 0$. Therefore, the system response to input 1 is

$$y[k] = 5(1^k) = 5 \tag{12.10}$$

(b) $f[k] = \cos\left[\frac{\pi}{6}k - 0.2\right]$

Here $\Omega = \frac{\pi}{6}$. According to Eqs. (12.8)

$$\left|H[e^{j\pi/6}]\right| = \frac{1}{\sqrt{1.64 - 1.6\cos\frac{\pi}{6}}} = 1.983$$

$$\angle H[e^{j\pi/6}] = -\tan^{-1}\left[\frac{0.8\sin\frac{\pi}{6}}{1 - 0.8\cos\frac{\pi}{6}}\right] = -0.916 \text{ rad.}$$

These values also can be read directly from Figs. 12.1a and 12.1b, respectively, corresponding to $\Omega = \frac{\pi}{6}$. Therefore

Properties of the ROC

➔The ROC is always defined by circles centered around the origin.

$h[k]r^{-k}$ is absolutely summable, where $r = |z|$.

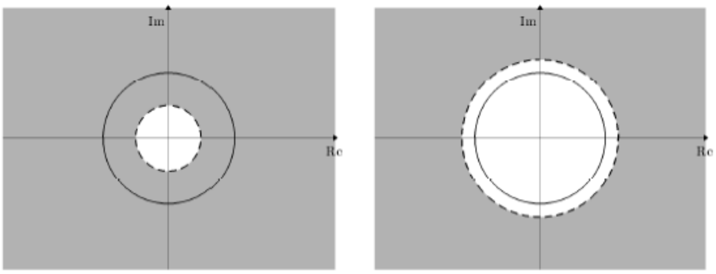
➔Right-sided signals have "outsided" ROCs.

if $\exists n_0$ such that $h[n] = 0\ \forall n < n_0$, then if $r_0 \in$ ROC, then $\forall r$ with $r_0 < r < \infty$ are also in the ROC.

➔Left-sided signals have "insided" ROCs.
(with ∀r within $0<r<r_0$)

Region of Convergence (ROC) Plots

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b}{1 - az^{-1}}, \quad |z| > |a|$$

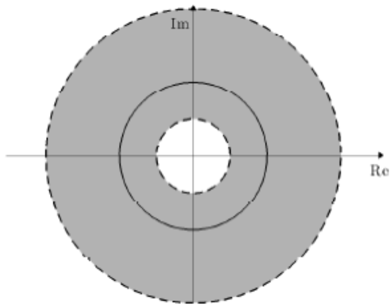


$a = .5$ $a = 1.2$

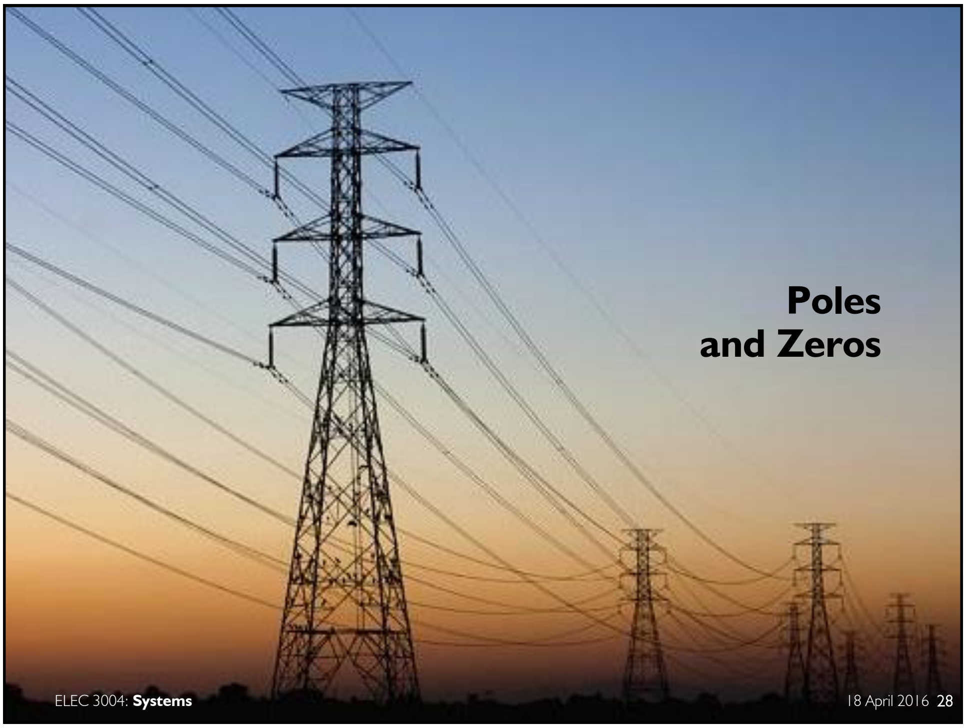
Combinations of Signals

$$y_1[n] = \begin{cases} ba^n & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad y_2[n] = \begin{cases} 0 & n \geq 0 \\ -ba^n & n < 0 \end{cases}$$

$a = .5$ $\qquad$ $a = 2$



ROC for $\alpha_1 y_1[n] + \alpha_2 y_2[n]$



Poles and Zeros

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n},$$

assume b and a have no common factors (cancel them out if they do . . .)

