Discrete Systems & Z-Transforms

ELEC 3004: **Digital Linear Systems**: Signals & Controls

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Lecture 5

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http://elec3004.org

Lecture Schedule:

Week	Date	Lecture Title
1	2-Mar	Introduction
	3-Mar	Systems Overview
2	9-Mar	Signals as Vectors & Systems as Maps
	10-Mar	[Signals]
3	16-Mar	Sampling & Data Acquisition & Antialiasing Filters
	17-Mar	[Sampling]
4	23-Mar	System Analysis & Convolution
	24-Mar	[Convolution & FT]
5	30-Mar	Discrete Systems & Z-Transforms
	31-Mar	[Z-Transforms]
6	13-Apr	Frequency Response & Filter Analysis
	14-Apr	[Filters]
7	20-Apr	Digital Filters
	21-Apr	[Digital Filters]
8	27-Apr	Introduction to Digital Control
	28-Apr	[Feedback]
9	4-May	Digital Control Design
	5-May	[Digitial Control]
10	11-May	Stability of Digital Systems
	12-May	[Stability]
11	18-May	State-Space
	19-May	Controllability & Observability
12	25-May	PID Control & System Identification
	26-May	Digitial Control System Hardware
13	31-May	Applications in Industry & Information Theory & Communications
	2-Jun	Summary and Course Review

Lecture Overview

- Course So Far:

ODE — $\mathcal{L}$: Laplace (**s**) — Transfer functions — Cascade of LCC ODE — **Convolution** — **Z-Transform**

- Lecture(s):

Convolution — $\mathcal{F}$: Fourier Series (Periodic functions) — $\mathcal{L} \rightarrow \mathcal{F}$: ($\xi = \sigma + i\tau$) ($\mathbb{R} \rightarrow \mathbb{C}$) — $\mathbb{C}$: Poles & Zeros — DFFT — Z-Transform

Graphical Understanding of Convolution

➔ For c(τ)= $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$:

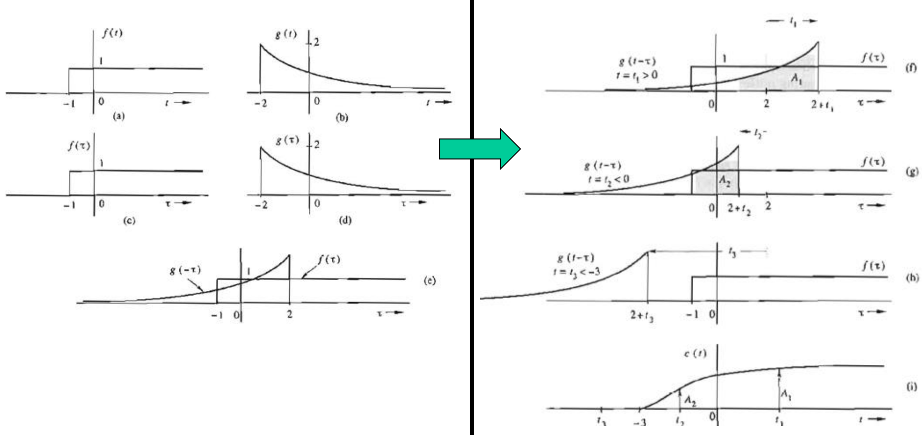
1. Keep the function ***f* (*τ*)** fixed
2. **Flip** (invert) the function ***g*(*τ*)** about the vertical axis (τ=0)
   = this is ***g*(-*τ*)**
3. **Shift** this frame (***g*(-*τ*)**) along τ (horizontal axis) by $\mathbf{t_0}$.
   = this is $\boldsymbol{g(t_0 - \tau)}$

➔ For c($\mathbf{t_0}$):

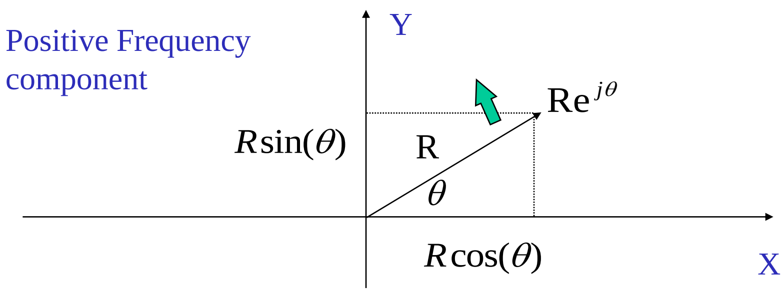
4. $\mathbf{c(t_0)}$ = the area under the product of $\boldsymbol{f(\tau)}$ **and** $\boldsymbol{g(t_0 - \tau)}$

5. Repeat this procedure, shifting the frame by different values (positive and negative) to obtain **<u>c(t) for all values of t</u>**.

Graphical Understanding of Convolution (Ex)



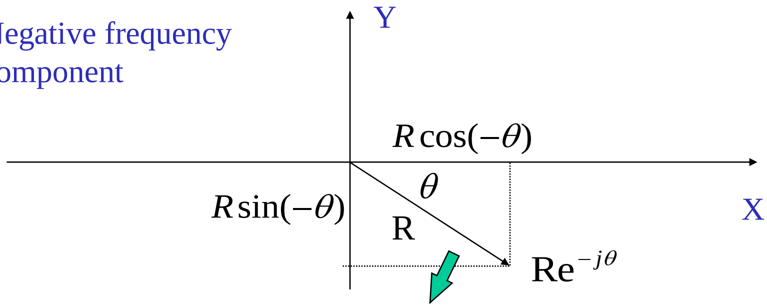
Complex Numbers and Phasors



$$
\begin{aligned}
Re^{j\theta} &= (R\cos\theta, R\sin\theta) \\
&= R\cos\theta + jR\sin\theta \\
&= R(\cos\theta + j\sin\theta)
\end{aligned}
$$

Complex Numbers and Phasors

Negative frequency component



$$
\begin{aligned}
\mathrm{R}e^{-j\theta} &= (R\cos(-\theta), R\sin(-\theta)) \\
&= R\cos(-\theta) + jR\sin(-\theta) \\
&= R(\cos\theta - j\sin\theta)
\end{aligned}
$$

Positive and Negative Frequencies

- Frequency is the derivative of phase
  more nuanced than :

$$\frac{1}{\tau} = repetition\ rate$$

- Hence both positive and negative frequencies are possible.

