B.1 PROPERTIES OF z-TRANSFORMS

Let $\mathcal{F}_i(s)$ be the Laplace transform of $f_i(t)$ and $F_i(z)$ be the z-transform of $f_i(kT)$.

Table B.1

Number	Laplace Transform	Samples	z-Transform	Comment
—	$\mathcal{F}_i(s)$	$f_i(kT)$	$F_i(z)$	
1	$\alpha\mathcal{F}_i(s) + \beta\mathcal{F}_2(s)$	$\alpha f_1(kT)\beta f_2(kT)$	$\alpha F_1(z)\beta F_2(z)$	The z-transform is linear
2	$\mathcal{F}_1(e^{Ts})\mathcal{F}_2(s)$	$\sum_{\ell=-\infty}^{\infty} f_1(\ell T)f_2(kT - \ell T)$	$F_1(z)F_2(z)$	Discrete convolution corresponds to product of z-transforms
3	$e^{+nTs}\mathcal{F}(s)$	$f(kT + nT)$	$z^n F(z)$	Shift in time
4	$\mathcal{F}(s + a)$	$e^{-akT}f(kT)$	$F(e^{aT}z)$	Shift in frequency
5	—	$\lim_{k\to\infty} f(kT)$	$\lim_{z\to 1}(z - 1)F(z)$	If all poles of $(z-1)F(z)$ are inside the unit circle and $F(z)$ converges for $1 \le |z|$
6	$\mathcal{F}(s/\omega_n)$	$f(\omega_n kT)$	$F(z; \omega_n T)$	Time and frequency scaling
7	—	$f_1(kT)f_2(kT)$	$\dfrac{1}{2\pi j} \oint_{c_3} F_1(\zeta)F_2(z/\zeta)\dfrac{d\zeta}{\zeta}$	Time product
8	$\mathcal{F}_3(s) = \mathcal{F}_1(s)\mathcal{F}_2(s)$	$\displaystyle\int_{-\infty}^{\infty} f_1(\tau)f_2(kT - \tau)d\tau$	$F_3(z)$	Continuous convolution does *not* correspond to product of z-transforms

B.2 TABLE OF z-TRANSFORMS

$\mathcal{F}(s)$ is the Laplace transform of $f(t)$ and $F(z)$ is the z-transform of $f(nT)$. Unless otherwise noted, $f(t) = 0$, $t < 0$ and the region of convergence of $F(z)$ is outside a circle $r < |z|$ such that all poles of $F(z)$ are inside r.

Table B.2

Number	$\mathcal{F}(s)$	$f(nT)$	$F(z)$
1	—	$1,\ n=0;\ 0\ n \neq 0$	1
2	—	$1,\ n=k;\ 0\ n \neq k$	z^{-k}
3	$\dfrac{1}{s}$	$1(nT)$	$\dfrac{z}{z-1}$
4	$\dfrac{1}{s^2}$	nT	$\dfrac{Tz}{(z-1)^2}$
5	$\dfrac{1}{s^3}$	$\dfrac{1}{2!}(nT)^2$	$\dfrac{T^2}{2}\dfrac{z(z+1)}{(z-1)^3}$
6	$\dfrac{1}{s^4}$	$\dfrac{1}{3!}(nT)^3$	$\dfrac{T^3}{6}\dfrac{z(z^2+4z+1)}{(z-1)^4}$
7	$\dfrac{1}{s^m}$	$\lim\limits_{a\to 0}\dfrac{(-1)^{m-1}}{(m-1)!}\dfrac{\partial^{m-1}}{\partial a^{m-1}}e^{-anT}$	$\lim\limits_{a\to 0}\dfrac{(-1)^{m-1}}{(m-1)!}\dfrac{\partial^{m-1}}{\partial a^{m-1}}\dfrac{z}{z-e^{-aT}}$
8	$\dfrac{1}{s+a}$	e^{-anT}	$\dfrac{z}{z-e^{-aT}}$
9	$\dfrac{1}{(s+a)^2}$	nTe^{-anT}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
10	$\dfrac{1}{(s+a)^3}$	$\dfrac{1}{2}(nT)^2e^{-anT}$	$\dfrac{T^2}{2}e^{-aT}\dfrac{z(z+e^{-aT})}{(z-e^{-aT})^3}$
11	$\dfrac{1}{(s+a)^m}$	$\dfrac{(-1)^{m-1}}{(m-1)!}\dfrac{\partial^{m-1}}{\partial a^{m-1}}(e^{-anT})$	$\dfrac{(-1)^{m-1}}{(m-1)!}\dfrac{\partial^{m-1}}{\partial a^{m-1}}\dfrac{z}{z-e^{-aT}}$
12	$\dfrac{a}{s(s+a)}$	$1-e^{-anT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