- the m roots of b are called the *zeros* of F; λ is a zero of F if $F(\lambda) = 0$
- the n roots of a are called the *poles* of F; λ is a pole of F if $\lim_{s \to \lambda} |F(s)| = \infty$

the *multiplicity* of a zero (or pole) λ of F is the multiplicity of the root λ of b (or a)

example: $\frac{6s+12}{s^2+2s+1}$ has one zero at $s = -2$, two poles at $s = -1$

Source: Boyd, EE102,5-12

Poles and Zeros

factored or *pole-zero* form of F:

$$F(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} = k \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

where

- $k = b_m / a_n$
- $z_1, \ldots, z_m$ are the zeros of F ($i.e.$, roots of b)
- $p_1, \ldots, p_n$ are the poles of F ($i.e.$, roots of a)

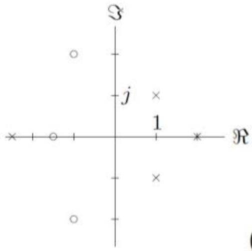
(assuming the coefficients of a and b are real) complex poles or zeros come in complex conjugate pairs

can also have *real factored form* . . .

Source: Boyd, EE102,5-13

Pole Zero Plot

poles & zeros of a rational functions are often shown in a *pole-zero plot*



(× denotes a pole; ∘ denotes a zero)

this example is for

$$F(s) = k\frac{(s+1.5)(s+1+2j)(s+1-2j)}{(s+2.5)(s-2)(s-1-j)(s-1+j)}$$

$$= k\frac{(s+1.5)(s^2+2s+5)}{(s+2.5)(s-2)(s^2-2s+2)}$$

(the plot doesn't tell us k)

Source: Boyd, EE102,5-14

Partial Fraction Expansion

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

let's assume (for now)

- no poles are repeated, $i.e.$, all roots of a have multiplicity one
- $m < n$

then we can write F in the form

$$F(s) = \frac{r_1}{s-\lambda_1} + \cdots + \frac{r_n}{s-\lambda_n}$$

called **partial fraction expansion** of F

- $\lambda_1, \ldots, \lambda_n$ are the poles of F
- the numbers $r_1, \ldots, r_n$ are called the **residues**
- when $\lambda_k = \overline{\lambda_l}$, $r_k = \overline{r_l}$

Source: Boyd, EE102,5-15

Partial Fraction Expansion Example

example:

$$\frac{s^2-2}{s^3+3s^2+2s} = \frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2}$$

let's check:

$$\frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2} = \frac{-1(s+1)(s+2) + s(s+2) + s(s+1)}{s(s+1)(s+2)}$$

in partial fraction form, **inverse Laplace transform** is **easy:**

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{r_1}{s-\lambda_1} + \cdots + \frac{r_n}{s-\lambda_n}\right) \\ &= r_1 e^{\lambda_1 t} + \cdots + r_n e^{\lambda_n t} \end{aligned}$$

(this is real since whenever the poles are conjugates, the corresponding residues are also)

Source: Boyd, EE102,5-16

2D DFT

$$\begin{aligned} \mathcal{F}(u,v) &= \frac{1}{N}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux+vy)/N} \\ f(x,y) &= \frac{1}{N}\sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \mathcal{F}(u,v)e^{j2\pi(ux+vy)/N} \end{aligned}$$