- Compare
  - velocity vs speed
  - frequency vs repetition rate

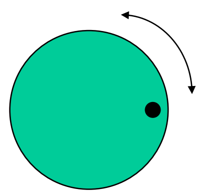
Negative Frequency

- Q: What is negative frequency?
- A: A mathematical convenience

- Trigonometrical FS
  – periodic signal is made up from
  – sum 0 to ∞ of sine and cosines ‘harmonics’

- Complex Fourier Series & the Fourier Transform
  – use exp(±jωt) instead of cos(ωt) and sin(ωt)
  – signal is sum from 0 to ∞ of exp(±jωt)
  – same as sum -∞ to ∞ of exp(-jωt)
  – which is more compact (i.e., less chalk!)

Another way to see Aliasing Too!

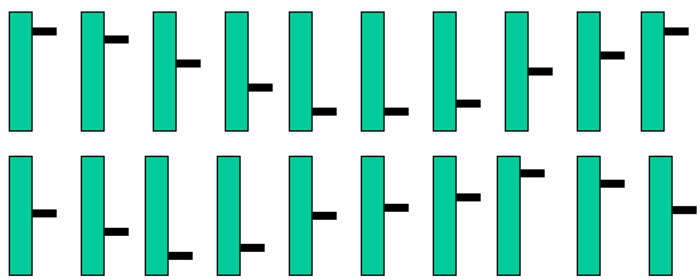


Rotating wheel and peg

Need both top and front view to determine rotation

Top View

Front View



Frequency Response

Fourier Series → Fourier Transforms

Typical Linear Processors

- Convolution $h(n,k)=h(n-k)$
- Cross Correlation $h(n,k)=h(n+k)$
- Auto Correlation $h(n,k)=x(k-n)$
- Cosine Transform $h(n,k)=\cos\left(\frac{2\pi}{N}nk\right)$
- Sine Transform $h(n,k)=\sin\left(\frac{2\pi}{N}nk\right)$
- Fourier Transform $h(n,k)=\exp\left(j\frac{2\pi}{N}nk\right)$

Transform Analysis

- Signal measured (or known) as a function of an independent variable
  - e.g., time: y = f(t)
- However, this independent variable may not be the most appropriate/informative
  - e.g., frequency: Y = f(w)
- Therefore, need to transform from one domain to the other
  - e.g., time ⇔ frequency
  - As used by the human ear (and eye)

Signal processing uses Fourier, Laplace, & *z* transforms etc

Sinusoids and Linear Systems

x(t) or x(n) → h(t) or h(n) → y(t) or y(n)

If $$x(t) = A\cos(\omega_0 t + \theta_0)$$

or $$x(n) = A\cos(\omega_0 nt + \theta_0)$$

then in steady state

$$y(t) = AC(\omega_0)\cos(\omega_0 t + \theta_0 + \theta(\omega_0))$$

$$y(n) = AC(\omega_0 T)\cos(\omega_0 nt + \theta_0 + \theta(\omega_0 T))$$

Sinusoids and Linear Systems

- The pair of numbers C(ω_0) and q(ω_0) are the complex gain of the system at the frequency ω_0 .
- They are respectively, the magnitude response and the phase response at the frequency ω_0 .

$$y(t) = AC(\omega_0)\cos(\omega_0 t + \theta_0 + \theta(\omega_0))$$
$$y(n) = AC(\omega_0 T)\cos(\omega_0 nt + \theta_0 + \theta(\omega_0 T))$$

Why Use Sinusoids?

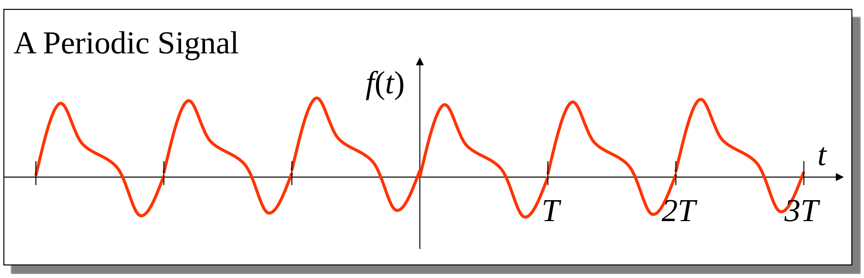
- Why probe system with sinusoids?
- Sinusoids are eigenfunctions of linear systems???
- What the hell does that mean?
- Sinusoid in implies sinusoid out
- Only need to know phase and magnitude (two parameters) to fully describe output rather than whole waveform
  - sine + sine = sine
  - derivative of sine = sine (phase shifted - cos)
  - integral of sine = sine (-cos)
- Sinusoids maintain orthogonality after sampling (not true of most orthogonal sets)

Frequency Response

Fourier Series → Fourier Transforms

Fourier Series

- Deal with continuous-time periodic signals.
- Discrete frequency spectra.