Number	$\mathcal{F}(s)$	$f(nT)$	$F(z)$
13	$\dfrac{a}{s^2(s+a)}$	$\dfrac{1}{a}(anT - 1 + e^{-anT})$	$\dfrac{z[(aT - 1 + e^{-aT})z + (1 - e^{-aT} - aTe^{-aT})]}{a(z-1)^2(z - e^{-aT})}$
14	$\dfrac{b-a}{(s+a)(s+b)}$	$(e^{-anT} - e^{-bnT})$	$\dfrac{(e^{-aT} - e^{-bT})z}{(z - e^{-aT})(z - e^{-bT})}$
15	$\dfrac{s}{(s+a)^2}$	$(1 - anT)e^{-anT}$	$\dfrac{z[z - e^{-aT}(1 + aT)]}{(z - e^{-aT})^2}$
16	$\dfrac{a^2}{s(s+a)^2}$	$1 - e^{-anT}(1 + anT)$	$\dfrac{z[z(1 - e^{-aT} - aTe^{-aT}) + e^{-2aT} - e^{-aT} + aTe^{-aT}]}{(z-1)(z - e^{-aT})^2}$
17	$\dfrac{(b-a)s}{(s+a)(s+b)}$	$be^{-bnT} - ae^{-anT}$	$\dfrac{z[z(b-a) - (be^{-aT} - ae^{-bT})]}{(z - e^{-aT})(z - e^{-bT})}$
18	$\dfrac{a}{s^2 + a^2}$	$\sin anT$	$\dfrac{z \sin aT}{z^2 - (2\cos aT)z + 1}$
19	$\dfrac{s}{s^2 + a^2}$	$\cos anT$	$\dfrac{z(z - \cos aT)}{z^2 - (2\cos aT)z + 1}$
20	$\dfrac{s+a}{(s+a)^2 + b^2}$	$e^{-anT}\cos bnT$	$\dfrac{z(z - e^{-aT}\cos bT)}{z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT}}$
21	$\dfrac{b}{(s+a)^2 + b^2}$	$e^{-anT}\sin bnT$	$\dfrac{ze^{-aT}\sin bT}{z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT}}$
22	$\dfrac{a^2 + b^2}{s((s+a)^2 + b^2)}$	$1 - e^{-anT}\left(\cos bnT + \dfrac{a}{b}\sin bnT\right)$	$\dfrac{z(Az + B)}{(z-1)(z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT})}$ $A = 1 - e^{-aT}\cos bT - \dfrac{a}{b}e^{-aT}\sin bT$ $B = e^{-2aT} + \dfrac{a}{b}e^{-aT}\sin bT - e^{-aT}\cos bT$

APPENDIX C

A Few Results from Matrix Analysis

Although we assume the reader has some acquaintance with linear equations and determinants, there are a few results of a more advanced character that even elementary control-system theory requires, and these are collected here for reference in the text. For further study, a good choice is Strang (1976).

C.1 DETERMINANTS AND THE MATRIX INVERSE

The determinant of a product of two square matrices is the product of their determinants:

$$\det \mathbf{AB} = \det \mathbf{A} \det \mathbf{B}. \tag{C.1}$$

If a matrix is diagonal, then the determinant is the product of the elements on the diagonal.

If the matrix is partitioned with square elements on the main diagonal, then an extension of this result applies, namely,

$$\det \begin{bmatrix} \mathbf{A} & 0 \\ \mathbf{B} & \mathbf{C} \end{bmatrix} = \det \mathbf{A} \det \mathbf{C} \quad \text{if } \mathbf{A} \text{ and } \mathbf{C} \text{ are square matrices.} \tag{C.2}$$

Suppose $\mathbf{A}$ is a matrix of dimensions $m \times n$ and $\mathbf{B}$ is of dimension $n \times m$. Let $\mathbf{I}_m$ and $\mathbf{I}_n$ be the identity matrices of size $m \times m$ and $n \times n$, respectively. Then

$$\det [\mathbf{I}_n + \mathbf{BA}] = \det [\mathbf{I}_m + \mathbf{AB}]. \tag{C.3}$$

