# CHAPTER 3

# Sampled-Data Systems

### 3.1 INTRODUCTION

The use of digital logic or digital computers to calculate a control action for a continuous, dynamic system introduces the fundamental operation of sampling. Samples are taken from the continuous, physical signals such as position, velocity, or temperature, and these samples are used in the computer to calculate the controls to be applied. Such digital controls are hybrids, where discrete signals appear in some places and continuous signals occur in other parts. Such systems are called sampled-data systems because some continuous data are sampled before being used. In many ways, the analysis of a purely continuous system or of a purely discrete system is simpler than is that of the hybrid case. However, in digital control much of the processing is done via digital logic on discrete signals, but the origin of the signals is in the continuous world, and the destination of our computed outputs is there as well. Thus the role of sampling and the conversion from continuous to discrete and back from discrete to continuous are very important to understanding digital control, and we must study the process of sampling and how to make mathematical models of analog-to-digital conversion because our real interest is in the hybrid, sampled-data case. This analysis will require some careful treatment via the Fourier transform, but the effort will be well rewarded with the understanding it will bring to later systems.

In this chapter, we introduce the analysis of the sampling process and describe both a time domain and a frequency domain representation. We also describe the companion process, that of sample extrapolation or holding to recover a continuous time signal from its samples. As part of this analysis, we show that, because a sampled-data system is made to be time varying by the introduction of sampling, it is not possible to describe such systems exactly by a transfer function. However, after sampling and holding, a continuous signal is recovered, and we can approximate the response of a sample and hold to a sinusoid by fitting another sinusoid of the same frequency to the complete response. We will show how to compute this best-fit sinusoidal response analytically and experimentally and thus have a good approximation for a transfer function. For those familiar with the idea, our approach is equivalent to the use of the "describing function" that is used to approximate a transfer function for simple nonlinear systems. This concept will be studied in Chapter 11.

Once the operations of sampling and holding are understood, we will show that we can always represent the relationship between the samples of the input and the samples of the output of a linear constant system by a *discrete* transfer function. Thus if we are willing to focus on the samples only, the entire power of linear, constant system theory is available to us. In fact, the analysis of discrete linear systems is in many ways simpler than the analysis of continuous linear systems in the way that subtraction is simpler than differentiation. We will see this when in the next chapter we use the z-transform to analyze difference equations; however, in the digital control of continuous dynamical systems we must understand the transitions from continuous to discrete and back again from discrete to continuous signals from the start. This is what we do in this chapter for the most elementary operations: sampling and holding.

### 3.2 ANALYSIS OF THE SAMPLE AND HOLD

To get samples of a physical signal such as a position or a velocity into digital form, we typically have a sensor that produces a voltage proportional to the physical variable and an analog-to-digital converter, commonly called an A/D converter or ADC, that transforms the voltage into a digital number. The physical conversion always takes a finite time, and in many instances this time is significant with respect to the sample time of the controls and with respect to the rate of change of the signal being sampled. In order to give the computer an accurate representation of the signal exactly at the sampling instants kT, the A/D converter is often preceded by a circuit called the Sample-and-Hold Circuit or SHC. A simple electronic schematic is sketched in Fig. 3.1, where the switch, S, is an electronic device driven by simple logic from a clock. Its operation is described below.

With the switch, S, in position 1, the amplifier output  $v_{out}(t)$  follows or tracks the input voltage  $v_{in}(t)$  through the transfer function 1/(RCs+1). The



Figure 3.1 Analog-to-digital converter with sample and hold.

circuit bandwidth of the SHC, 1/RC, is selected to be high compared to the input signal bandwidth. Typical values are R = 1000 ohms,  $C = 30 \times 10^{-12}$ farads for a bandwidth of  $f = 1/2\pi RC = 5.3$  MHz. During this "tracking time," the ADC is turned off and ignores  $v_{out}$ . When a sample is to be taken at t = kT the switch S is set to position 2 and the capacitor C holds the output of the operational amplifier frozen from that time at  $v_{out}(kT)$ . The ADC is now signaled to begin conversion and has the constant input from the SHC to work on, so the resulting digital number is a true representation of the input voltage at the sample time. When the conversion is completed, the digital number is presented to the digital computer at which time the calculations based on this sample value can begin. The SHC switch is now moved to position 1, and the circuit is again tracking, waiting for the next command to freeze a sample. For example, the conversion time of the Burr-Brown ADC803 is 1.5 microseconds for 12 bits of accuracy. The SHC needs only to hold the voltage for this short time in order for the conversion to be completed before it is started tracking again. The value taken is held inside the computer for the entire sampling period of the system, so the combination of the electronic SHC plus the ADC operate as a sample-and-



Figure 3.2 The sampler.

hold for the sampling period, T seconds, which may be many milliseconds long.

For the purpose of the analysis we separate the sample and hold into two mathematical operations: a sampling operation represented as impulse modulation and a hold operation represented as a linear filter. The symbol or schematic of the ideal sampler is shown in Fig. 3.2; its role is to give a mathematical representation of the process of taking periodic samples from r(t) to produce r(kT) and to do this in such a way that we can include the sampled signals in the analysis of continuous signals using the Laplace transform.<sup>1</sup> The technique is to use *impulse modulation* as the mathematical representation of sampling. Thus, from Fig. 3.2, we picture the output of the sampler as a string of impulses,

$$r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT).$$
(3.1)

The impulse can be visualized as the limit of a pulse of unit area that has growing amplitude and shrinking duration. The essential property of the impulse is the sifting property that

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$$
(3.2)

for all functions f that are continuous at a. The integral of the impulse is the unit step

$$\int_{-\infty}^{t} \delta(\tau) d\tau = 1(t), \qquad (3.3)$$

and the Laplace transform of the unit impulse is one, because

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(\tau) e^{-s\tau} d\tau = 1.$$
(3.4)

<sup>&</sup>lt;sup>1</sup>We assume that the reader has some familiarity with Fourier and Laplace transform analysis. For a general reference, see Bracewell (1978).