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Two Forms for Fourier Series

Sinusoidal Form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt$$

Complex Form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t} dt$$

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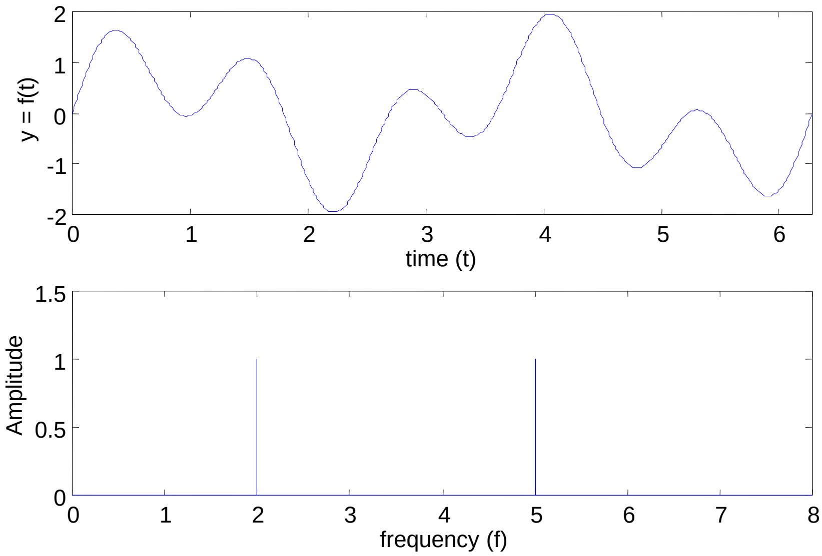
Fourier Series

- Any finite power, periodic, signal x(t)
  - period T
- can be represented as (∞) summation of
  - sine and cosine waves
- Called: Trigonometrical Fourier Series

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)$$

- Fundamental frequency $\omega_0 = 2\pi/T$ rad/s or $1/T$ Hz
- DC (average) value $A_0/2$



Frequency representation (spectrum) shows signal contains:
• 2Hz and 5Hz components (sinewaves) of equal amplitude

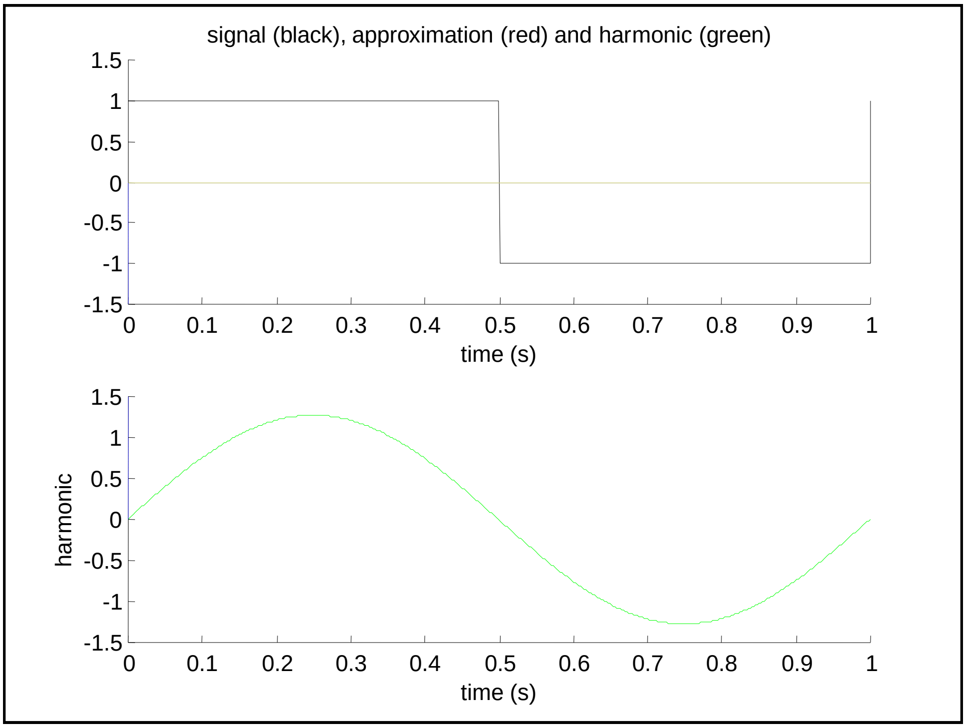
Fourier Series Coefficients

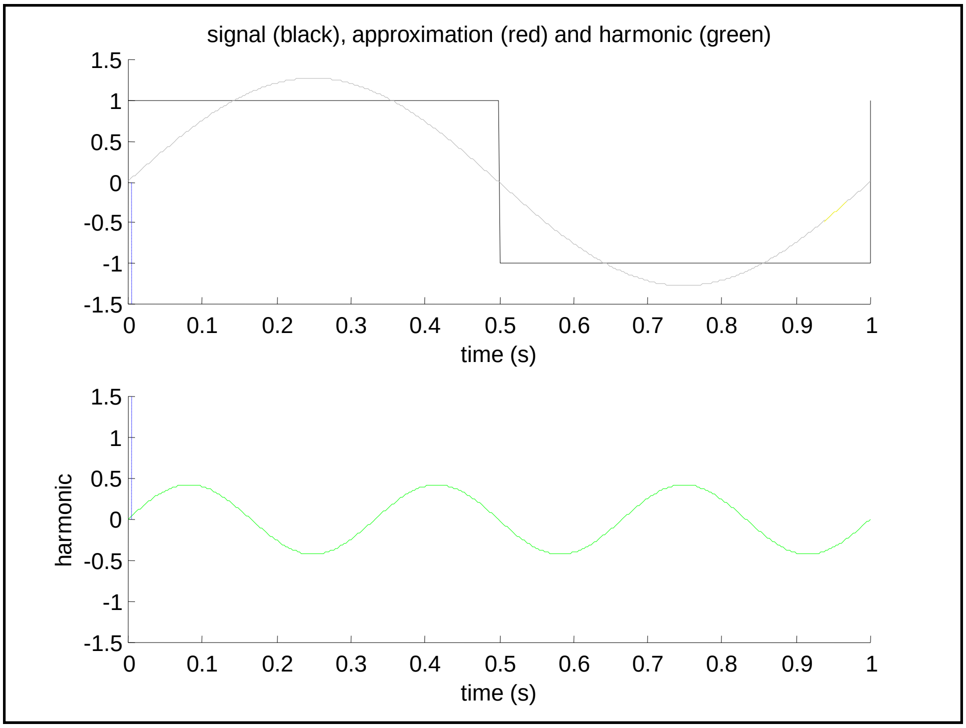
- An & Bn calculated from the signal, x(t)
  - called: Fourier coefficients

