Digital Filters

ELEC 3004: **Digital Linear Systems**: Signals & Controls

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Lecture 7

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Schedule

Week	Date	Lecture Title
1	2-Mar	Introduction
	3-Mar	Systems Overview
2	9-Mar	Signals as Vectors & Systems as Maps
	10-Mar	[Signals]
3	16-Mar	Sampling & Data Acquisition & Antialiasing Filters
	17-Mar	[Sampling]
4	23-Mar	System Analysis & Convolution
	24-Mar	[Convolution & FT]
5	30-Mar	Discrete Systems & Z-Transforms
	31-Mar	[Z-Transforms]
6	13-Apr	Frequency Response & Filter Analysis
	14-Apr	[Filters]
7	20-Apr	Digital Filters
	21-Apr	[Digital Filters]
8	27-Apr	Introduction to Digital Control
	28-Apr	[Feedback]
9	4-May	Digital Control Design
	5-May	[Digitial Control]
10	11-May	Stability of Digital Systems
	12-May	[Stability]
11	18-May	State-Space
	19-May	Controllability & Observability
12	25-May	PID Control & System Identification
	26-May	Digitial Control System Hardware
13	31-May	Applications in Industry & Information Theory & Communications
	2-Jun	Summary and Course Review

➔ Digital Filters

- Wikipedia Says:

A **digital filter** is a system that performs mathematical operations on a sampled, discrete-time signal to reduce or enhance certain aspects of that signal.

- Basically we have a transfer function or
  … a difference equation

In the Z-domain:

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_M z^{-M}}$$

- This is a recursive form with inputs (Numerator) and outputs (Denominator)
  ➔ "IIR infinite impulse response" behaviour
- If the denominator is made equal to unity (i.e. no feedback)

➔ then this becomes an FIR or finite impulse response filter.

Flashback: LCC Difference Equations

- Start with: $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$
  - **N**: Highest derivative of the output y(t)
  - **M**: Highest derivative of the input x(t)
- $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$
- Solution strategy: Find the particular solution to:
  $\sum_{k=0}^{N} a_k y[n-k] = 0$

- Leads to a Recursive Equation:

$$y[n-k] = \frac{1}{a_0}\{ \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=0}^{N} a_k y[n-k]\}\backslash$$

➔ If N=0: Non-Recursive Solution:

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x[n-k]$$

Flashback: Two Types of Systems

- Linear shift-invariant:

$$y = \sum_{k=0}^{N-1} u[k] Z^k h$$

Z: Shift operator

$$Z \cdot [u_0, u_1, u_2, u_3, \ldots, u_{n-1}]^T = [u_{n-1}, u_0, u_1, u_2, \ldots, u_{n-2}]^T$$

- Linear time-invariant system

$$y = \sum_{k=-\infty}^{\infty} u[k] R^k h$$

R: Unit delay operator

$$R \cdot [\ldots, u_0, u_1, u_2, u_3, \ldots]^T = [\ldots, u_{-1}, u_0, u_1, \ldots]^T$$

Flashback: Impulse Response of Both Types

$$y[n] = \frac{1}{2}u[n-1] + \frac{1}{2}u[n]$$

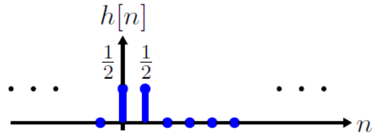
$$y[-1] = 0$$
$$y[0] = \frac{1}{2}$$
$$y[1] = \frac{1}{2}$$
$$y[2] = 0$$
$$\vdots$$

$$y[n] = \frac{1}{2}y[n-1] + u[n]$$

$$h[-1] = 0$$
$$h[0] = 1$$
$$h[1] = \frac{1}{2}$$
$$h[2] = \frac{1}{4}$$
$$\vdots$$
$$h[n] = \begin{cases} 0 & n < 0 \\ (\frac{1}{2})^n & n \geq 0 \end{cases}$$

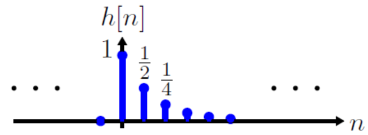
Flashback: Impulse Response of Both Types

$$y[n] = \frac{1}{2}u[n-1] + \frac{1}{2}u[n]$$



'Finite impulse response" (FIR)

$$y[n] = \frac{1}{2}y[n-1] + u[n]$$



"Infinite impulse response" (IIR)

→ Digital Filters Types

FIR

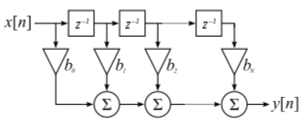
From H(z):

→ $$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-\mathrm{i}\omega} + \cdots + h_{n-1} e^{-\mathrm{i}(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega - \mathrm{i} \sum_{t=0}^{n-1} h_t \sin t\omega \end{aligned}$$

→ Filter becomes a "multiply, accumulate, and delay" system:

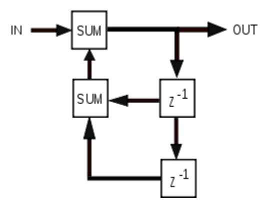
$$y(t) = \sum_{\tau=0}^{n-1} h_\tau u(t-\tau)$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-N]$$



IIR

- Impulse response function that is non-zero over an infinite length of time.



FIR Properties

- Require no feedback.
- Are inherently stable.
- They can easily be designed to be linear phase by making the coefficient sequence symmetric
- Flexibility in shaping their magnitude response
- Very Fast Implementation (based around FFTs)

- The main disadvantage of FIR filters is that considerably more computation power in a general purpose processor is required compared to an IIR filter with similar sharpness or selectivity, especially when low frequency (relative to the sample rate) cutoffs are needed.

FIR as a class of LTI Filters

- Transfer function of the filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

- Finite Impulse Response (FIR) Filters: (N = 0, no feedback)

➔From H(z):

$$\begin{aligned} H(\omega) &= h_0 + h_1 e^{-i\omega} + \cdots + h_{n-1} e^{-i(n-1)\omega} \\ &= \sum_{t=0}^{n-1} h_t \cos t\omega - i \sum_{t=0}^{n-1} h_t \sin t\omega \end{aligned}$$

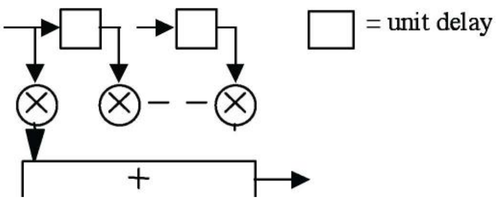
∵ H(ω) is periodic and conjugate

∴ Consider ω ∈ [0, π]

FIR Filters

- Let us consider an FIR filter of length *M*
- Order *N*=*M*-1 **(watch out!)**
- Order → number of delays

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k) = \sum_{k=0}^{M-1} h(k) x(n-k)$$



FIR Impulse Response

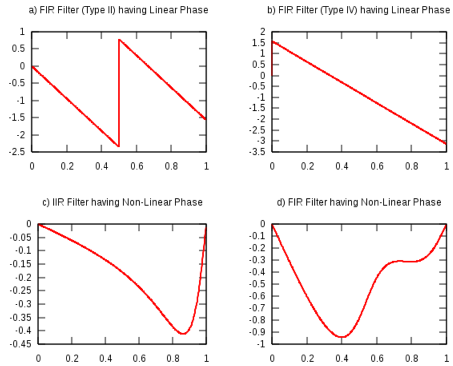
Obtain the impulse response immediately with *x*(*n*)= δ(*n*):

$$h(n) = y(n) = \sum_{k=0}^{M-1} b_k \delta(n-k) = b_n$$

- The impulse response is of finite length *M* (good!)
- FIR filters have only zeros (no poles) (as they must, N=0 !!)
  - Hence known also as **all-zero** filters
- FIR filters also known as **feedforward** or **non-recursive,** or **transversal** filters

FIR & Linear Phase

- The phase response of the filter is a linear function of frequency
- Linear phase has constant group delay, all frequency components have equal delay times. ∴ No distortion due to different time delays of different frequencies