Using these properties we can see that  $r^*(t)$ , defined in (3.1), depends only on the discrete sample values r(kT). The Laplace transform of  $r^*(t)$  can be computed as follows:

$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r^*(\tau) e^{-s\tau} d\tau.$$

If we substitute (3.1) for  $r^*(t)$ , we get

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r(\tau) \delta(\tau - kT) e^{-s\tau} d\tau,$$

and now, exchanging integration and summation and using (3.2), we have

$$R^*(s) = \sum_{k=-\infty}^{\infty} r(kT)e^{-skT}.$$
(3.5)

The notation  $R^*(s)$  is used to symbolize the (Laplace) transform of  $r^*(t)$ , the sampled or impulse-modulated r(t).<sup>2</sup>

Having a model of the sampling operation as impulse modulation, to complete the description of the physical sample-and-hold we need to model

$$1^{*}(t) = \sum_{k=0}^{\infty} \delta(t - kT),$$
 (a)

and, using (3.2), we obtain

$$\mathcal{L}\{1^*(t)\} = 1/(1 - e^{-Ts}).$$
 (b)

The reader should be warned that the Fourier integral converges to the *average* value of a function at a discontinuity and not the value approached from the right as we assume. Because our use of the transform theory is elementary and the convenience of equation (b) above is substantial, we have selected the continuous-from-theright convention. In case of doubt, the discontinuous term should be separated and treated by special analysis, perhaps in the time domain.

<sup>&</sup>lt;sup>2</sup>It will be necessary, from time to time, to consider sampling a signal that is not continuous. The only case we will consider will be equivalent to applying a step function, 1(t), to a sampler. For the purposes of this book we will define the unit step to be continuous from the right and assume that the impulse,  $\delta(t)$ , picks up the full value of unity. By this convention and (3.1) we compute

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the hold operation, which will take the impulses that are produced by the mathematical sampler and produce the piecewise constant output of the device. Typical signals are sketched on Fig. 3.3. Once the samples are taken, as represented by  $r^*(t)$  in (3.1), the hold is defined as the means whereby these impulses are extrapolated to the piecewise constant signal  $r_h$ , defined as

$$r_h(t) = r(kT)$$
  $kT \le t < kT + T$ , (3.6)

Because  $r_h$  is composed of zero-order polynomials passed through the samples of r(kT), this hold operation is called the zero-order hold or ZOH and has the transfer function ZOH(s). We can compute ZOH(s) by determining its impulse response. The hold filter will receive one unit-size impulse if the input signal is zero at every sample time except t = 0 and is equal to one there. In that case,  $r^*(t) = \delta(t)$  and  $r_h(t)$ , which is now the impulse response of ZOH, is a pulse of height 1 and duration T seconds. The mathematical representation of the impulse response is, using the unit step function,

p(t) = 1(t) - 1(t - T).



Figure 3.3 The sample and hold, showing typical signals. (a) Input signal r; (b) sampled signal  $r^*$ ; (c) output signal  $r_h$ ; (d) sample and hold.

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The required transfer function is the Laplace transform of p(t) as

$$ZOH(s) = \mathcal{L}\{p(t)\}\$$
  
=  $\int_0^\infty [1(t) - 1(t - T)]e^{-st}dt$   
=  $(1 - e^{-sT})/s.$  (3.7)

Thus the linear behavior of an A/D converter with sample and hold can be modeled by Fig. 3.3. We must emphasize that the impulsive signal  $r^*(t)$  in Fig. 3.3 is not expected to represent a physical signal in the A/D converter circuit; rather it is a hypothetical signal introduced to allow us to obtain a transfer-function model of the hold operation and to give an input-output model of the sample-and-hold suitable for transform and other linear systems analysis.

## 3.3 SPECTRUM OF A SAMPLED SIGNAL AND ALIASING

We can get further insight into the process of sampling by an alternative representation of the transform of  $r^*(t)$ , using Fourier analysis. From (3.1) we see that  $r^*(t)$  is a product of r(t) and the train of impulses,  $\Sigma \delta(t - kT)$ . The latter series, being periodic, can be represented by a Fourier series,

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{n=-\infty}^{\infty} C_n e^{j(2\pi n/T)t},$$

where the Fourier coefficients,  $C_n$ , are given by the integral over one period as

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t - kT) e^{-jn(2\pi t/T)} dt.$$

The only term in the sum of impulses that is in the range of the integral is the one at the origin  $\delta(t)$ , so the integral reduces to

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn(2\pi t/T)} dt;$$

but the sifting property from (3.2) makes this easy to integrate, with the result

$$C_n = \frac{1}{T}.$$

Thus we have derived the representation for the sum of impulses as a Fourier series:

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(2\pi n/T)t}.$$
(3.8)

We define  $\omega_s = 2\pi/T$  as the radian sampling frequency and now substitute (3.8) into (3.1) using  $\omega_s$ , and we take the Laplace transform of the output of the mathematical sampler,

$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r(t) \left\{ \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \right\} e^{-st} dt;$$

we integrate the sum, term by term

$$R^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} r(t) e^{jn\omega_s t} e^{-st} dt;$$

and if we combine the exponentials,

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} r(t) e^{-(s-jn\omega_s)t} dt.$$

The integral here is the Laplace transform of r(t) with only a change of variable where the frequency goes. The result can therefore be written as

$$R^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} R(s - jn\omega_s), \qquad (3.9)$$

where R(s) is the transform of r(t). In communication or radio engineering terms, (3.8) expresses the fact that the impulse train corresponds to an infinite sequence of carrier frequencies at integral values of  $2\pi/T$ , and (3.9) shows that when r(t) modulates all these carriers, it produces a never-ending train of sidebands. A sketch of the elements in the sum given in (3.9) is shown in Fig. 3.4.