To show this result, we consider the determinant of the matrix product

$$\det \begin{bmatrix} \mathbf{I}_m & \mathbf{0} \\ \mathbf{B} & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{A} \\ -\mathbf{B} & \mathbf{I}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{I}_m & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_n + \mathbf{BA} \end{bmatrix} = \det [\mathbf{I}_n + \mathbf{BA}].$$

But this is also equal to

$$\det \begin{bmatrix} \mathbf{I}_m & -\mathbf{A} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix} \begin{bmatrix} \mathbf{I}_m & \mathbf{A} \\ -\mathbf{B} & \mathbf{I}_n \end{bmatrix} = \det \begin{bmatrix} \mathbf{I}_m + \mathbf{AB} & \mathbf{0} \\ -\mathbf{B} & \mathbf{I}_n \end{bmatrix} = \det [\mathbf{I}_m + \mathbf{AB}],$$

and therefore these two determinants are equal to each other, which is (C.3).

If the determinant of a matrix $\mathbf{A}$ is not zero, then we can define a related matrix $\mathbf{A}^{-1}$, called "$\mathbf{A}$ inverse," which has the property that

$$\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{C.4}$$

According to property (C.1) we have

$$\det \mathbf{AA}^{-1} = \det \mathbf{A} \cdot \det \mathbf{A}^{-1} = 1,$$

or

$$\det \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}.$$

It can be shown that there is an $n \times n$ matrix called the *adjugate of* $\mathbf{A}$ with elements composed of sums of products of the elements of $\mathbf{A}^1$ and having the property that

$$\mathbf{A} \cdot \text{adj} \, \mathbf{A} = \det \mathbf{A} \cdot \mathbf{I}. \tag{C.5}$$

Thus, if the determinant of $\mathbf{A}$ is not zero, the inverse of $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \frac{\text{adj} \, \mathbf{A}}{\det \mathbf{A}}.$$

A famous and useful formula for the inverse of a combination of matrices has come to be called the *matrix inversion lemma* in the control literature. It arises in the development of recursive algorithms for estimation, as found

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[1]If $\mathbf{A}^{ij}$ is the $n-1 \times n-1$ matrix (minor) found by deleting row i and column j from $\mathbf{A}$, then the entry in row i and column j of the adj $\mathbf{A}$ is $(-1)^{i+j}\det \mathbf{A}^{ji}$.

in Chapter 8. The formula is as follows: If det $\mathbf{A}$, det $\mathbf{C}$, and det $(\mathbf{A}+\mathbf{BCD})$ are different from zero, then we have the matrix inversion lemma:

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}. \qquad (C.6)$$

The truth of (C.6) is readily confirmed if we multiply both sides by $\mathbf{A}+\mathbf{BCD}$ to obtain

$$\begin{aligned}
\mathbf{I} &= \mathbf{I} + \mathbf{BCDA}^{-1} - \mathbf{B}(\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1} \\
&\quad - \mathbf{BCDA}^{-1}\mathbf{B}(\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1} \\
&= \mathbf{I} + \mathbf{BCDA}^{-1} - [\mathbf{B} + \mathbf{BCDA}^{-1}\mathbf{B}][\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}.
\end{aligned}$$

If we subtract $\mathbf{I}$ from both sides and factor $\mathbf{BC}$ from the left on the third term, we find

$$0 = \mathbf{BCDA}^{-1} - \mathbf{BC}[\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B}][\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B}]^{-1}\mathbf{DA}^{-1},$$

which is

$$0 = 0 \text{ which was to be demonstrated.}$$

C.2 EIGENVALUES AND EIGENVECTORS

We consider the discrete dynamic system

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k, \qquad (C.7)$$

where, for purposes of illustration, we will let

$$\mathbf{\Phi} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ 1 & 0 \end{bmatrix}. \qquad (C.8)$$

If we assume that it is possible for this system to have a motion given by a geometric series such as z^k, we can assume that there is a vector $\mathbf{v}$ so that $\mathbf{x}_k$ can be written

$$\mathbf{x}_k = \mathbf{v}z^k. \qquad (C.9)$$

Substituting (C.9) into (C.7), we must find the vector $\mathbf{v}$ and the number z such that