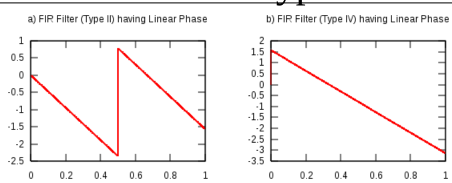


Ref: Wikipedia (Linear Phase)

- FIR Filters with:

$$\sum_{n=-\infty}^{\infty} h[n] \cdot \sin(\omega \cdot (n-\alpha)+\beta)=0$$

FIR & Linear Phase → Four Types



Ref: Wikipedia (Linear Phase)

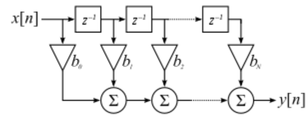
Impulse response	# coefs	$H(\omega)$	Type
$h(n)=h(M-1-n)$	Odd	$e^{-j\omega(M-1)/2}\left(h\left(\frac{M-1}{2}\right)+2\sum_{k=1}^{(M-3)/2} h\left(\frac{M-1}{2}-k\right)\cos(\omega k)\right)$	1
$h(n)=h(M-1-n)$	Even	$e^{-j\omega(M-1)/2} 2\sum_{k=1}^{(M-3)/2} h\left(\frac{M}{2}-k\right)\cos\left(\omega\left(k-\frac{1}{2}\right)\right)$	2
$h(n)=-h(M-1-n)$	Odd	$e^{-j[\omega(M-1)/2-\pi/2]}\left(2\sum_{k=1}^{(M-1)/2} h\left(\frac{M-1}{2}-k\right)\sin(\omega k)\right)$	3
$h(n)=-h(M-1-n)$	Even	$e^{-j[\omega(M-1)/2-\pi/2]} 2\sum_{k=1}^{(M-1)/2} h\left(\frac{M}{2}-k\right)\sin\left(\omega\left(k-\frac{1}{2}\right)\right)$	4

- Type 1: most versatile
- Type 2: frequency response is always 0 at ω=π
  (not suitable as a high-pass)
- Type 3 and 4: introduce a π/2 phase shift, 0 at ω=0
  (not suitable as a high-pass)

FIR Filter Design

- How to get all these coefficients?

$$H(\omega) = h_0 + h_1 e^{-i\omega} + \cdots + h_{n-1} e^{-i(n-1)\omega}$$

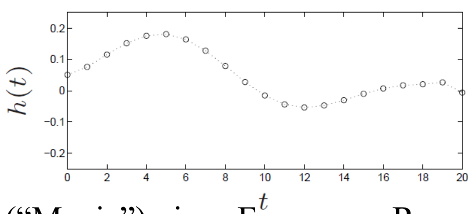


FIR Design Methods:

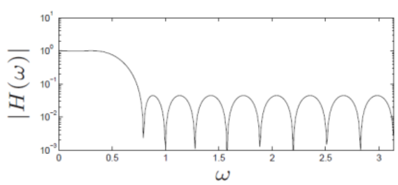
1. Impulse Response Truncation
   + Simplest
   – Undesirable frequency domain-characteristics, not very useful
2. Windowing Design Method
   + Simple
   – Not optimal (not minimum order for a given performance level)
3. Optimal filter design methods
   + "More optimal"
   – Less simple…

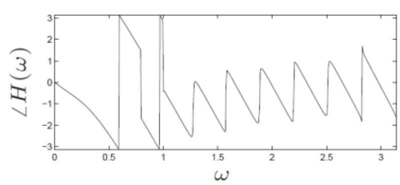
FIR Filter Design & Operation
Ex: Lowpass FIR filter

- Set Impulse response (order n = 21)
- "Determine" h(t)
  – h(t) is a 20 element vector that we'll use to as a weighted sum



- FFT ("Magic") gives Frequency Response & Phase





Why is this “hard”? Looking at the Low-Pass Example

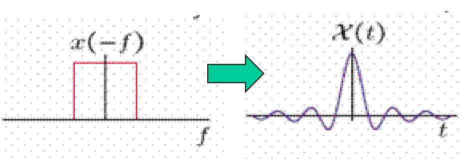
$$H_d(\omega) = \begin{cases} 1 \text{ if } |\omega| \leq \omega_c \\ 0 \text{ if } \omega_c < |\omega| < \pi \end{cases}$$

- Why is this hard?
  - Shouldn’t it be “easy” ??
    … just hit it with some FFT “magic” and then keep the bands we want and then hit it with some Inverse-FFT “supermagic”???
  - Remember we need a “system” that does this “rectangle function” in frequency
  - Let’s consider what that means…
    - It basically suggests we need an **Inverse FFT** of a **“rectangle function”**

Flashback: Fourier Series & Rectangular Functions

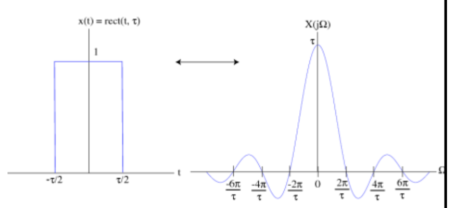
$\mathfrak{F}$: Fourier Tranform

$$\mathfrak{F}^{-1}\left\{rect\left(\frac{\omega}{2}\right)\right\} = \frac{sinc\,(t)}{\pi}$$



Ref: http://cnx.org/content/m26719/1.1/
http://www.wolframalpha.com/input/?i=IFFT%28sinc%28f%29%29

$$\mathfrak{F}\{rect\,(t)\} = sinc\left(\frac{\omega}{2}\right)$$



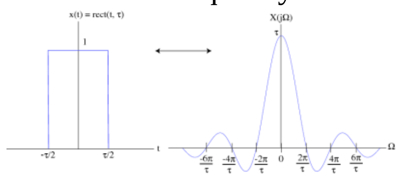
Ref: http://cnx.org/content/m32899/1.8/
http://www.thefouriertransform.com/pairs/box.php

See:

- Table 7.1 (p. 702) Entry 17 & Table 9.1 (p. 852) Entry 7

Flashback: Fourier Series & Rectangular Functions [2]

- The sinc function might look familiar
  - This is the frequency content of a square wave (box)



Ref: http://www.wolframalpha.com/input/?i=FFT%28rect%28t%29%29
http://cnx.org/content/m32899/1.8/

- This also applies to **signal reconstruction!**

➔ Whittaker–Shannon interpolation formula
  - This says that the "better way" to go from Discrete to Continuous (i.e. D to A) is not ZOH, but rather via the sinc!

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

∴ FIR and Low Pass Filters…

∴

$$H_d(\omega) = \begin{cases} 1 \text{ if } |\omega| \leq \omega_c \\ 0 \text{ if } \omega_c < |\omega| < \pi \end{cases}$$

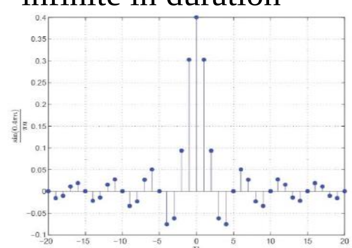
Has impulse response:

$$h_d(n) = \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}$$

Thus, to filter an impulse train with an ideal low-pass filter use:

$$x(t) = \left(\sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t-nT)\right) * \mathrm{sinc}\left(\frac{t}{T}\right)$$

- **However**!!
  a sinc is non-causal and infinite in duration



And, this **cannot** be implemented **in practice** ☹

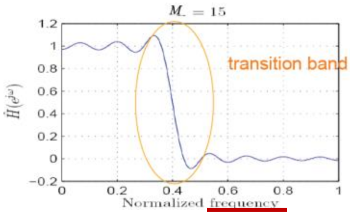
∵ we need to know all samples of the input, both in the **past** and in the **future**

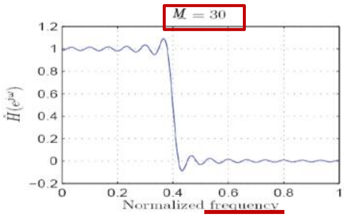
Plan 0: Impulse Response Truncation

Maybe we saw this coming…

∴ Clip off the sinc at some large *n*

$$\hat{h}(n) = \frac{\sin(n\omega_c)}{\pi n} \text{ for } |n| \leq M \text{ and } 0 \text{ otherwise}$$





- *Ripples* in both passband/stopband
  and the transition not abrupt (i.e., a *transition band*).
- As M→∞, transition band→ 0 (as expected!)