Figure 3.4 (a) Sketch of a spectrum amplitude and (b) the components of the spectrum after sampling, showing aliasing.

An important feature of sampling, shown in Fig. 3.4, is illustrated at the frequency marked  $\omega_1$ . Two curves are drawn comprising two of the elements of the sum at  $\omega_1$ . One of these, the larger amplitude in the figure located at the frequency  $\omega_1$ , is the value of  $R(j\omega_1)$ . The smaller component at  $\omega_1$  comes from the spectrum centered at  $2\pi/T$  and is  $R(j\omega_0)$ , where  $\omega_0$  is such that  $\omega_0 = \omega_1 - 2\pi/T$ . This frequency,  $\omega_0$ , which shows up at  $\omega_1$  after sampling, is called in the trade an "alias" of  $\omega_1$ ; the phenomenon is called *aliasing*.

The phenomenon of aliasing has a clear meaning in time. Two continuous sinusoids of different frequencies appear at the same frequency when sampled. We cannot, therefore, distinguish between them based on their samples alone. Fig. 3.5 shows a plot of a sinusoid at  $\frac{1}{8}$  Hz and of a sinusoid at  $\frac{7}{8}$  Hz. If we sample these waves at 1 Hz, as indicated by the dots, then we get the same sample values from both signals and would continue to get the same sample values for all time. Note that the sampling frequency is 1, and, if  $f_1 = \frac{1}{8}$ , then

$$f_0 = \frac{1}{8} - 1 = -\frac{7}{8}.$$

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Figure 3.5 Plot of two sinusoids that have identical values at unit sampling intervals—an example of aliasing.

The significance of the negative frequency is that the  $\frac{7}{8}$ -Hz sinusoid in Fig. 3.5 is a negative sine function.

Thus, as a direct result of the sampling operation, when data are sampled at frequency  $2\pi/T$  the total harmonic content that shows up at a given frequency  $\omega_1$  is to be found not only from the original signal at  $\omega_1$  but also from all those frequencies that are aliases of  $\omega_1$ , namely, components from all frequencies  $\omega_1 + n2\pi/T = \omega_1 + n\omega_s$  as shown in the formula of (3.9) and sketched in Fig. 3.4. The errors caused by this aliasing can be very severe, especially if a substantial amplitude of high-frequency noise is present with the signal to be sampled. To minimize the error caused by this effect, it is standard practice to *precede* the sampling operation (such as the sample-and-hold circuit discussed earlier) by a low-pass antialias filter that will remove (almost) all spectral content above the half-sampling frequency, i.e., above  $\pi/T$ . A sketch of the result is drawn in Fig. 3.6.

If all spectral content above the frequency  $\pi/T$  is removed, then no aliasing is introduced by sampling and the signal spectrum is undistorted, even though it is repeated endlessly, centered at  $n2\pi/T$ . The critical frequency,  $\pi/T$ , was first reported by H. Nyquist, and is called the Nyquist frequency.



Figure 3.6 (a) Sketch of a spectrum amplitude and (b) the components of the spectrum after sampling, showing removal of aliasing with an antialiasing filter.

Band-limited signals that have no components above the Nyquist frequency are represented unambiguously by their samples.

A corollary to the aliasing issue is the sampling theorem. We have seen that if  $R(j\omega)$  has components above the Nyquist frequency  $\omega_s/2$  or  $\pi/T$ , then overlap and aliasing will occur. Conversely, we noticed that if  $R(j\omega)$  is zero for  $|\omega| \ge \pi/T$ , then sampling at intervals of T sec will produce no aliasing and the original spectrum can be recovered exactly from  $R^*$ , the spectrum of the samples. Once the spectrum is recovered, by inverse transform we can calculate the original signal itself. This is the sampling theorem: One can recover a signal from its samples, if the sampling frequency ( $\omega_s = 2\pi/T$ ) is at least twice the highest frequency ( $\pi/T$ ) in the signal. Notice that the sampling theorem requires that  $R(j\omega)$  is exactly zero for all frequencies above  $\pi/T$ .

A phenomenon somewhat related to aliasing is that of hidden oscillations. If signal frequencies only up to  $\pi/T$  can be sampled without confusion, there is the possibility that a signal could contain some frequencies that the samples do not show at all. Such signals, when they occur in a digital control, are called "hidden oscillations," an example of which is shown in Fig. 5.29.

# 3.4 DATA EXTRAPOLATION AND IMPOSTORS

The sampling theorem states that under the right conditions, it is possible to recover a signal from its samples; we now consider a formula for doing so. From Fig. 3.6 we can see that the spectrum of  $R(j\omega)$  is contained in the low-frequency part of  $R^*(j\omega)$ . Therefore, to recover  $R(j\omega)$  we need only process  $R^*(j\omega)$  through a low-pass filter and multiply by T to regain R. As a matter of fact, if  $R(j\omega)$  has zero energy for frequencies in the bands above  $\pi/T$  (such an R is said to be band-limited), then an *ideal low-pass filter* with gain T for  $-\pi/T \leq \omega \leq \pi/T$  and zero elsewhere would recover  $R(j\omega)$  from  $R^*(j\omega)$  exactly. Suppose we define this ideal low-pass filter characteristic as  $L(j\omega)$ . Then we have the result

$$R(j\omega) = L(j\omega)R^*(j\omega). \tag{3.10}$$

The signal r(t) is the inverse transform of  $R(j\omega)$ , and because by (3.10)  $R(j\omega)$  is the *product* of two transforms, its inverse transform r(t) must be the convolution of the time functions  $\ell(t)$  and  $r^*(t)$ . The form of the filter impulse response can be computed by using the definition of  $L(j\omega)$  from which the inverse transform gives<sup>3</sup>

$$\ell(t) = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} Te^{j\omega t} d\omega$$
  

$$= \frac{T}{2\pi} \frac{e^{j\omega t}}{jt} \Big|_{-\pi/T}^{\pi/T}$$
  

$$= \frac{T}{2\pi j t} (e^{j(\pi t/T)} - e^{-j(\pi t/T)})$$
  

$$= \frac{1}{\pi t/T} \sin \frac{\pi t}{T_{p}}$$
  

$$= \operatorname{sinc} \frac{\pi t}{T}.$$
(3.11)

Using (3.1) for  $r^*(t)$  and (3.11) for  $\ell(t)$ , we find that their convolution is

$$r(t) = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kT) \operatorname{sinc} \frac{\pi(t-\tau)}{T} d\tau.$$

<sup>3</sup> "Sinc" is the name given to the function defined by  $\operatorname{sinc}(\theta) = \sin(\theta)/\theta$ .