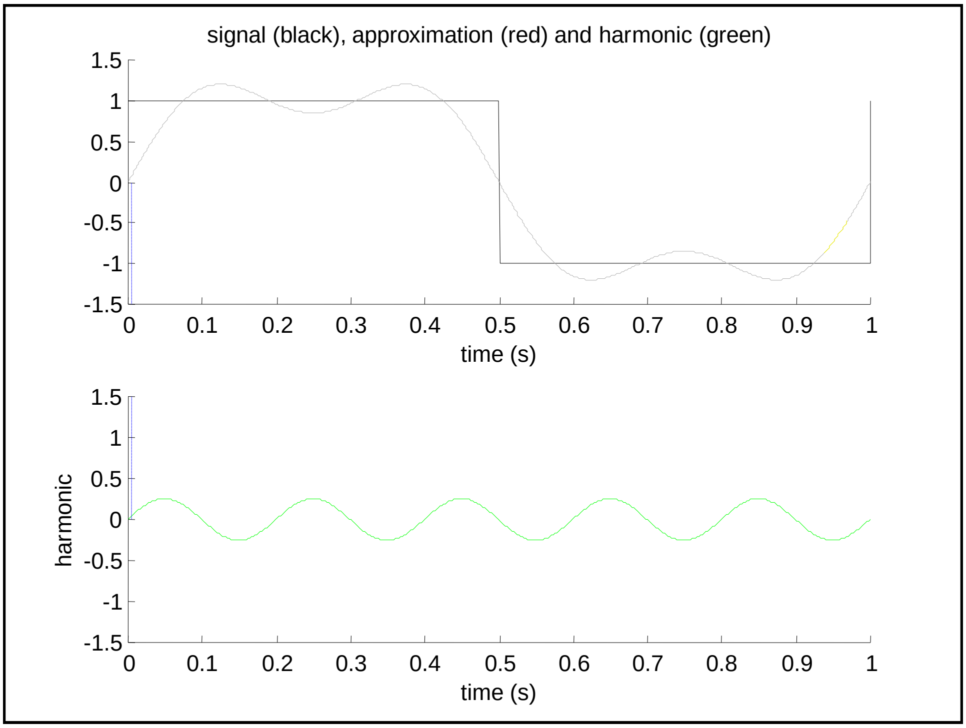
$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(nw_0 t)dt \qquad n = 0,1,2,\cdots$$

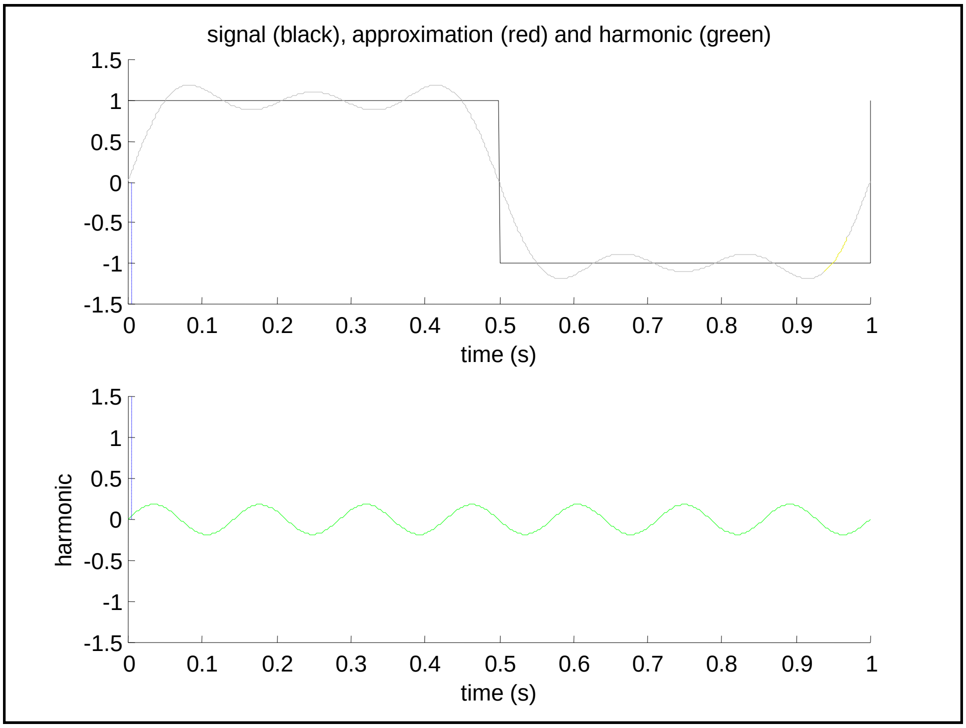
$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(nw_0 t)dt \qquad n = 1,2,3,\cdots$$