➔ FIR Filters: Window Function Design Method

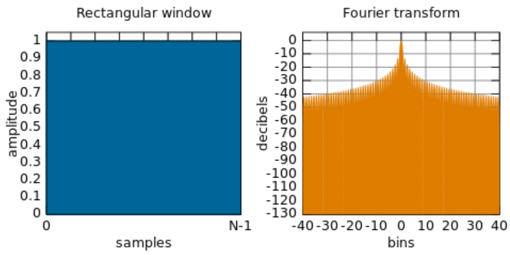
- Windowing: a generalization of the truncation idea

- There many, many "window" functions:
  - Rectangular
  - Triangular
  - Hanning
  - Hamming
  - Blackman
  - Kaiser
  - Lanczos
  - Many More … (see: http://en.wikipedia.org/wiki/Window_function)

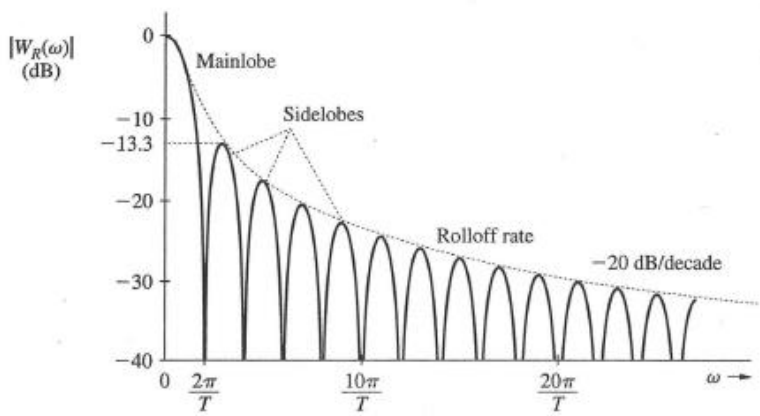
Some Window Functions [1]

1. Rectangular

$$w(n) = 1$$



Windowing and its effects/terminology

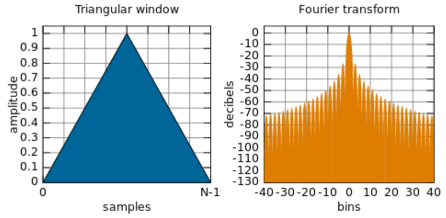


Lathi, Fig. 7.45

Some More Window Functions …

2. Triangular window

$$w(n) = 1 - \left| \frac{n - \frac{N-1}{2}}{\frac{N+1}{2}} \right|$$



- And Bartlett Windows
  - A slightly narrower variant with zero weight at both ends:

$$w(n) = 1 - \left| \frac{n - \frac{N-1}{2}}{\frac{N-1}{2}} \right|$$

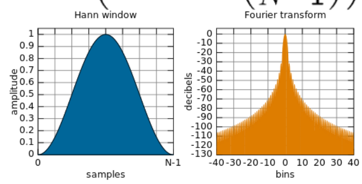
Some More Window Functions…

3. Generalized Hamming Windows

$$w(n) = \alpha - \beta \cos\left(\frac{2\pi n}{N-1}\right)$$

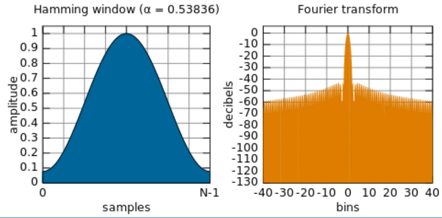
→Hanning Window

$$\rightarrow w(n) = 0.5\left(1 - \cos\left(\frac{2\pi n}{N-1}\right)\right)$$



→Hamming's Window

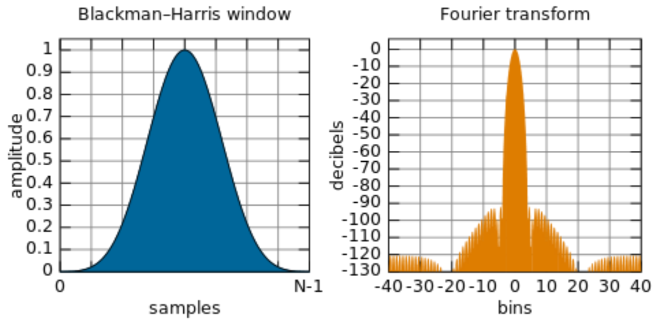
$$\rightarrow \alpha = 0.54,\ \beta = 1 - \alpha = 0.46$$



Some More Window Functions…

4. Blackman–Harris Windows
   - A generalization of the Hamming family,
   - Adds more shifted sinc functions for less side-lobe levels

$$w(n) = a_0 - a_1 \cos\left(\frac{2\pi n}{N-1}\right) + a_2 \cos\left(\frac{4\pi n}{N-1}\right) - a_3 \cos\left(\frac{6\pi n}{N-1}\right)$$

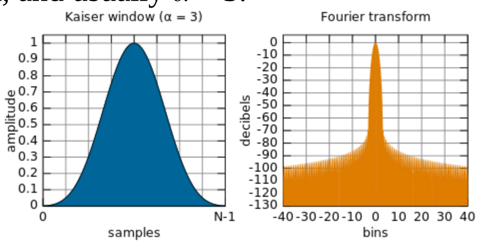


Some More Window Functions…

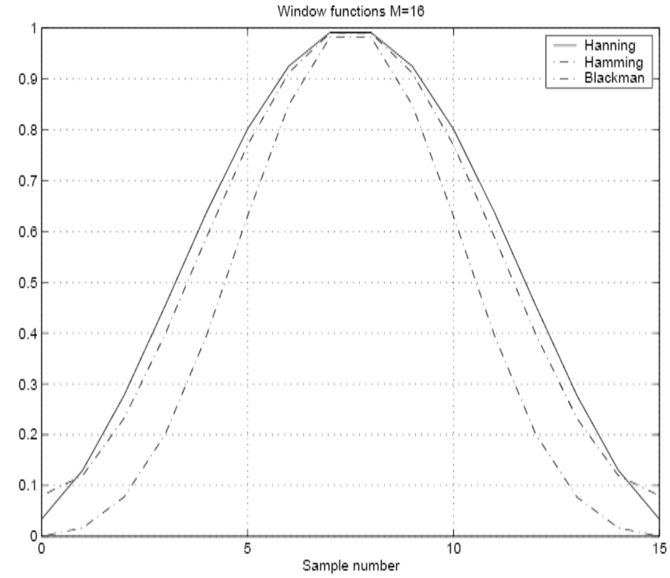
5. Kaiser window
   - A DPSS (discrete prolate spheroidal sequence)
   - Maximize the energy concentration in the main lobe

$$\rightarrow w(n) = \frac{I_0\left(\pi\alpha\sqrt{1-\left(\frac{2n}{N-1}-1\right)^2}\right)}{I_0(\pi\alpha)}$$

   - Where: I_0 is the zero-th order modified Bessel function of the first kind, and usually $\alpha = 3$.