Figure 3.7 Plot of the impulse response of the ideal low-pass filter.

Using the sifting property of the impulse, we have

$$r(t) = \sum_{k=-\infty}^{\infty} r(kT) \operatorname{sinc} \frac{\pi(t-kT)}{T}.$$
(3.12)

Equation (3.12) is a constructive statement of the sampling theorem: It shows explicitly how to construct the (by assumption) band-limited function r(t) from its samples. The sinc functions are the interpolators that fill in the time gaps between samples with a wave that has no frequencies above  $\pi/T$ . A plot of the impulse response of this "ideal" hold filter is drawn in Fig. 3.7 from the formula of (3.11).

There is one serious drawback to the extrapolating signal given by (3.11). Because  $\ell(t)$  is the impulse response of the ideal low-pass filter  $L(j\omega)$ , it follows that this filter is noncausal because  $\ell(t)$  is nonzero for t < 0.  $\ell(t)$ starts at  $t = -\infty$  when the impulse that triggers it does not occur until t = 0! In many communications problems the interpolated signal is not needed until well after the samples are acquired, and the noncausality can be overcome by adding a phase lag,  $e^{-j\omega\lambda}$ , to  $L(j\omega)$ , which adds a *delay* to the filter and to the signals processed through it. In feedback control systems, a large delay is usually disastrous for stability, so we avoid such approximations to this function and use something else, like the polynomial

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holds, of which the zero-order hold already mentioned in connection with the ADC is the most elementary.

In Section 3.2 we introduced the zero-order hold as a model for the storage register in an A/D converter that maintains a constant signal value between samples. We showed in (3.7) that it has the transfer function

$$ZOH(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega}.$$
(3.13)

We can discover the frequency properties of ZOH by expressing (3.13) in magnitude and phase form. To do this, we factor out  $e^{-j\omega T/2}$  and multiply and divide by 2j to write the transfer function in the form

$$ZOH(j\omega) = e^{-j\omega T/2} \left\{ \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right\} \frac{2j}{j\omega}.$$

The term in brackets is recognized as the sine, so this can be written

$$ZOH(j\omega) = Te^{-j\omega T/2} \frac{\sin(\omega T/2)}{\omega T/2}$$

and, using the definition of the sinc function,

$$ZOH(j\omega) = e^{-j\omega T/2} T\operatorname{sinc}(\omega T/2).$$
(3.14)

Thus the effect of the zero-order hold is to introduce a phase shift of  $\omega T/2$ , corresponding to a time delay of T/2 seconds, and to multiply the gain by a function with the magnitude of  $\operatorname{sinc}(\omega T/2)$ . A plot of the magnitude is shown in Fig. 3.8, which illustrates the fact that although the zero-order hold is a low-pass filter, it has a cut-off frequency well beyond the Nyquist frequency. The magnitude function is

$$|ZOH(j\omega)| = T \left| \operatorname{sinc} \frac{\omega T}{2} \right|, \qquad (3.15)$$

which slowly gets smaller as  $\omega$  gets larger until it is zero for the first time at  $\omega = \omega_s = 2\pi/T$ . The phase is

$$\angle ZOH(j\omega) = \frac{-\omega T}{2}, \qquad (3.16)$$

plus the 180° shifts where the sinc function changes sign.



Figure 3.8 (a) Magnitude and (b) phase of polynomial hold filters.

We can now give a complete analysis of the sample-and-hold circuit of Fig. 3.3 for a sinusoidal input r(t) in both the time and the frequency domains. We consider first the time domain, which is simpler, being just an exercise in construction. For purposes of illustration, we will use r(t) = $3\sin(50t + \pi/6)$  as plotted in Fig. 3.9.

If we sample r(t) at the instants kT where the sampling frequency is  $\omega_s = 2\pi/T = 200\pi$  and T = 0.01, then the plot of the resulting  $r_h(kT)$  is as shown in Fig. 3.9. Notice that although the input is a single sinusoid, the output is clearly *not* sinusoidal. Thus it is not possible to describe this system by a transfer function, because the fundamental property of linear, constant systems is that a sinusoid input produces an output that is a sinusoid of the same frequency, and the complex ratio of the amplitudes and phases is the transfer function. The sample-and-hold system is linear but time varying. In the frequency domain, it is clear that the output  $r_h(t)$  contains more than one frequency, and a complete analysis requires that we compute the amplitudes and phases of them all. However, in the application to control systems, the output of the hold will typically be applied to a dynamical system that is of low-pass character; thus the most important component in  $r_h(t)$  is the fundamental harmonic, at  $\omega_o = 50$  rad/sec in this case. Also, in the important field of digital filtering, one is usually using the digital filter to



Figure 3.9 Plot of  $3\sin(50t+\pi/3)$  and the output of a sample-and-hold with sample period T = 0.01.

replace an analog filter, and it is the fundamental that represents the signal component in the output. The other harmonics that appear in the output are *impostors*, posing as signal when they are really unwanted consequences of the sample-and-hold process. In any event, we can proceed to analyze  $r_h(t)$  for all its harmonics and select out the fundamental component when it makes sense, either by analysis or, in the implementation, by a low-pass anti-impostor filter.