$$\mathbf{v}z^{k+1} = \mathbf{\Phi}\mathbf{v}z^k,$$

or, multiplying by z^{-k} yields

$$\mathbf{v}z = \mathbf{\Phi v}. \tag{C.10}$$

If we collect both the terms of (C.10) on the left, we find

$$(z\mathbf{I} - \mathbf{\Phi})\mathbf{v} = 0. \tag{C.11}$$

These linear equations have a solution for a nontrivial $\mathbf{v}$ if and only if the determinant of the coefficient matrix is zero. This determinant is a polynomial of degree n in z ($\mathbf{\Phi}$ is an $n \times n$ matrix) called the *characteristic polynomial* of $\mathbf{\Phi}$, and values of z for which the characteristic polynomial is zero are roots of the characteristic equation and are called *eigenvalues* of $\mathbf{\Phi}$. For example, for the matrix given in (C.8) the characteristic polynomial is

$$\det \left\{ \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ 1 & 0 \end{bmatrix} \right\}.$$

Adding the two matrices, we find

$$\det \left\{ \begin{matrix} z - \frac{5}{6} & +\frac{1}{6} \\ -1 & z \end{matrix} \right\},$$

which can be evaluated to give

$$z\left(z - \tfrac{5}{6}\right) + \tfrac{1}{6} = \left(z - \tfrac{1}{2}\right)\left(z - \tfrac{1}{3}\right). \tag{C.12}$$

Thus the characteristic roots of this $\mathbf{\Phi}$ are $\frac{1}{2}$ and $\frac{1}{3}$. Associated with these characteristic roots are solutions to (C.11) for vectors $\mathbf{v}$, called the *characteristic* or *eigenvectors*. If we let $z = \frac{1}{2}$, then (C.11) requires

$$\left\{ \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ 1 & 0 \end{bmatrix} \right\} \begin{bmatrix} v_{11} \\ v_21 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{C.13}$$

Adding the matrices, we find that these equations become

$$\begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{C.14}$$

Equations (C.14) are satisfied by any v_{11} and v_{21} such that

$$v_{21} = 2v_{11},$$

from which we conclude that the eigenvector corresponding to $z_1 = \frac{1}{2}$ is given by

$$\mathbf{v}_1 = \begin{bmatrix} a \\ 2a \end{bmatrix}. \tag{C.15}$$

We can arbitrarily select the scale factor a in (C.15). Some prefer to make the length[2] of eigenvectors equal to one. Here we make the largest component of $\mathbf{v}$ have unit magnitude. Thus the scaled $\mathbf{v}_1$ is

$$\mathbf{v}_1 = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}. \tag{C.16}$$

In similar fashion, the eigenvector $\mathbf{v}^2$ associated with $z_2 = \frac{1}{3}$ can be computed to be

$$\mathbf{v}_2 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}.$$

Note that even if all elements of $\mathbf{\Phi}$ are real, it is possible for characteristic values and characteristic vectors to be complex.

C.3 SIMILARITY TRANSFORMATIONS

If we make a change of variables in (C.7) according to $\mathbf{x} = \mathbf{T}\boldsymbol{\xi}$, where $\mathbf{T}$ is an $n \times n$ matrix, then we start with the equations

$$\mathbf{x}_{k+1} = \mathbf{\Phi}\mathbf{x}_k,$$

and, substituting for $\mathbf{x}$, we have

$$\mathbf{T}\boldsymbol{\xi}_{k+1} = \mathbf{\Phi}\mathbf{T}\boldsymbol{\xi}_k.$$

Then, if we multiply on the left by $\mathbf{T}^{-1}$, we get the equation in $\boldsymbol{\xi}$,

$$\boldsymbol{\xi}_{k+1} = \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}\boldsymbol{\xi}_k. \tag{C.17}$$

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[2]Usually we define the length of a vector as the square root of the sum of squares of its components or, if $||\mathbf{v}||$ is the symbol for length, then $||\mathbf{v}||^2 = \mathbf{v}^T\mathbf{v}$. If $\mathbf{v}$ is complex, as will happen if z_i is complex, then we must take a conjugate, and we define $||\mathbf{v}||^2 = (\mathbf{v}^*)^T\mathbf{v}$, where $\mathbf{v}^*$ is the complex conjugate of $\mathbf{v}$.