Comparison of Alternative Windows –Time Domain

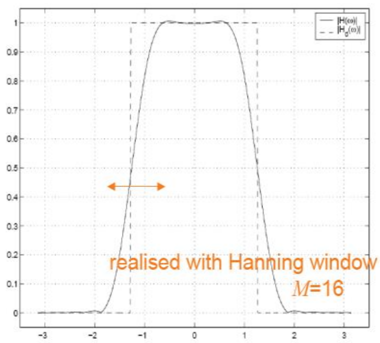


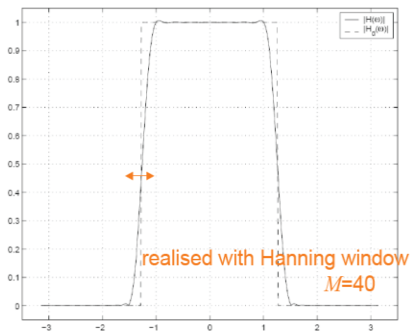
Punskaya, Slide 90

Adding Order

\+ Transition and Smoothness

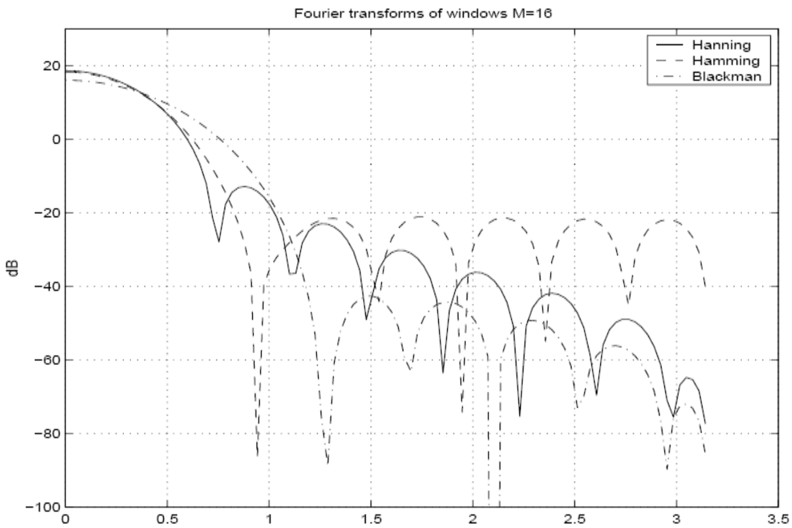
– Increased Size





Punskaya, Slide 94

Comparison of Alternative Windows
Frequency Domain



Punskaya, Slide 91

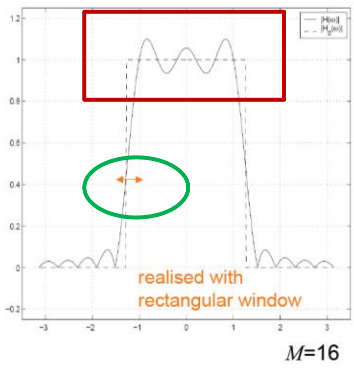
Summary Characteristics of Common Window Functions

No.	Window $w(t)$	Mainlobe Width	Rolloff Rate (dB/oct)	Peak Sidelobe level (dB)	Peak $20\log_{10}\delta$
1	Rectangular: $\text{rect}\left(\frac{t}{T}\right)$	$\frac{4\pi}{T}$	-6	-13.3	-21dB
2	Bartlett: $\Delta\left(\frac{t}{2T}\right)$	$\frac{8\pi}{T}$	-12	-26.5	
3	Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$	$\frac{8\pi}{T}$	-18	-31.5	-44dB
4	Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$	$\frac{8\pi}{T}$	-6	-42.7	-53dB
5	Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$	$\frac{12\pi}{T}$	-18	-58.1	-74dB
6	Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{t}{T}\right)^2}\right]}{I_0(\alpha)}$ $0 \le \alpha \le 10$	$\frac{11.2\pi}{T}$	-6	-59.9 $(\alpha = 8.168)$	

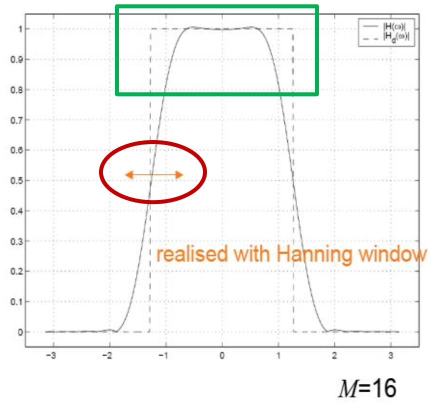
Lathi, Table 7.3
Punskaya, Slide 92

FIR: Rectangular & Hanning Windows

- Rectangular



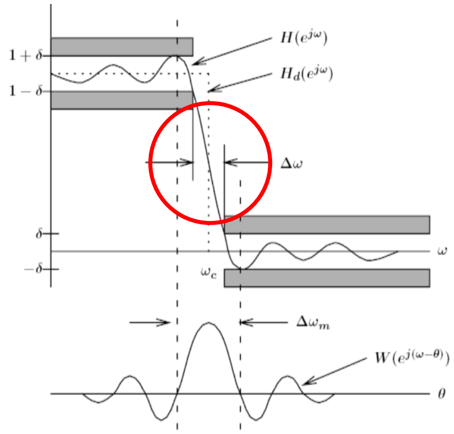
- Hanning



➔ Hanning: Less ripples, but wider transition band

Punskaya, Slide 93

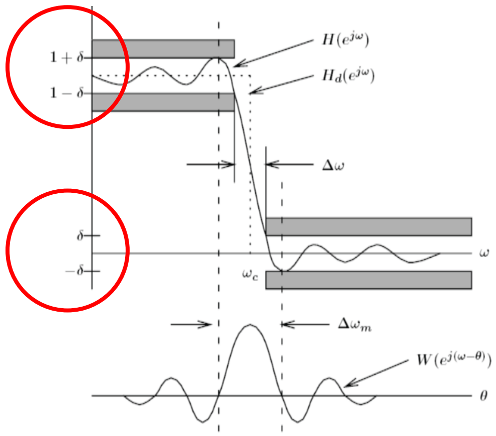
Windowed FIR Property 1: Equal transition bandwidth



Punskaya, Slide 96

- Equal transition bandwidth on both sides of the ideal cutoff frequency

Windowed FIR Property 2: Peak Errors same in Passband & Stopband



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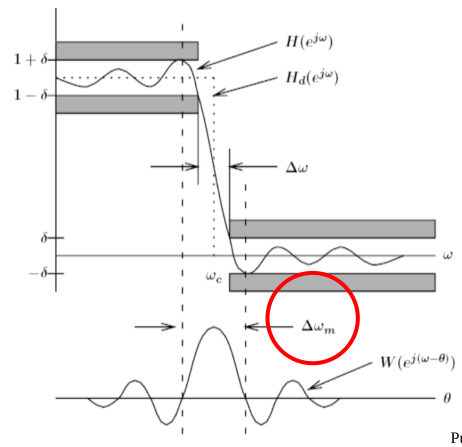
- Peak approximation error in the passband (1+δ → 1-δ) is equal to that in the stopband (δ → -δ)

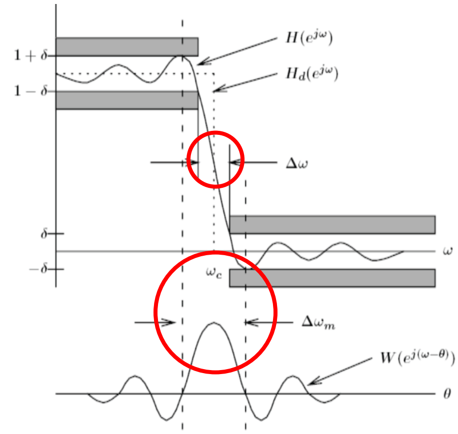
Windowed FIR Property 3: Mainlobe Width



Punskaya, Slide 99

- The distance between approximation error peaks is approximately equal to the width of the mainlobe Δw_m

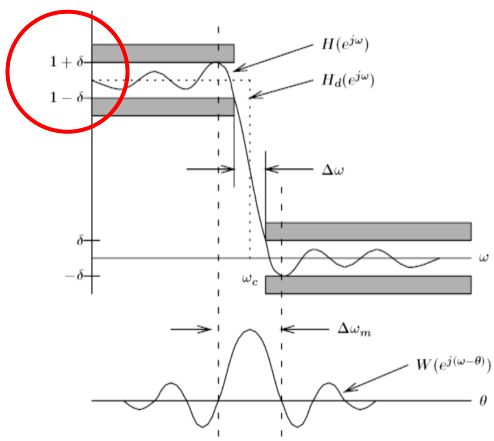
Windowed FIR Property 4: Mainlobe Width [2]