Note: Limits of integration can vary,
provided they cover one period


signal (black), approximation (red) and harmonic (green)
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time (s)
harmonic
time (s)


signal (black), approximation (red) and harmonic (green)
1.5
1
0.5
0
-0.5
-1
-1.5
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
time (s)
harmonic
time (s)


signal (black), approximation (red) and harmonic (green)
time (s)
harmonic
time (s)


signal (black), approximation (red) and harmonic (green)
time (s)
harmonic
time (s)

Example: Square wave

$$x(t) = \begin{cases} 1, & 0 < t < 1; \\ -1, & 1 < t < 2; \\ x(t+2). & \end{cases}$$ ← periodic! i.e., x(t + 2) = x(t)

$$A_n = \int_0^2 x(t)\cos(n\pi t)dt = \int_0^1 \cos(n\pi t)dt - \int_1^2 \cos(n\pi t)dt$$

$$A_n = \left[\frac{-\sin(n\pi t)}{n\pi}\right]_0^1 - \left[\frac{-\sin(n\pi t)}{n\pi}\right]_1^2 = 0$$

No cos terms as sin(nπ) = 0 ∀ n
x(t) has odd symmetry

$$B_n = \int_0^2 x(t)\sin(n\pi t)dt = \int_0^1 \sin(n\pi t)dt - \int_1^2 \sin(n\pi t)dt$$

cos(2nπ) = 1 ∀ n

$$B_n = \left[\frac{-\cos(n\pi t)}{n\pi}\right]_0^1 - \left[\frac{-\cos(n\pi t)}{n\pi}\right]_1^2 = -\frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi} + \frac{1}{n\pi} - \frac{\cos(n\pi)}{n\pi}$$

$$B_n = \frac{2}{n\pi}(1 - \cos(n\pi))$$

Sin terms only

Example: Square wave

Therefore, Trigonometric Fourier series is,

$$x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - \cos(n\pi))\sin(n\pi t)$$

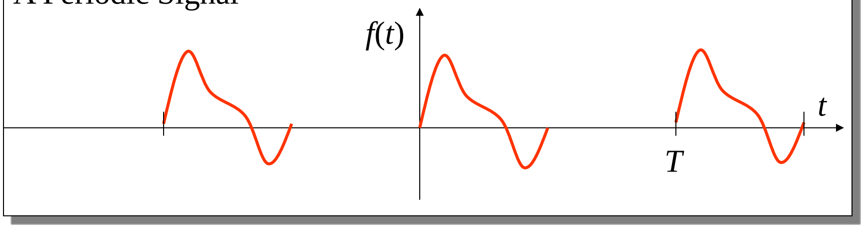
Expanding the terms gives,

$$\begin{aligned} x(t) = \quad & \frac{4}{\pi}\sin(\pi t) && \text{(fundamental)} \\ & +0 && \text{(second harmonic)} \\ & +\frac{4}{3\pi}\sin(3\pi t) && \text{(third harmonic)} \\ & +0 && \text{(fourth harmonic)} \\ & +\frac{4}{5\pi}\sin(5\pi t) && \text{(fifth harmonic)} \\ & +\text{etc} && \end{aligned}$$

- Only odd harmonics;
- In proportion 1,1/3,1/5,1/7,…
- Higher harmonics contribute less;
- Therefore, converges

How to Deal with Aperiodic Signal?

A Periodic Signal



If $T \rightarrow \infty$, what happens?

Source: URI ELE436

Fourier Integral

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau} d\tau \right] e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau} d\tau \right] \omega_0 e^{jn\omega_0 t}$$

Let $\Delta\omega = \omega_0 = \frac{2\pi}{T}$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau} d\tau \right] e^{jn\omega_0 t} \Delta\omega$$

$$T \rightarrow \infty \Rightarrow d\omega = \Delta\omega \approx 0$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_T(\tau) e^{-j\omega \tau} d\tau \right] e^{j\omega t} d\omega$$

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Fourier Integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(\tau)e^{-j\omega\tau}d\tau\right]}_{F(j\omega)}e^{j\omega t}d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$ Synthesis

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ Analysis

Source: URI ELE436

Fourier Series vs. Fourier Integral

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$ Period Function

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)e^{-jn\omega_0 t}dt$$ Discrete Spectra

Fourier Integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega$$ Non-Period Function

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$ Continuous Spectra

Source: URI ELE436

Complex Fourier Series (CFS)

- Also called Exponential Fourier series
  - As it uses Euler's relation
- FS as a Complex phasor summation

$$A\exp(jw_0t) = A\cos(w_0t) + jA\sin(w_0t)$$

which implies,

$$\cos(nw_0t) = \frac{\exp(jnw_0t)+\exp(-jnw_0t)}{2}$$

$$\sin(nw_0t) = \frac{\exp(jnw_0t)-\exp(-jnw_0t)}{2j}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jnw_0t)$$

Where X_n are the CFS coefficients

Complex Fourier Coefficients

- Again, Xn calculated from x(t)
- Only one set of coefficients, Xn
  - but, generally they are complex

$$X_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\exp(-jnw_0t)dt$$

Remember: fundamental $\omega_0 = 2\pi/T$!