First, we need the spectrum of r(t). Because a sinusoid can be decomposed into two exponentials, we consider the Fourier transform of  $v(t) = e^{j\omega_o t + j\phi}$ . For this we have

$$V(j\omega) = \int_{-\infty}^{\infty} e^{(j\omega_0 t + j\phi)} e^{-j\omega t} dt.$$
(3.17)

This integral does not converge in any obvious way, but we can approach it from the back door, as it were. Consider again the impulse,  $\delta(t)$ . The direct transform of this object is easy, considering the sifting property, as follows:

$$\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1,$$

Now the general form of the inverse Fourier transform is given by the expression

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega.$$

If we apply the inverse transform integral to the impulse and its transform, we take  $f(t) = \delta$  and  $F(j\omega) = 1$  with the result

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega.$$

However, except for notation and a simple change of variables, this is exactly the integral we needed to evaluate the spectrum of the single exponential. If we exchange t with  $\omega$  the integral reads

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} dt.$$

Equation 3.17 is of this form

$$V(j\omega) = \int_{-\infty}^{\infty} e^{(j\omega_o t + j\phi)} e^{-j\omega t} dt$$
$$= e^{j\phi} \int_{-\infty}^{\infty} e^{jt(\omega_o - \omega)} dt$$
$$= 2\pi e^{j\phi} \delta(\omega - \omega_o).$$
(3.18)

At the last step in this development, the sign of the argument in the delta function was changed, because  $\delta$  is an even function and the order is more natural as  $(\omega - \omega_o)$  rather than the opposite. We can now express the spectrum of  $r(t) = A \cos(\omega_o t + \phi)$  in terms of impulse functions. It is

$$R(j\omega) = \pi A[e^{j\phi}\delta(\omega - \omega_o) + e^{-j\phi}\delta(\omega + \omega_o)]$$

and consists of two impulses at  $\omega_o$  and  $-\omega_o$  of intensity  $\pi A$  and phase  $\phi$  and  $-\phi$ , respectively. A sketch of this spectrum is shown in Fig 3.10(a) for  $A = 1/\pi$ . We represent the impulses by arrows whose heights are proportional to the intensities of the impulses.

After sampling, as we saw in (3.9), the spectrum of  $R^*$  is directly derived from that of R as the sum of multiple copies of that of R shifted by  $n2\pi/T$ for all integers n and multiplied by 1/T. A plot of the result is shown in Fig.



Figure 3.10 Plot of the spectra of (a)R, (b) $R^*$ , and (c) $R_h$ .

3.10(b) normalized by T. Finally, to find the spectrum of  $R_h$ , we need only multiply the spectrum of  $R^*$  by the transfer function of the ZOH, which is

$$ZOH(j\omega) = Te^{-j\omega T/2} \operatorname{sinc}(\omega T/2).$$

Thus the spectrum of  $R_h$  is also a sum of an infinite number of terms, but now with intensities modified by the sinc function and phases shifted by the delay function  $\omega T/2$ . These intensities are plotted in Fig. 3.10(c). Naturally, when all the harmonics included in  $R_h$  are converted to their time functions and added, they sum to the piecewise, constant, staircase function plotted earlier in Fig. 3.9.

If we want a best approximation to  $r_h$  using only one sinusoid, we need only take out the first or fundamental harmonic from the components of  $R^*$ . This component has phase shift  $\phi$  and amplitude  $A \operatorname{sinc}(\omega T/2)$ . In the time domain, the corresponding sinusoid is given by

$$v_1(t) = A[\operatorname{sinc}(\omega T/2)]\sin[\omega_o(t - \frac{1}{2})].$$

A plot of this approximation for the signal from Fig 3.9 is given in Fig. 3.11 along with both the original input and the sampled-and-held output to show the nature of the approximation.



Figure 3.11 Plot of the output of the sample and hold and the first harmonic approximation.

In control design, we can frequently achieve a satisfactory design for a sampled-data system by approximating the sample and hold with a continuous transfer function corresponding to the delay of T/2. The controller design is then done in the continuous domain but is implemented by computing a discrete equivalent. More discussion of this technique; sometimes called *emulation*, will be given in Chapter 5, where some examples illustrate the results.

**First-Order Hold.** A more complex circuit that might be preferred to the zero-order hold is the first-order hold, which, as suggested by its name, extrapolates data between sampling periods by a first-order polynomial, a straight line. We can, as with the zero-order hold, compute the transfer function of the filter that, acting with impulse sampling, produces the action of a first-order hold. A sketch of the response to a unit pulse will be helpful and is shown in Fig. 3.12. The first line, rising from 1 to 2 over the period from t = 0 to t = T, is an extrapolation of the line between the points at t = -T (where the sample was, by assumption, equal to zero) and at t = 0, where the sample was unity. Likewise, the line going in the negative

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direction from 0 at t = T to -1 at t = 2T is the extrapolation of the line between the points at t = 0 and t = T. The Laplace transform of this h(t)is

$$FOH(s) = T\left(\frac{1-e^{-Ts}}{Ts}\right)^2 (Ts+1).$$
 (3.19)

The magnitude and phase of  $FOH(j\omega)$  were plotted in Fig. 3.8 to permit comparison with the characteristics of ZOH. Note that for low frequencies (below  $\pi/2T$ ) the first-order hold has significantly less phase lag than does the zero-order hold. However, the FOH has much more amplitude distortion than the ZOH does. No clear guidelines seem to exist indicating that one is preferred to the other from the standpoint of an ideal control-system response; however, the increased hardware complexity of the FOH implementation almost always dictates that the ZOH be used. In some cases (e.g., hydraulic systems), the steps from the ZOH have been found to be detrimental to the control actuator, and the solution has been to simply add a low-pass filter to the ZOH output with a time constant on the order of the sample period. Following our earlier definition, this would be called an anti-imposter filter; some prefer the term "smoothing filter." The filter is to be considered part of the plant, and its effects are taken into account in the design.



Figure 3.12 Impulse response of first-order-hold filter.