Punskaya, Slide 96

- The width of the mainlobe is wider than the transition bandwidth

Windowed FIR Property 5: Peak Δδ is determined by the window shape



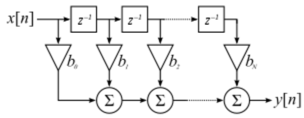
Punskaya, Slide 96

- peak approximation error is determined by the window shape, <u>independent</u> of the filter order

FIR Filter Design

- How to get all these coefficients?

$$H(\omega) = h_0 + h_1 e^{-i\omega} + \cdots + h_{n-1} e^{-i(n-1)\omega}$$

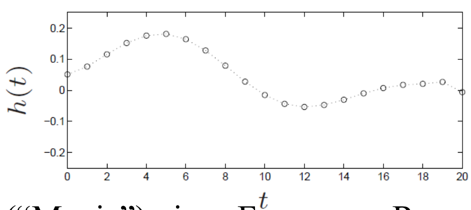


FIR Design Methods:

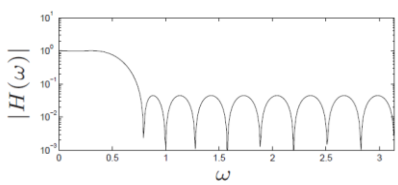
1. Impulse Response Truncation
   + Simplest
   – Undesirable frequency domain-characteristics, not very useful
2. Windowing Design Method
   + Simple
   – Not optimal (not minimum order for a given performance level)
3. Optimal filter design methods
   + "More optimal"
   – Less simple…

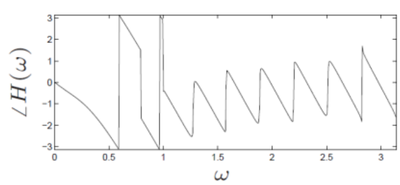
FIR Filter Design & Operation
Ex: Lowpass FIR filter

- Set Impulse response (order n = 21)
- "Determine" h(t)
  – h(t) is a 20 element vector that we'll use to as a weighted sum



- FFT ("Magic") gives Frequency Response & Phase





Why is this "hard"? Looking at the Low-Pass Example

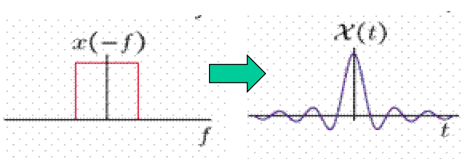
$$H_d(\omega) = \begin{cases} 1 \text{ if } |\omega| \leq \omega_c \\ 0 \text{ if } \omega_c < |\omega| < \pi \end{cases}$$

- Why is this hard?
  - Shouldn't it be "easy" ??
    … just hit it with some FFT "magic" and then keep the bands we want and then hit it with some Inverse-FFT "supermagic"???
  - Remember we need a "system" that does this "rectangle function" in frequency
  - Let's consider what that means…
    - It basically suggests we need an **Inverse FFT** of a **"rectangle function"**

Flashback: Fourier Series & Rectangular Functions

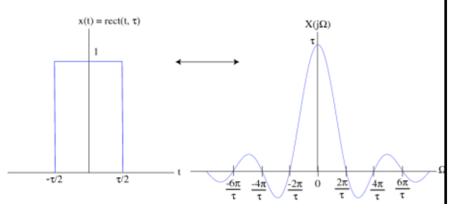
$\mathfrak{F}$: Fourier Tranform

$$\mathfrak{F}^{-1}\left\{rect\left(\frac{\omega}{2}\right)\right\} = \frac{sinc(t)}{\pi}$$



Ref: http://cnx.org/content/m26719/1.1/
http://www.wolframalpha.com/input/?i=IFFT%28sinc%28f%29%29

$$\mathfrak{F}\{rect(t)\} = sinc\left(\frac{\omega}{2}\right)$$



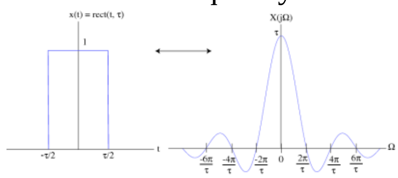
Ref: http://cnx.org/content/m32899/1.8/
http://www.thefouriertransform.com/pairs/box.php

See:

- Table 7.1 (p. 702) Entry 17 & Table 9.1 (p. 852) Entry 7

Flashback: Fourier Series & Rectangular Functions [2]

- The sinc function might look familiar
  - This is the frequency content of a square wave (box)



Ref: http://www.wolframalpha.com/input/?i=FFT%28rect%28t%29%29
http://cnx.org/content/m32899/1.8/

- This also applies to **<u>signal reconstruction!</u>**

➔ Whittaker–Shannon interpolation formula

  - This says that the "better way" to go from Discrete to Continuous (i.e. D to A) is not ZOH, but rather via the sinc!

$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \text{sinc}\left(\frac{t-nT}{T}\right)$$

∴ FIR and Low Pass Filters…

∴

$$H_d(\omega) = \begin{cases} 1 \text{ if } |\omega| \le \omega_c \\ 0 \text{ if } \omega_c < |\omega| < \pi \end{cases}$$

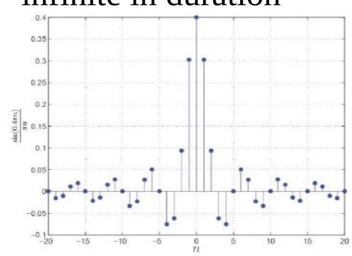
Has impulse response:

$$h_d(n) = \frac{\omega_c}{\pi} \frac{\sin \omega_c n}{\omega_c n}$$

Thus, to filter an impulse train with an ideal low-pass filter use:

$$x(t) = \left(\sum_{n=-\infty}^{\infty} x[n] \cdot \delta(t - nT)\right) * \text{sinc}\left(\frac{t}{T}\right)$$

- **<u>However</u>**!!
  a sinc is non-causal and infinite in duration



And, this **cannot** be implemented **in practice** ☹

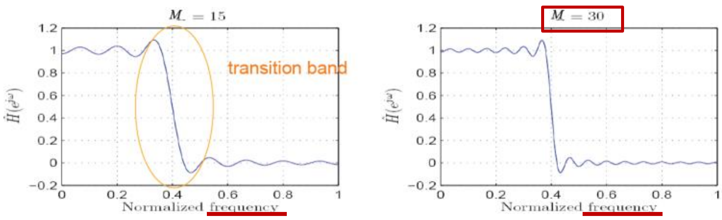
∵ we need to know all samples of the input, both in the **past** and in the **future**

Plan 0: Impulse Response Truncation

Maybe we saw this coming…

∴ Clip off the `sinc` at some large *n*

$$\hat{h}(n) = \frac{\sin(n\omega_c)}{\pi n} \text{ for } |n| \leq M \text{ and } 0 \text{ otherwise}$$



- *Ripples* in both passband/stopband and the transition not abrupt (i.e., a *transition band*).
- As M→∞, transition band→ 0 (as expected!)