If we define the new system matrix as $\mathbf{\Psi}$, then the new states satisfy the equations

$$\boldsymbol{\xi}_{k+1} = \mathbf{\Psi}\boldsymbol{\xi}_k,$$

where

$$\mathbf{\Psi} = \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}. \tag{C.18}$$

If we now seek the characteristic polynomial of $\mathbf{\Psi}$, we find

$$\det\left[z\mathbf{I} - \mathbf{\Psi}\right] = \det\left[z\mathbf{I} - \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}\right].$$

Because $\mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$, we can write this polynomial as

$$\det\left[z\mathbf{T}^{-1}\mathbf{T} - \mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}\right],$$

and the $\mathbf{T}^{-1}$ and $\mathbf{T}$ can be factored out on the left and right to give

$$\det\left[\mathbf{T}^{-1}[z\mathbf{I} - \mathbf{\Phi}]\mathbf{T}\right].$$

Now, using property (C.1) for the determinant, we compute

$$\det\mathbf{T}^{-1} \cdot \det\left[z\mathbf{I} - \mathbf{\Phi}\right] \cdot \det\mathbf{T},$$

which, by the equation following (C.4), gives us the final result

$$\det\left[z\mathbf{I} - \mathbf{\Psi}\right] = \det\left[z\mathbf{I} - \mathbf{\Phi}\right]. \tag{C.19}$$

From (C.19) we see that $\mathbf{\Psi}$ and $\mathbf{\Phi}$ have the same characteristic polynomials. The matrices are said to be "similar," and the transformation (C.18) is a similarity transformation.

A case of a similarity transformation of particular interest is one for which the resulting matrix $\mathbf{\Psi}$ is diagonal. As an attempt to find such a matrix, suppose we *assume* that $\mathbf{\Psi}$ is diagonal and write the transformation $\mathbf{T}$ in terms of its columns, $\mathbf{t}_i$. Then (C.18) can be expressed as

$$\mathbf{T}\mathbf{\Psi} = \mathbf{\Phi}\mathbf{T},$$
$$[\mathbf{t}_1\,\mathbf{t}_2\ldots\mathbf{t}_n]\mathbf{\Psi} = \mathbf{\Phi}[\mathbf{t}_1\,\mathbf{t}_2\ldots\mathbf{t}_n]$$
$$= [\mathbf{\Phi}\mathbf{t}_1\,\mathbf{\Phi}\mathbf{t}_2\ldots\mathbf{\Phi}\mathbf{t}_n]. \tag{C.20}$$

If we *assume* that $\boldsymbol{\Psi}$ is diagonal with elements $\lambda_1, \lambda_2, \ldots, \lambda_n$, then (C.20) can be written as

$$[\mathbf{t}_1 \, \mathbf{t}_2 \, \ldots \, \mathbf{t}_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & & & \\ \cdot & & & \\ \cdot & & \ddots & \\ \cdot & & & \lambda_n \end{bmatrix} = [\boldsymbol{\Phi}\mathbf{t}_1 \ \ \boldsymbol{\Phi}\mathbf{t}_2 \, \ldots \, \boldsymbol{\Phi}\mathbf{t}_n].$$

Multiplying the matrices on the left, we find

$$[\lambda_1 \mathbf{t}_1 \lambda_2 \mathbf{t}_2 \ldots \lambda_n \mathbf{t}_n] = [\boldsymbol{\Phi}\mathbf{t}_1 \ldots \boldsymbol{\Phi}\mathbf{t}_n]. \tag{C.21}$$

Because the two sides of (C.21) are equal, they must match up column by column, and we can write the equation for column j as

$$\lambda_j \mathbf{t}_j = \boldsymbol{\Phi}\mathbf{t}_j. \tag{C.22}$$

Comparing (C.22) with (C.10), we see that $\mathbf{t}_j$ is an eigenvector of $\boldsymbol{\Phi}$ and λ_j is an eigenvalue. We conclude that if the transformation $\mathbf{T}$ converts $\boldsymbol{\Phi}$ into a diagonal matrix $\boldsymbol{\Psi}$, then the columns of $\mathbf{T}$ must be eigenvectors of $\boldsymbol{\Phi}$ and the diagonal elements of $\boldsymbol{\Psi}$ are the eigenvalues of $\boldsymbol{\Phi}$ [which are also the eigenvalues of $\boldsymbol{\Psi}$, by (C.19)]. It turns out that if the eigenvalues of $\boldsymbol{\Phi}$ are distinct, then there are exactly n eigenvectors and they are independent; that is, we can construct a nonsingular transformation $\mathbf{T}$ from the n eigenvectors.

In the example given above, we would have

$$\mathbf{T} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 1 \end{bmatrix},$$

for which

$$\mathbf{T}^{-1} = \begin{bmatrix} 6 & -2 \\ -6 & 3 \end{bmatrix},$$

and the new diagonal system matrix is

$$\mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T} = \begin{bmatrix} 6 & -2 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} \frac{5}{6} & -\frac{1}{6} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

as advertised!