Obviously many other, more sophisticated, data extrapolators can and will be designed. However, rarely is additional complexity justified in terms of improved performance in feedback control.

### 3.5 BLOCK-DIAGRAM ANALYSIS

We have thus far talked mainly about discrete, continuous, and sampled signals. To analyze a feedback system that contains a digital computer, we need to be able to compute the transforms of output signals of systems that contain sampling operations in various places, including feedback loops, in the block diagram. The technique for doing this is a simple extension of the ideas of block-diagram analysis of systems that are all continuous or all discrete, but one or two rules need to be carefully observed to assure success. First, we should review the facts of sampled-signal analysis.

We represent the process of sampling a continuous signal and holding it by impulse modulation followed by low-pass filtering. For example, the system of Fig. 3.13 leads to

$$E(s) = R^*(s)H(s),$$
  
 $U(s) = E^*(s)G(s).$  (3.20)

The result of impulse modulation of continuous-time signals like e(t) and u(t) is to produce a series of sidebands as given in (3.9) and plotted in Fig. 3.4, which result in periodic functions of frequency. If the transform of the signal to be sampled is a product of a transform that is already periodic of period  $2\pi/T$ , and one that is not, as in  $U(s) = E^*(s)G(s)$ , where  $E^*(s)$  is periodic and G(s) is not, we can show that  $E^*(s)$  comes out as a factor of the result. This is the most important relation for the block-diagram analysis of sampled-data systems, namely,<sup>4</sup>

$$U^*(s) = (E^*(s)G(s))^* = E^*(s)G^*(s).$$
(3.21)

We can prove (3.21) either in the frequency domain, using (3.9), or in the time domain, using (3.1) and convolution. We will use (3.9) here. If U(s) =

<sup>&</sup>lt;sup>4</sup>We of course assume the existence of  $U^*(s)$ , which is assured if G(s) tends to zero as s tends to infinity at least as fast as 1/s. We must be careful to avoid impulse modulation of impulses, for  $\delta(t)\delta(t)$  is undefined.



Figure 3.13 A cascade of samplers and filters.

 $E^*(s)G(s)$ , then by definition we have

$$U^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} E^*(s - jn\omega_s)G(s - jn\omega_s); \qquad (3.22)$$

but  $E^*(s)$  is

$$E^*(s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} E(s - jk\omega_s),$$

so that

$$E^*(s - jn\omega_s) = \frac{1}{T} \sum_{k=-\infty}^{\infty} E(s - jk\omega_s - jn\omega_s).$$
(3.23)

Now in (3.23) we can let  $k = \ell - n$  to get

$$E^*(s - jn\omega_s) = \frac{1}{T} \sum_{\ell = -\infty}^{\infty} E(s - j\ell\omega_s)$$
$$= E^*(s).$$
(3.24)

In other words, because  $E^*$  is already periodic, shifting it an integral number of periods leaves it unchanged. Substituting (3.24) into (3.22) yields

$$U^*(s) = E^*(s) \frac{1}{T} \sum_{-\infty}^{\infty} G(s - jn\omega_s)$$
  
=  $E^*(s)G^*(s)$ . QED (3.25)

Note especially what is not true. If U(s) = E(s)G(s), then  $U^*(s) \neq E^*(s)G^*(s)$  but rather  $U^*(s) = (EG)^*(s)$ . The periodic character of  $E^*$  in (3.21) is crucial.

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Figure 3.14 Block diagram of digital control as a sampled-data system.

The final result we require is that, given a sampled-signal transform such as  $U^*(s)$ , we can find the corresponding z-transform simply by letting  $e^{sT} = z$  or

$$U(z) = U^*(s) |_{e^{sT} = z}.$$
(3.26)

There is an important time-domain reflection of (3.26). The inverse Laplace transform of  $U^*(s)$  is the sequence of *impulses* with intensities given by u(kT); the inverse z-transform of U(z) is the sequence of values u(kT). Conceptually, sequences of values and the corresponding z-transforms are easy to think about as being processed by a computer program, whereas the model of sampling as a sequence of impulses is what allows us to analyze a discrete system embedded in a continuous world. Of course, the impulse modulator must *always* be followed by a low-pass circuit (hold circuit) in the physical world. Note that (3.26) can also be used in the other direction to obtain  $U^*(s)$ , the Laplace transform of the train of impulses, from a given U(z).

These rules of analysis can be illustrated by example. Consider the block diagram given in Fig. 3.14 taken from Fig. 1.1. In Fig. 3.14 we have modeled the A/D converter plus computer program plus D/A converter as an impulse modulator [which takes the samples from e(t)], a computer program that processes these samples, and a zero-order hold that constructs the piecewise, constant output of the D/A converter from the impulses of  $m^*$ . In the actual computer we assume that the samples of e(t) are manipulated by a difference equation whose input-output effect is described by the z-transform D(z). These operations are represented in Fig. 3.14 as if they were performed on impulses, and hence the transfer function is  $D^*(s)$  according to (3.26). Finally, the manipulated impulses,  $m^*(t)$ , are applied to the zero-order hold from which the piecewise-constant-control signal u(t) comes. In reality, of course, the computer operates on the sample values of e(t) and the piecewise-constant output is generated via a storage register and a D/A converter. The

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impulses provide us with a convenient, consistent, and effective model of the processes to which Laplace-transform methods can be applied.