Relationships

- There is a simple relationship between
  - trigonometrical and
  - complex Fourier coefficients,

$$X_0 = \frac{A_0}{2}$$

$$X_n = \begin{cases} \dfrac{A_n - jB_n}{2}, & n > 0; \\ \dfrac{A_n + jB_n}{2}, & n < 0. \end{cases}$$

Constrained to be symmetrical, i.e., complex conjugate

$$X_{-n} = X_n^*$$

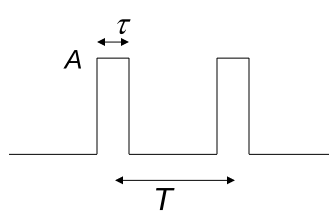
Therefore, can calculate simplest form and convert

Example: Complex FS

- Consider the pulse train signal

$$x(t) = \begin{cases} A, & 0 \le |t| \le \dfrac{\tau}{2}; \\ 0, & \dfrac{\tau}{2} < |t| \le T; \\ x(t+T). & \end{cases}$$

- Has complex Fourier series:



$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp\left(-jn\omega_0 t\right)dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\exp\left(-jn\omega_0 t\right)dt$$

$$= \frac{-A\tau}{jn\omega_0 T\tau}\left[\exp\left(\frac{-jn\omega_0\tau}{2}\right) - \exp\left(\frac{jn\omega_0\tau}{2}\right)\right]$$

Note: × by τ/τ …

Note: n is the ind. variable

Example: Complex FS

- Which using Euler's identity reduces to:

$$X_n = \frac{A\tau}{T}\frac{\sin(nw_0\,\tau/2)}{nw_0\,\tau/2} = \frac{A\tau}{T}\mathrm{sa}(nw_0\,\tau/2)$$

$$w_0 = \frac{2\pi}{T}$$

Note: letting $\theta = \frac{n\omega_0\tau}{2}$

$$\exp(-j\theta) - \exp(j\theta)$$
$$= \cos(-\theta) + j\sin(-\theta) - (\cos(\theta) + j\sin(\theta))$$
$$= \cos(\theta) - j\sin(\theta) - \cos(\theta) - j\sin(\theta) \quad = \quad -2j\sin(\theta)$$

Note:
cos(-θ) = cos(θ): even
sin(-θ) = -sin(θ): odd

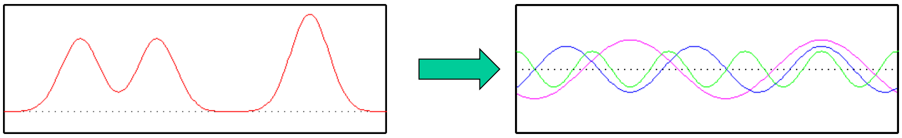
Frequency Response

Fourier Series → **Fourier Transforms**

Fourier Transform

- A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

1-D Example:



- When you let these three waves interfere with each other you get your original wave function!

Source: Tufts Uni Sykes Group

Fourier Series

- What we have produced is a processor to calculate one coefficient of the complex Fourier Series
- Fourier Series Coefficients = Heterodyne and average over observation interval T

$$C_k = \frac{1}{T}\int_0^T h(t)e^{-j\frac{2\pi}{T}kt}\,dt$$

Fourier Transform

- If we change the limits of integration to the entire real line, remove the division by T, and make the frequency variable continuous, we get the Fourier Transform

$$C(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$

Fourier Transform (is not the Fourier Series per se)

	Continuous Time	Discrete Time
Periodic	Fourier Series	Discrete Fourier Transform
Aperiodic	Continuous Fourier Transform	Fourier Transform

Source: URI ELE436

Fourier Transform

- Fourier series
  - Only applicable to periodic signals
- Real world signals are rarely periodic
- Develop Fourier transform by
  - Examining a periodic signal
  - Extending the period to infinity

Fourier Transform

- Problem: as $T \to \infty$, $X_n \to 0$
  - i.e., Fourier coefficients vanish!
- Solution: re-define coefficients
  - $X_n' = T \times X_n$
- As $T \to \infty$
  - (harmonic frequency) $n\omega_0 \to \omega$ (continuous freq.)
  - (discrete spectrum) $X_n' \to X(\omega)$ (continuous spect.)
  - ω0 (fundamental freq.) reduces $\to d\omega$ (differential)
    - Summation becomes integration

Fourier Transform Pair

Inverse Fourier Transform:

$$\boxed{f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega}$$

Synthesis

Fourier Transform:

$$\boxed{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt}$$

Analysis

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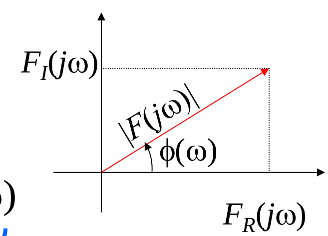
Continuous Spectra

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$F(j\omega) = F_R(j\omega) + jF_I(j\omega)$$

$$= \underbrace{|F(j\omega)|}_{\text{Magnitude}}\, e^{\overbrace{j\phi(\omega)}^{}}$$