If the elements of $\mathbf{\Phi}$ are real and an eigenvalue is complex, say $\lambda_1 = \alpha + j\beta$, then the conjugate, $\lambda_1^* = \alpha - j\beta$, is also an eigenvalue because the characteristic polynomial has real coefficients. In such a case, the respective eigenvectors will be conjugate. If $\mathbf{v}_1 = \mathbf{r} + j\mathbf{i}$, then $\mathbf{v}_2 - \mathbf{v}_1^* = \mathbf{r} - j\mathbf{i}$, where $\mathbf{r}$ and $\mathbf{i}$ are matrices of real elements representing the real and imaginary parts of the eigenvectors. In such cases, it is common practice to use the real matrices $\mathbf{r}$ and $-\mathbf{i}$ as columns of the transformation matrix $\mathbf{T}$ rather than go through the complex arithmetic required to deal directly with $\mathbf{v}_1$ and $\mathbf{v}_1^*$. The resulting transformed equations are not diagonal, but rather the corresponding variables appear in the coupled equations

$$\dot{\eta} = \alpha\eta - \beta\nu, \quad \dot{\nu} = \beta\eta + \alpha\nu. \tag{C.23}$$

C.4 THE CALEY-HAMILTON THEOREM

A very useful property of a matrix $\mathbf{\Phi}$ follows from consideration of the inverse of $z\mathbf{I} - \mathbf{\Phi}$. As we saw in (C.5), we can write

$$(z\mathbf{I} - \mathbf{\Phi})\,\mathrm{adj}\,(z\mathbf{I} - \mathbf{\Phi}) = \mathbf{I}\det\,(z\mathbf{I} - \mathbf{\Phi}). \tag{C.24}$$

The coefficient of $\mathbf{I}$ on the right-hand side of (C.24) is the characteristic polynomial of $\mathbf{\Phi}$, which we can write as

$$a(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n.$$

The adjugate of $z\mathbf{I} - \mathbf{\Phi}$, on the other hand, is a *matrix* of polynomials in z, found from the determinants of the minors of $z\mathbf{I} - \mathbf{\Phi}$. If we collect the constant matrix coefficients of the powers of z, it is clear that we can write

$$\mathrm{adj}\,(z\mathbf{I} - \mathbf{\Phi}) = \mathbf{B}_1 z^{n-1} + \mathbf{B}_2 z^{n-2} + \cdots + \mathbf{B}_n,$$

and (C.24) becomes a polynomial equation with matrix coefficients. Written out, it is

$$[z\mathbf{I} - \mathbf{\Phi}][\mathbf{B}_1 z^{n-1} + \mathbf{B}_2 z^{n-2} + \cdots + \mathbf{B}_n] = z^n \mathbf{I} + a_1 \mathbf{I} z^{n-1} + \cdots + a_n \mathbf{I}. \quad (\text{C.25})$$

If we now multiply the two matrices on the left and equate coefficients of equal powers of z, we find

$$\mathbf{B}_1 = \mathbf{I},$$
$$\mathbf{B}_2 = \mathbf{\Phi}\mathbf{B}_1 + a_1\mathbf{I} = \mathbf{\Phi} + a_1\mathbf{I},$$
$$\mathbf{B}_3 = \mathbf{\Phi}\mathbf{B}_2 + a_2\mathbf{I} = \mathbf{\Phi}^2 + a_1\mathbf{\Phi} + a_2\mathbf{I},$$
$$\vdots$$
$$\mathbf{B}_n = \mathbf{\Phi}\mathbf{B}_{n-1} + a_{n-1}\mathbf{I} = \mathbf{\Phi}^{n-1} + a_1\mathbf{\Phi}^{n-2} + \cdots + a_{n-1}\mathbf{I},$$
$$0 = \mathbf{\Phi}\mathbf{B}_n + a_n\mathbf{I} = \mathbf{\Phi}^n + a_1\mathbf{\Phi}^{n-1} + a_2\mathbf{\Phi}^{n-1} + \cdots + a_n\mathbf{I}. \quad (\text{C.26})$$

Equation (C.26) is a statement that the matrix obtained when matrix $\mathbf{\Phi}$ is substituted for z in the characteristic polynomial, $a(z)$, is exactly zero! In other words, we have the Cayley-Hamilton theorem according to which

$$a(\mathbf{\Phi}) = 0. \quad (\text{C.27})$$