2D DFT

- Each DFT coefficient is a complex value
  - There is a single DFT coefficient for each spatial sample
  - A complex value is expressed by two real values in either Cartesian or polar coordinate space.
    - Cartesian: R(u,v) is the *real* and I(u, v) the *imaginary* component
    - Polar: |F(u,v)| is the *magnitude* and phi(u,v) the *phase*

$$\mathcal{F}(u,v) = R(u,v) + jI(u,v)$$

$$\mathcal{F}(u,v) = |F(u,v)|e^{j\phi(u,v)}$$

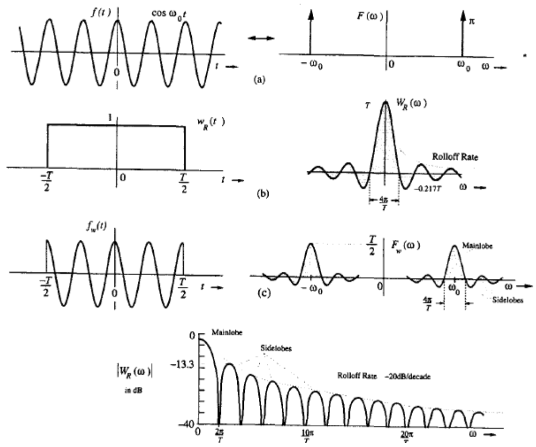
2D DFT

- Representing the DFT coefficients as magnitude and phase is a more useful for processing and reasoning.
  - The magnitude is a measure of strength or length
  - The phase is a direction and lies in [-pi, +pi]
- The magnitude and phase are easily obtained from the real and imaginary values

$$\begin{aligned} |\mathcal{F}(u,v)| &= \sqrt{R^2(u,v) + I^2(u,v)} \\ \phi(u,v) &= \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]. \end{aligned}$$

Windowing for the DFT

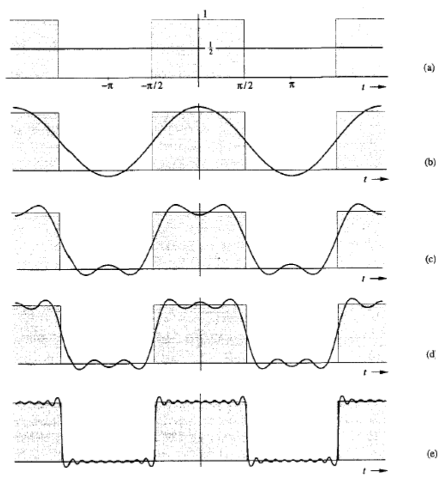
$$f_w(t) = f(t)w(t) \qquad \text{and} \qquad F_w(\omega) = \frac{1}{2\pi}F(\omega) * W(\omega)$$

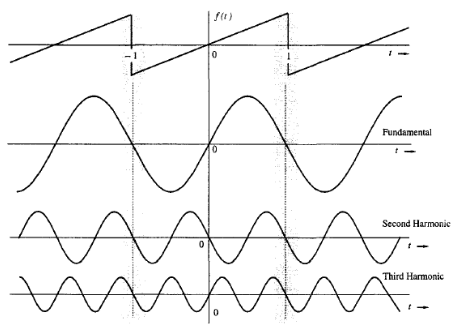


Source: Lathi, p.303

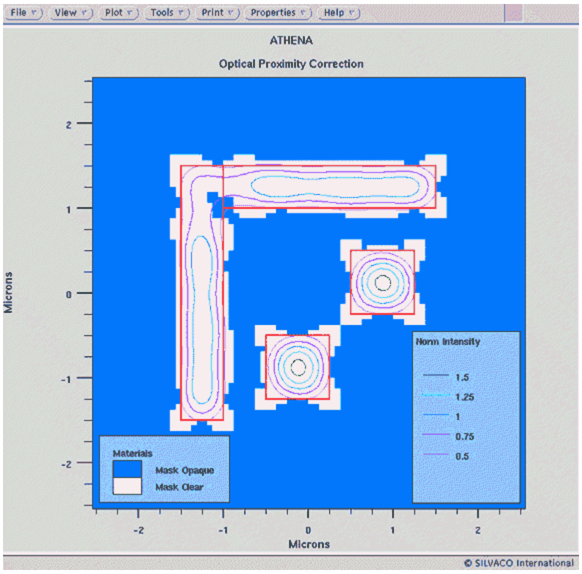
Harmonics

- Synthesis of a square pulse: periodic signal by successive addition of its harmonics (Lathi, p. 202-3)





Optical Proximity Correction



Next Time…

- **Digital Filters**

- Review:
  - Chapter 12 of Lathi
  - § 10. 3 of Strang on FFTs (cached on Course Website)

- Ponder?

$$y[k] = f[k] * h[k] \qquad Y(\Omega) = F(\Omega)H(\Omega)$$

where $F(\Omega)$, $Y(\Omega)$, and $H(\Omega)$ are DTFTs of $f[k]$, $y[k]$, and $h[k]$, respectively; that is,

$$f[k] \Longleftrightarrow F(\Omega), \quad y[k] \Longleftrightarrow Y(\Omega), \quad \text{and} \quad h[k] \Longleftrightarrow H(\Omega)$$