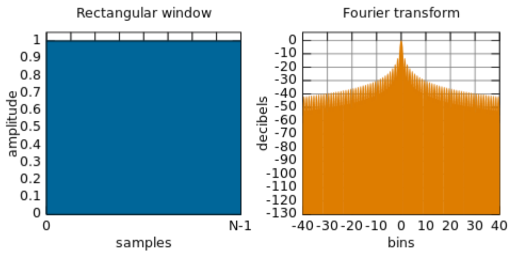
→ FIR Filters: Window Function Design Method

- Windowing: a generalization of the truncation idea
- There many, many “window” functions:
  - Rectangular
  - Triangular
  - Hanning
  - Hamming
  - Blackman
  - Kaiser
  - Lanczos
  - Many More … (see: http://en.wikipedia.org/wiki/Window_function)

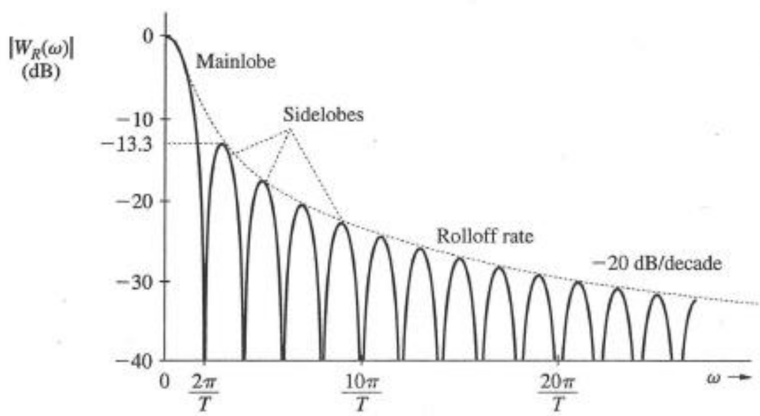
Some Window Functions [1]

1. Rectangular

$$w(n) = 1$$



Windowing and its effects/terminology

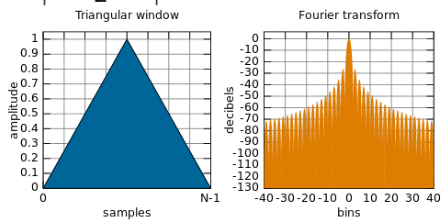


Lathi, Fig. 7.45

Some More Window Functions …

2. Triangular window

$$w(n) = 1 - \left|\frac{n - \frac{N-1}{2}}{\frac{N+1}{2}}\right|$$



- And Bartlett Windows
  – A slightly narrower variant with zero weight at both ends:

$$w(n) = 1 - \left|\frac{n - \frac{N-1}{2}}{\frac{N-1}{2}}\right|$$

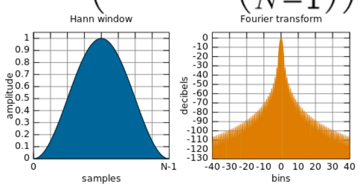
Some More Window Functions…

3. Generalized Hamming Windows

$$w(n) = \alpha - \beta \cos\left(\frac{2\pi n}{N-1}\right)$$

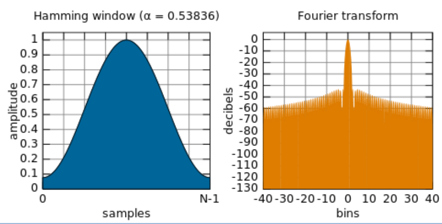
→ Hanning Window

$$\rightarrow w(n) = 0.5\left(1 - \cos\left(\frac{2\pi n}{N-1}\right)\right)$$



→ Hamming's Window

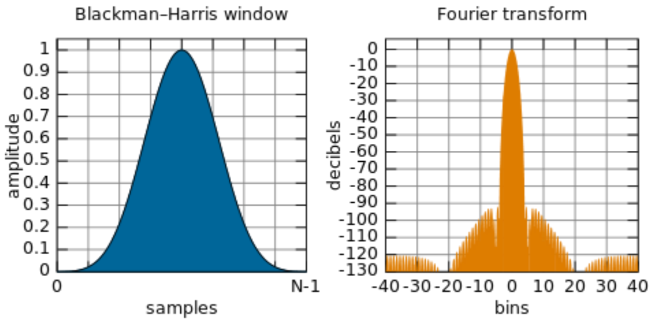
$$\rightarrow \alpha = 0.54,\ \beta = 1 - \alpha = 0.46$$



Some More Window Functions…

4. Blackman–Harris Windows
   - A generalization of the Hamming family,
   - Adds more shifted sinc functions for less side-lobe levels

$$w(n) = a_0 - a_1 \cos\left(\frac{2\pi n}{N-1}\right) + a_2 \cos\left(\frac{4\pi n}{N-1}\right) - a_3 \cos\left(\frac{6\pi n}{N-1}\right)$$

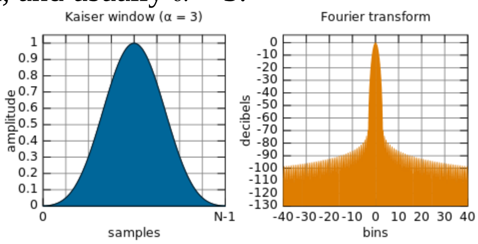


Some More Window Functions…

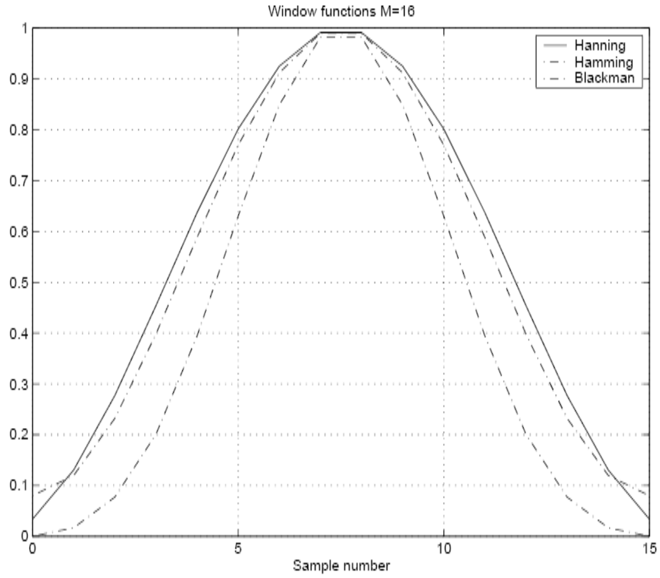
5. Kaiser window
   - A DPSS (discrete prolate spheroidal sequence)
   - Maximize the energy concentration in the main lobe

$$\rightarrow w(n) = \frac{I_0\left(\pi\alpha\sqrt{1-\left(\frac{2n}{N-1}-1\right)^2}\right)}{I_0(\pi\alpha)}$$

   - Where: I_0 is the zero-th order modified Bessel function of the first kind, and usually $\alpha = 3$.



Comparison of Alternative Windows –Time Domain



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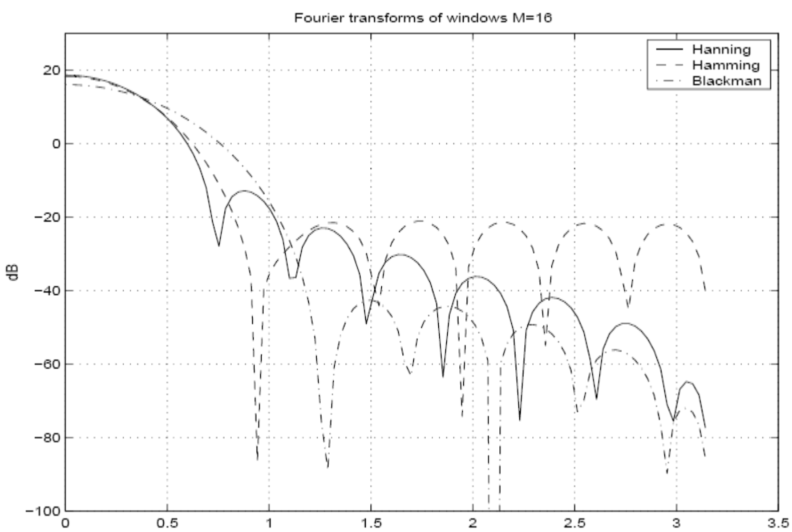
Comparison of Alternative Windows
Frequency Domain