From the results given thus far, we can write relations among Laplace transforms as

$$E(s) = R - Y, \tag{3.27}$$

$$M^*(s) = E^*D^*, (3.28)$$

$$U = M^* \left[ \frac{1 - e^{Ts}}{s} \right], \tag{3.29}$$

$$Y = GU. \tag{3.30}$$

The usual idea is to relate the discrete output,  $Y^*$ , to the discrete input,  $R^*$ . Suppose we sample each of these equations by using the results of (3.5) to "star" each transform. The equations are<sup>5</sup>

$$E^* = R^* - Y^*, (3.31)$$

$$M^* = E^* D^*, (3.32)$$

$$U^* = M^*, (3.33)$$

$$Y^* = [GU]^*. (3.34)$$

Now (3.34) indicates that we need U, not  $U^*$ , to compute  $Y^*$ , so we must back up to substitute (3.29) into (3.34):

$$Y^* = \left[ GM^* \left( \frac{1 - e^{Ts}}{s} \right) \right]^*. \tag{3.35}$$

Taking out the periodic parts, which are those in which s appears only as  $e^{sT}$  [which include  $M^*(s)$ ], we have

$$Y^* = (1 - e^{-T_s}) M^* \left(\frac{G}{s}\right)^*.$$
(3.36)

Substituting from (3.28) for  $M^*$  gives

$$Y^* = (1 - e^{-Ts})E^*D^*(G/s)^*.$$
(3.37)

<sup>5</sup>In sampling (3.29) we obtain (3.33) by use of the convention given in footnote 2, which follows (3.5) for impulse modulation of discontinuous functions. From the time-domain operation of the zero-order hold, it is clear that the samples of u and m are the same, and then from this (3.33) follows.

And substituting (3.31) for  $E^*$  yields

$$Y^* = (1 - e^{-Ts})D^*(G/s)^*[R^* - Y^*].$$
(3.38)

If we call

 $(1 - e^{-Ts})D^*(G/s)^* = H^*,$ 

then we can solve (3.38) for  $Y^*$ , obtaining

$$Y^* = \frac{H^*}{1+H^*}R^*.$$
 (3.39)

**Example 3.1:** These equations can be illustrated with a simple example. Suppose that our plant has the first-order transfer function

$$G(s) = \frac{a}{s+a}, \qquad (3.40)$$

that the computer program corresponds to a discrete integrator

$$u(kT) = u(kT - T) + K_0 e(kT), \qquad (3.41)$$

and that the computer D/A holds the output constant so that the zero-order hold is the correct model. Suppose we select the sampling period T so that  $e^{-aT} = \frac{1}{2}$ . We wish to compute the components of  $H^*$  given in (3.39). For the computer program we have the transfer function of (3.41), which in terms of z is

$$D(z) = \frac{U(z)}{E(z)} = \frac{K_0}{1 - z^{-1}} = \frac{K_0 z}{z - 1}.$$

Using (3.26), we get the Laplace-transform form

$$D^*(s) = \frac{K_0 e^{sT}}{e^{sT} - 1}.$$
(3.42)

For the plant and zero-order-hold we require<sup>6</sup>

$$(1 - e^{-Ts})(G(s)/s)^* = (1 - e^{-Ts})\left(\frac{a}{s(s+a)}\right)^*$$
$$= (1 - e^{-Ts})\left(\frac{1}{s} - \frac{1}{s+a}\right)^*.$$

Using (3.5), we have

$$(1 - e^{-Ts})(G(s)/s)^* = (1 - e^{-Ts})\left(\frac{1}{1 - e^{-Ts}} - \frac{1}{1 - e^{-aT}e^{-Ts}}\right).$$

Because we assumed (for simplicity) that  $e^{-aT} = \frac{1}{2}$ , this reduces to

$$(1 - e^{-Ts})(G(s)/s)^* = \frac{(1/2)e^{-Ts}}{1 - (1/2)e^{-Ts}}$$
$$= \frac{1/2}{e^{Ts} - 1/2}.$$
 (3.43)

Combining (3.43) and (3.42), then, in this case, we obtain

$$H^*(s) = \frac{K_0}{2} \frac{e^{sT}}{(e^{sT} - 1)(e^{sT} - 1/2)}.$$
 (3.44)

Equation (3.44) can now be used in (3.39) to find the closed-loop transfer function from which the dynamic and static responses can be studied, as a function of  $K_0$ , the program gain. We note also that beginning with (3.27), we can readily calculate that

$$Y(s) = R^* \frac{D^*}{1 + H^*} \frac{(1 - e^{-Ts})}{s} G(s).$$
(3.45)

Equation (3.45) shows how to compute the response of this system in between sampling instants. For a given r(t), the starred terms in (3.45) and the  $(1 - e^{-Ts})$ -term correspond to a train of impulses whose individual values can be computed by expanding in powers of  $e^{-Ts}$ . These impulses are applied to G(s)/s, which is the step response of the plant. Thus, between sampling instants, we will see segments of the plant step response.

<sup>&</sup>lt;sup>6</sup>Notice the similarity with (2.39) and Example 2.8.



Figure 3.15 A simple system that does not have a transfer function.

With the exception of the odd-looking forward transfer function, (3.39) looks like the familiar feedback formula: forward-over-one-plus-feedback. Unfortunately, the sequence of equations by which (3.39) was computed was a bit haphazard, and such an effort might not always succeed. Another example will further illustrate the problem.

**Example 3.2:** Consider the block diagram of Fig. 3.15, which has only one sampling operation. This situation can arise if the error sensor has significant dynamics that precede the sampling action of the A/D converter. In this case H(s) represents the sensor dynamics. Again, we write the equations (all symbols are Laplace transforms):

$$E = R - Y, \tag{3.46}$$

$$U = HE, \tag{3.47}$$

$$Y = U^*G; \tag{3.48}$$

and again we sample

$$E^* = R^* - Y^*, \tag{3.49}$$

$$U^* = (HE)^*, (3.50)$$

$$Y^* = U^* G^*. (3.51)$$

How do we solve? In (3.50) we need E, not  $E^*$ . So we must go back to (3.46):

$$U^* = (H(R - Y))^*$$
  
= (HR)\* - (HY)\*.