Punskaya, Slide 91

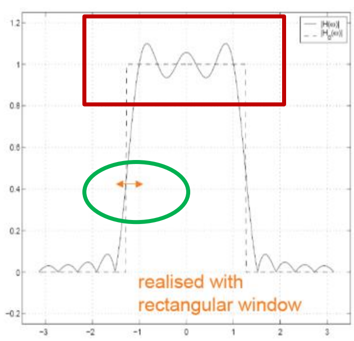
Summary Characteristics of Common Window Functions

No.	Window $w(t)$	Mainlobe Width	Rolloff Rate (dB/oct)	Peak Sidelobe level (dB)	Peak $20\log_{10}\delta$
1	Rectangular: $\text{rect}\left(\frac{t}{T}\right)$	$\frac{4\pi}{T}$	-6	-13.3	-21dB
2	Bartlett: $\Delta\left(\frac{t}{2T}\right)$	$\frac{8\pi}{T}$	-12	-26.5	
3	Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$	$\frac{8\pi}{T}$	-18	-31.5	-44dB
4	Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$	$\frac{8\pi}{T}$	-6	-42.7	-53dB
5	Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$	$\frac{12\pi}{T}$	-18	-58.1	-74dB
6	Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{t}{T}\right)^2}\right]}{I_0(\alpha)}$ $\quad 0 \le \alpha \le 10$	$\frac{11.2\pi}{T}$	-6	-59.9 ($\alpha = 8.168$)	

Lathi, Table 7.3
Punskaya, Slide 92

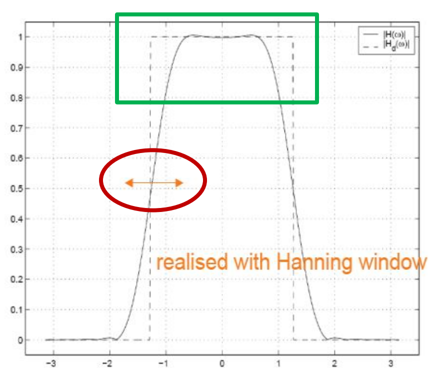
FIR: Rectangular & Hanning Windows

- Rectangular



M=16

- Hanning



M=16

➔ Hanning: Less ripples, but wider transition band

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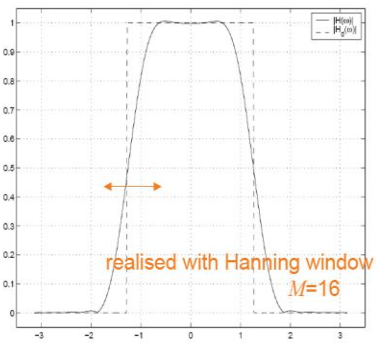
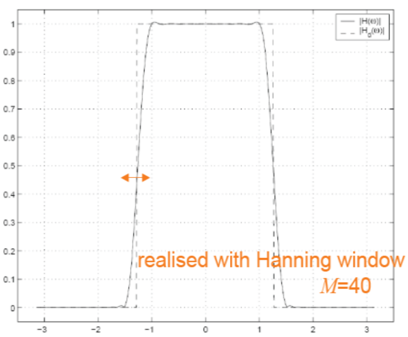
Adding Order

+ Transition and Smoothness
– Increased Size





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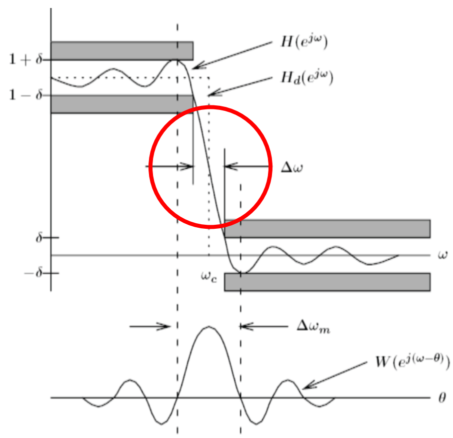
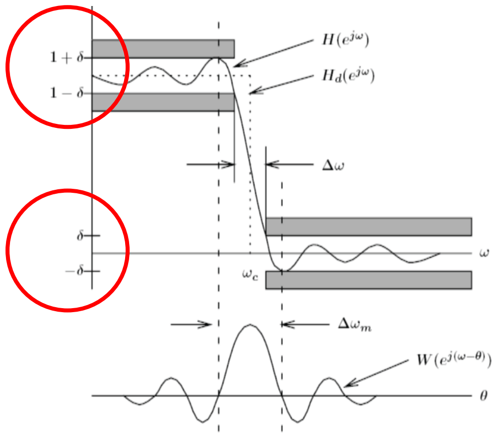
Windowed FIR Property 1: Equal transition bandwidth



Punskaya, Slide 96

- Equal transition bandwidth on both sides of the ideal cutoff frequency

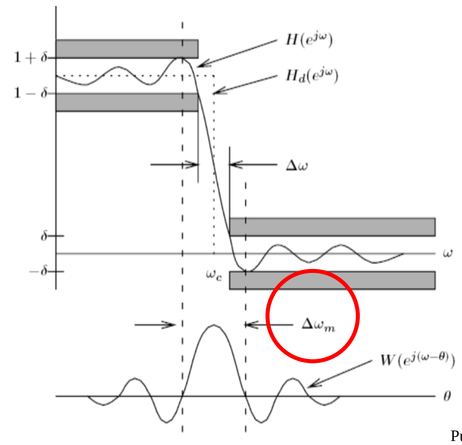
Windowed FIR Property 2: Peak Errors same in Passband & Stopband



Punskaya, Slide 96

- Peak approximation error in the passband (1+δ → 1-δ) is equal to that in the stopband (δ → -δ)

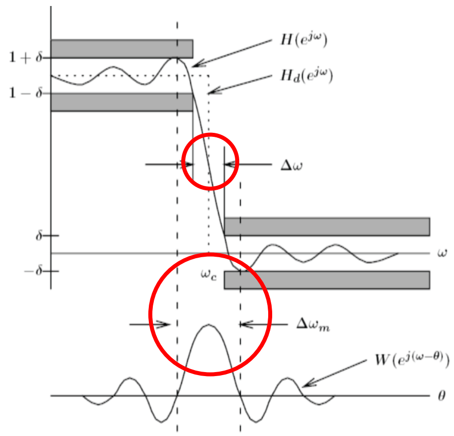
Windowed FIR Property 3: Mainlobe Width



Punskaya, Slide 99

- The distance between approximation error peaks is approximately equal to the width of the mainlobe Δw_m

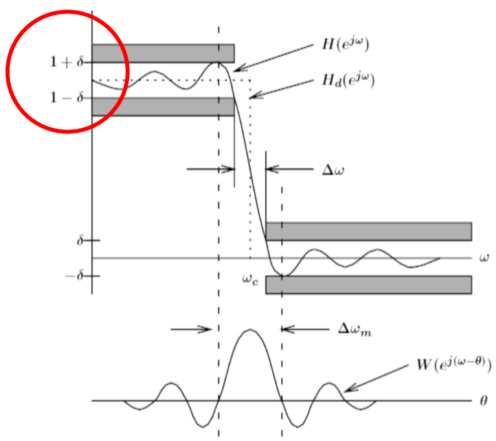
Windowed FIR Property 4: Mainlobe Width [2]



Punskaya, Slide 96

- The width of the mainlobe is wider than the transition bandwidth

Windowed FIR Property 5: Peak Δδ is determined by the window shape



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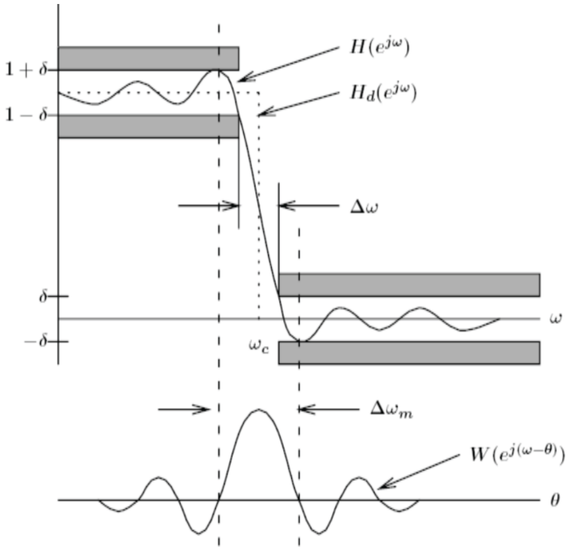
- peak approximation error is determined by the window shape, <u>independent</u> of the filter order

Window Design Method Design Terminology



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Where:

- ω_c: cutoff frequency
- δ: maximum passband ripple
- $\Delta\omega$: transition bandwidth
- $\Delta\omega_m$: width of the window mainlobe

Passband / stopband ripples

ω_s and ω_p: Corner Frequencies

Passband / stopband ripples are often expressed in dB:

- passband ripple = 20 $\log_{10}(1+\delta_p)$ dB
- peak-to-peak passband ripple ≅ 20 $\log_{10}(1+2\delta_p)$ dB
- minimum stopband attenuation = -20 $\log_{10}(\delta_s)$ dB

Passband / stopband ripples

ω_s and ω_p: Corner Frequencies

Passband / stopband ripples are often expressed in dB:

- passband ripple = ~~$20\ log_{10}\ (1+\delta_p\ )$ dB~~ $= 20\ log_{10}\ (\delta_p\ )$ dB
- peak-to-peak passband ripple $\cong$ ~~$20\ log_{10}\ (1+2\delta_p)$ dB~~
  $\cong 20\ log_{10}\ (2\delta_p)$ dB
- minimum stopband attenuation = ~~$-20\ log_{10}\ (\delta_s)$ dB~~
  $=20\ log_{10}\ (\delta_s)$ dB

Summary of Design Procedure

1. Select a suitable window function
2. Specify an ideal response $H_d(\omega)$
3. Compute the coefficients of the ideal filter $h_d(n)$
4. Multiply the ideal coefficients by the window function to give the filter coefficients
5. Evaluate the frequency response of the resulting filter and iterate if necessary (e.g. by increasing M if the specified constraints have not been satisfied).

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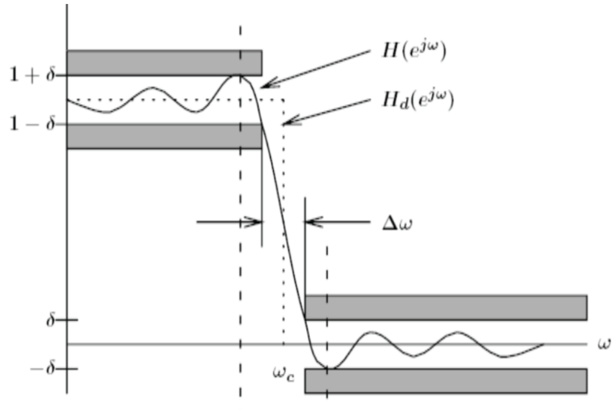
Windowed Filter Design Example

- Design a type I low-pass filter with:
  - $\omega_p = \mathbf{0.2\pi}$
  - $\omega_s = \mathbf{0.3\pi}$
  - $\delta = \mathbf{0.01}$



Windowed Filter Design Example:
Step 1: Select a suitable Window Function

No.	Window $w(t)$	Mainlobe Width	Rolloff Rate (dB/oct)	Peak Sidelobe level (dB)	Peak $20\log_{10}\delta$
1	Rectangular: $\text{rect}\left(\frac{t}{T}\right)$	$\frac{4\pi}{T}$	−6	−13.3	-21dB
2	Bartlett: $\Delta\left(\frac{t}{2T}\right)$	$\frac{8\pi}{T}$	−12	−26.5	
3	Hanning: $0.5\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]$	$\frac{8\pi}{T}$	−18	−31.5	-44dB
4	Hamming: $0.54+0.46\cos\left(\frac{2\pi t}{T}\right)$	$\frac{8\pi}{T}$	−6	−42.7	-53dB
5	Blackman: $0.42+0.5\cos\left(\frac{2\pi t}{T}\right)+0.08\cos\left(\frac{4\pi t}{T}\right)$	$\frac{12\pi}{T}$	−18	−58.1	-74dB
6	Kaiser: $\frac{I_0\left[\alpha\sqrt{1-4\left(\frac{t}{T}\right)^2}\right]}{I_0(\alpha)}$ $0 \le \alpha \le 10$	$\frac{11.2\pi}{T}$	−6	−59.9 ($\alpha = 8.168$)	

- LP with: $\omega_p = \mathbf{0.2\pi}$, $\omega_s = \mathbf{0.3\pi}$, $\delta = \mathbf{0.01}$
- $\delta = \mathbf{0.01}$**:** The required peak error spec: $-20\log10(\delta) = -40$ dB
  } Hanning Window
- Main-lobe width:

$\omega_s - \omega_p = 0.3\pi - 0.2\pi = 0.1\pi \rightarrow 0.1\pi = 8\pi / M$

→ Filter length $M \geq 80$ & Filter order $N \geq 79$

- BUT, Type-I filters have even order so $N = 80$

Windowed Filter Design Example: Step 2: Specify the Ideal Response

- From Property 1 (Midpoint rule)

$\omega_c = (\omega_s + \omega_p)/2 = (0.2\pi+0.3\pi)/2 = 0.25\pi$

∴ An ideal response will be:

$$H_d(\omega) = \begin{cases} 1 & \text{if} \quad |\omega| \leq 0.25\pi \\ 0 & \text{if} \quad 0.25\pi < |\omega| < \pi \end{cases}$$

Windowed Filter Design Example: Step 3: Compute the coefficients of the ideal filter

- The ideal filter coefficients h_d are given by the Inverse **Discrete time** Fourier transform of $H_d(\omega)$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$
$$= \frac{\omega_c}{\pi}\frac{\sin \omega_c n}{\omega_c n}.$$

\+ Delayed impulse response (to make it causal)

$$\widetilde{h}(n) = \widehat{h}\left(n - \frac{N-1}{2}\right)$$

- Coefficients of the ideal filter (via equation or IFFT):

$$h(n) = \frac{\sin(0.5\pi(n-40))}{\pi(n-40)}$$

Windowed Filter Design Example:
Step 4: Multiply to obtain the filter coefficients

$$h(n) = \frac{\sin(0.5\pi(n-40))}{\pi(n-40)}$$

- Multiply by a Hamming window function for the passband:

$$w(n) = 0.54 - 0.46\ \cos\left(\frac{2\pi n}{M}\right)$$

Windowed Filter Design Example:
Step 5: Evaluate the Frequency Response and Iterate

- The frequency response is computed as the DFT of the filter coefficient vector
- **If** the resulting filter does not meet the specifications, **then**:
  - Adjust the ideal filter frequency response (for example, move the band edge) and repeat (step 2)
  - Adjust the filter length and repeat (step 4)
  - change the window (& filter length) (step 4)
- And/Or consult with Matlab:
  - **`FIR1`** and **`FIR2`**
  - **`B=FIR2(N,F,M):`** Designs a Nth order FIR digital filter with

Windowed Filter Design Example: Consulting Matlab:

- `FIR1` and `FIR2`
  - `B=FIR2(N,F,M):` Designs a Nth order FIR digital filter
  - `F` and `M` specify frequency and magnitude breakpoints for the filter such that **plot(N,F,M)** shows a plot of desired frequency
  - Frequencies `F` must be in increasing order between 0 and Fs/2, with Fs corresponding to the sample rate.
  - **B** is the vector of length N+1, it is real, has linear phase and symmetric coefficients
  - Default window is Hamming – others can be specified

In Conclusion

- FIR Filters are digital (can not be implemented in analog) and exploit the difference and delay operators
- A window based design builds on the notion of a truncation of the "ideal" box-car or rectangular low-pass filter in the Frequency domain (which is a sinc function in the time domain)
- Other Design Methods exist:
  - Least-Square Design
  - Equiripple Design
  - Remez method
  - The Parks-McClellan Remez algorithm
  - Optimisation routines …