Using (3.48) for Y, we have

$$U^* = (HR)^* - (HU^*G)^*.$$

Taking out the periodic  $U^*$  in the second term on the right gives

$$U^* = (HR)^* - U^*(HG)^*.$$

Solving, we get

$$U^* = \frac{(HR)^*}{1 + (HG)^*},$$
(3.52)

From (3.45), we can solve for  $Y^*$ :

$$Y^* = \frac{(HR)^*}{1 + (HG)^*} G^*.$$
(3.53)

Equation (3.53) displays a curious fact. The transform of the input is bound up with H(s) and cannot be divided out to give a transfer function! This system displays an important fact that all our facile manipulations of samples, and so on, might cause us to neglect: A sampled-data system is time varying. The response depends on the time relative to the sampling instant at which the signal is applied. Only when the input samples alone are required to generate the output samples can we obtain a discrete transfer function. The time variation occurs on the taking of samples. In general, as in Fig. 3.15, the entire input signal r(t) is involved in the system response, and the transfer-function concept fails. Even in the absence of a transfer function, however, the techniques developed here permit study of stability and response to specific inputs such as step, ramp, or sinusoidal signals.

We need to know the general rules of block-diagram analysis. In solving Fig. 3.15 we found ourselves working with U, the signal that was sampled. This is in fact the key to the problem. Given a block diagram with several samplers, always select the variables at the inputs to the samplers as the unknowns. Being sampled, these variables have periodic transforms and will always "come free" after the equation sampling process and give a set of starred variables for which we can solve.

**Example 3.3:** Consider as a final example the block diagram drawn in Fig. 3.16. We select E and M as independent variables



Figure 3.16 A final example for transfer-function analysis of sampled-data systems.

and write

$$E(s) = R - M^* G_2, (3.54)$$

$$M(s) = E^* HG_1. (3.55)$$

Next we sample these signals, and use the "if periodic, then out" rule from (3.21):

$$E^* = R^* - M^* G_2^*, (3.56)$$

$$M^* = E^* (HG_1)^*. (3.57)$$

We solve these equations by substituting for  $M^*$  in (3.56) from (3.57):

$$E^* = R^* - E^* (HG_1)^* G_2^*$$
  
=  $\frac{R^*}{1 + (HG_1)^* G_2^*}$ . (3.58)

To obtain Y we use the equation

$$Y = E^* H$$
  
=  $\frac{R^* H}{1 + (HG_1)^* G_2^*},$  (3.59)

and

$$Y^* = \frac{R^* H^*}{1 + (HG_1)^* G_2^*}.$$
(3.60)

In this case we have a transfer function. Why? Because only the samples of the external input are used to cause the output. To obtain

the z-transform of the samples of the output, we would let  $e^{sT} = z$  in (3.60). From (3.59) we can solve for the continuous output, which consists of impulses applied to H(s) in this case.

### 3.6 SUMMARY

In this chapter we have considered the analysis of mixed systems that are partly discrete and partly continuous, taking the continuous point of view. We used impulse modulation to represent the sampling process, and we derived the transfer functions of filters that would represent zero-order and first-order hold action. We showed that the transform of a sampled signal is periodic and that sampling introduces aliasing, which may be interpreted in both the frequency and the time domains. From the condition of no aliasing we derived the sampling theorem.

Finally, we presented the block-diagram analysis of sampled-data systems, showing that proper techniques, including the treatment of the sampler inputs as unknowns, would lead to solution for the output transforms. However, we also found that not every sampled-data system has a transfer function.

# PROBLEMS AND EXERCISES

**3.1** Derive (3.45).

è

3.2 Sketch a signal that shows hidden oscillations.

**3.3** Consider the circuit of Fig. 3.17. By plotting the response to a signal that is zero for all sample instants except t = 0 and that is 1.0 at t = 0, show that this circuit implements a first-order hold.

**3.4** Sketch the step response y(t) of the system shown in Fig. 3.18 for  $k = \frac{1}{2}$ , 1, and 2.



Figure 3.17 Block diagram of a first-order hold.



Figure 3.18 A sampled-data system.



Figure 3.19 A general hold circuit.

**3.5** Sketch the response of a *second*-order hold circuit to a step unit. What might be the major disadvantage of this data extrapolator? See Fig. 3.19.

**3.6** Find the transform of the output Y(s) and its samples  $Y^*(s)$  for the block diagrams shown in Fig. 3.20. Indicate whether a transfer function exists in each case.

**3.7** Assume the following transfer functions are preceded by a zero-order hold and compute the resulting discrete transfer functions.

a) 
$$G_1(s) = 1/s^2$$

b) 
$$G_2(s) = e^{-1.5s}/(s+1)$$





Figure 3.20 Block diagrams of sampled data systems. (a) Single loop; (b) multiple loop.



Figure 3.21 Block diagrams showing the modified z-transform.

c)  $G_3(s) = 1/s(s+1)$ d)  $G_4(s) = e^{-1.5s}/s(s+1)$ e)  $G_5(s) = 1/(s^2-1)$ 

**3.8** One technique for examining the response of a sampled-data system between sampling instants is to shift the response a fraction of a sample period to the left and sample the result. The effect is as shown in the block diagram of Fig. 3.21 and is described by the equation

$$Y^*(s,m) = R^*(s)\mathcal{Z}\{G(s)e^{mTs}\}.$$

The function  $\mathcal{Z}{G(s)e^{mTs}}$  is called the modified z-transform of G(s). Let

$$G(s) = \frac{1}{s+1}, \ T = 1, \ R(s) = \frac{1}{s},$$

- a) Compute y(t) by constructing the samples y(kT) from  $Y^*(s)$  and observing that with this plant, y(t) is an exponential decay with unit time constant over the intersample interval. Sketch the response for five sample intervals.
- b) Let  $m = \frac{1}{2}$  and compute the samples corresponding to  $Y^*(s;m)$  [or Y(z;m)]. Plot these on the same sketch as the samples of part (a) and verify that the midway points have been found.

**3.9** Show how to construct a signal of "hidden oscillations," even one that grows in an unstable fashion but whose sample values are zero. Where in the *s*-plane are the poles of the transforms of your signal(s)?