Phase



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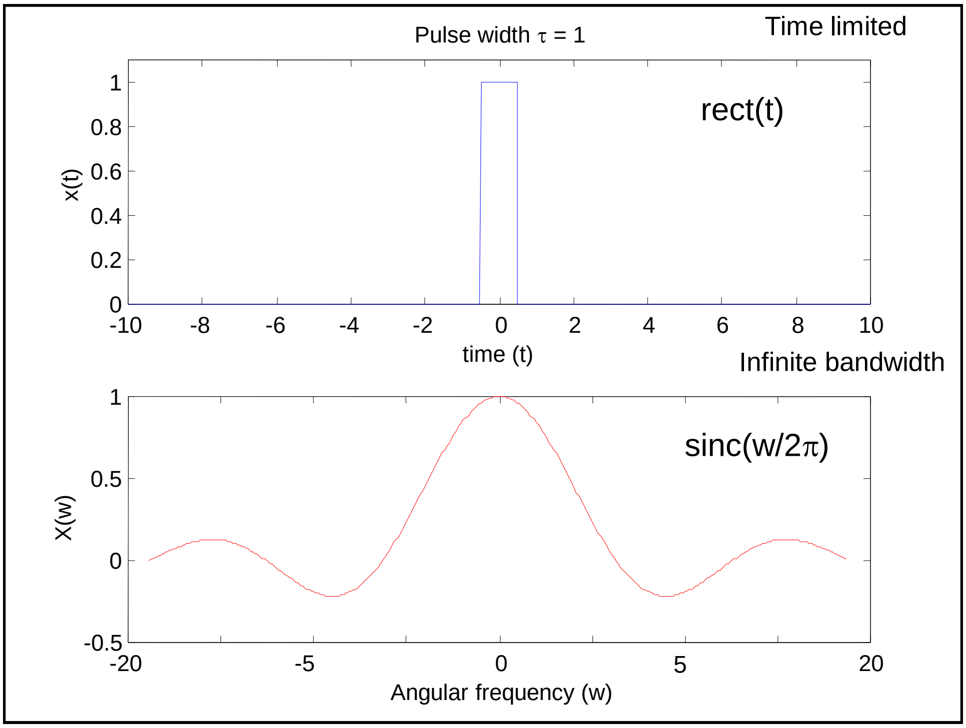
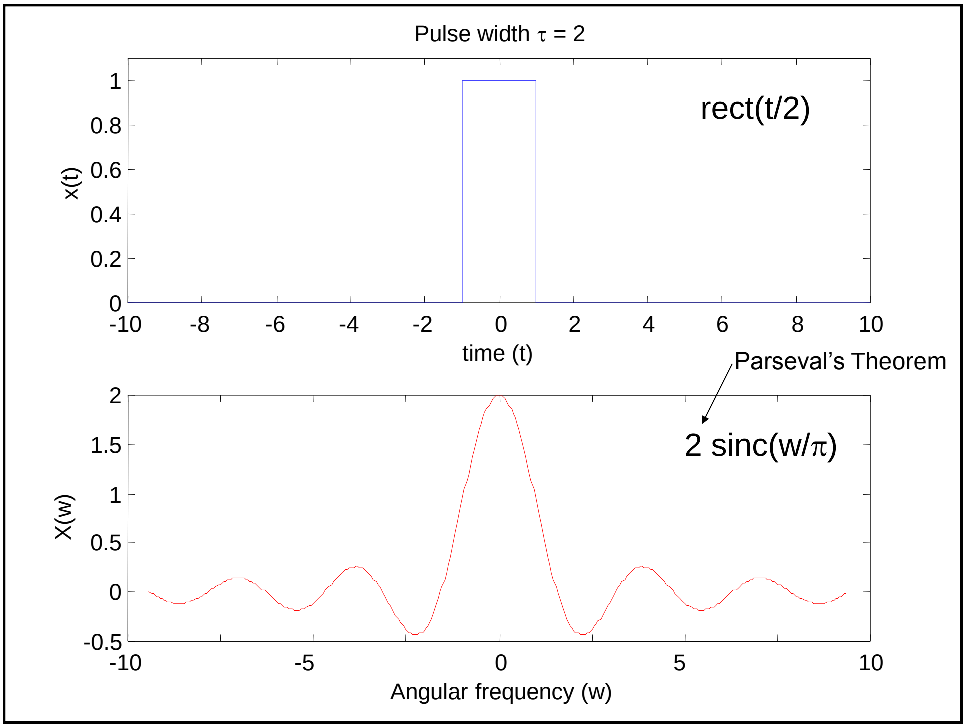
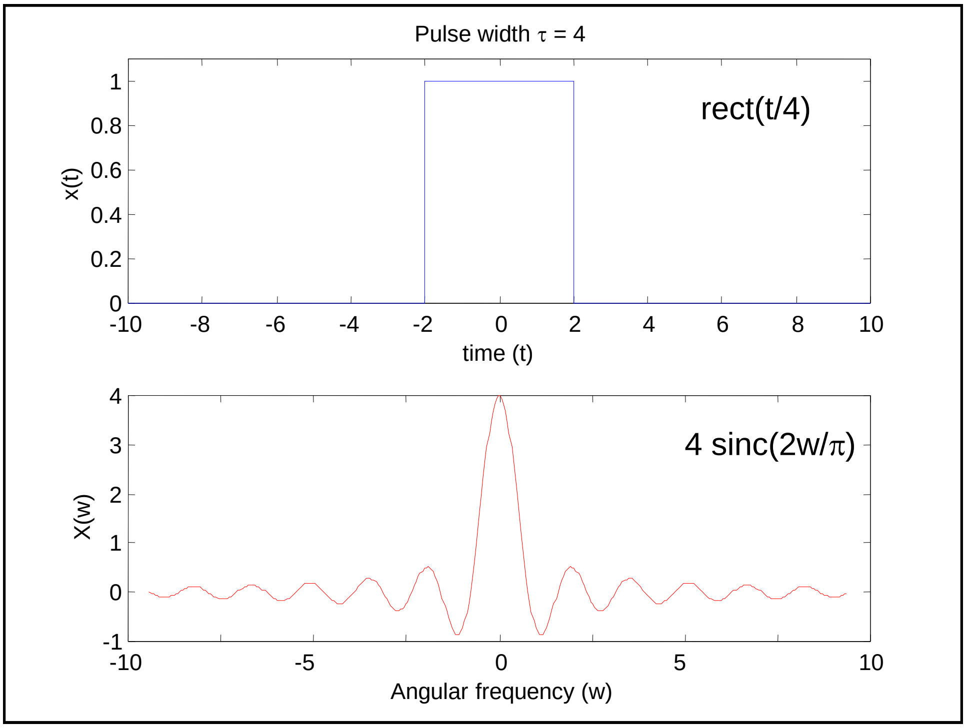
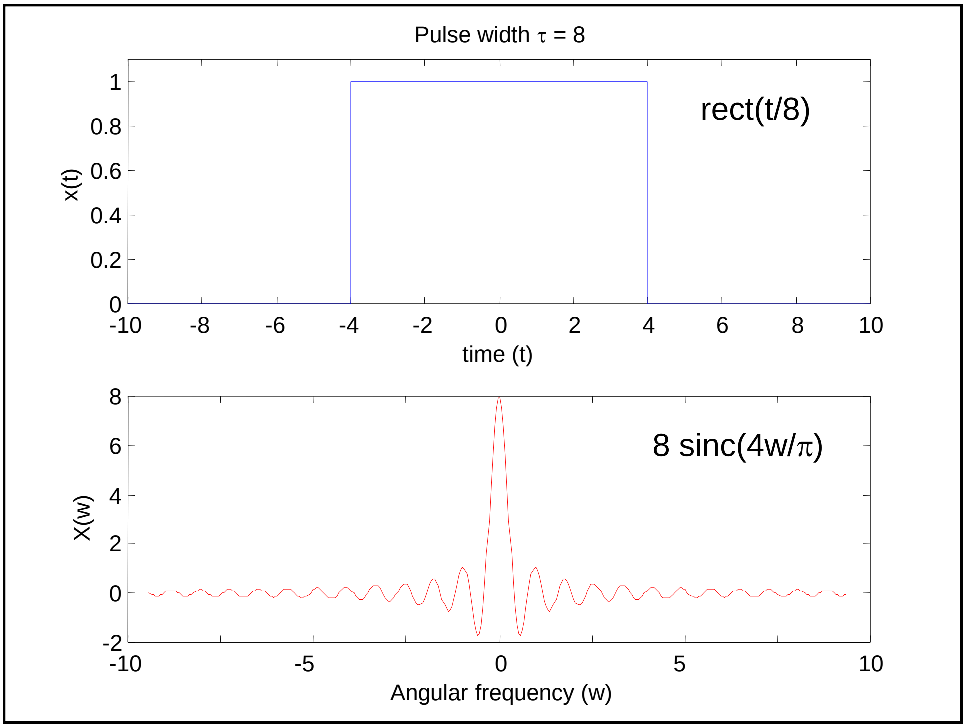
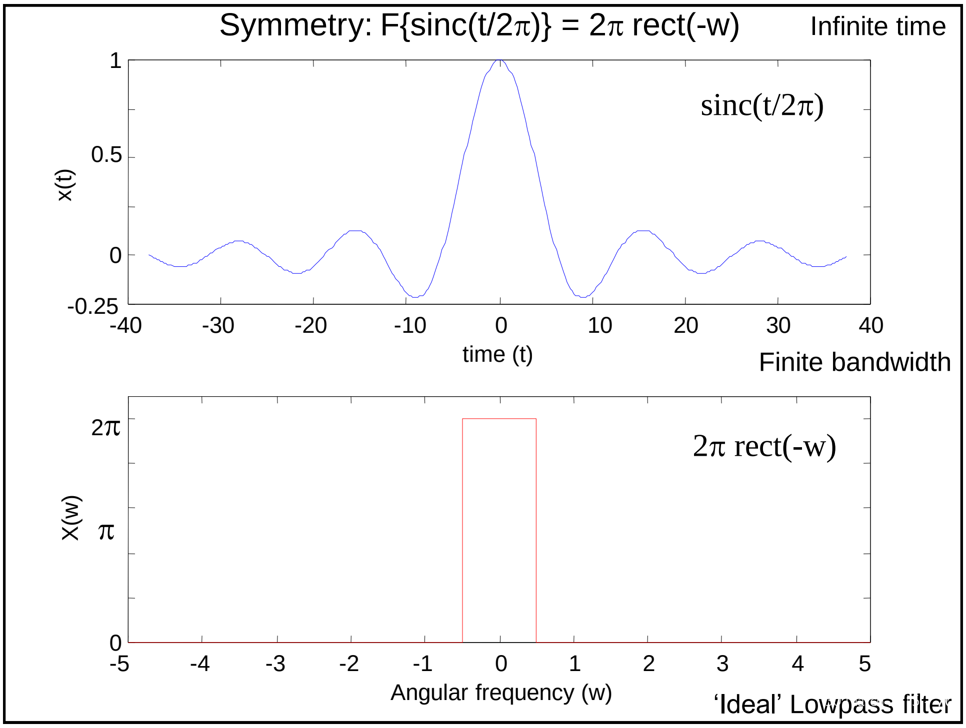

Pulse width τ = 1
Time limited
rect(t)
x(t)
time (t)
Infinite bandwidth
sinc(w/2π)
X(w)
Angular frequency (w)


Pulse width τ = 2
rect(t/2)
x(t)
time (t)
Parseval's Theorem
2 sinc(w/π)
X(w)
Angular frequency (w)


Pulse width τ = 4
rect(t/4)
x(t)
time (t)
4 sinc(2w/π)
X(w)
Angular frequency (w)


Pulse width τ = 8
rect(t/8)
x(t)
time (t)
8 sinc(4w/π)
X(w)
Angular frequency (w)



Properties of Fourier Transform

- Linearity
  - F {a x(t) + b y(t)} = a X(ω) + b Y(ω)
- Time and frequency scaling
  - F {x(at)} = 1/a X(ω/a)
  - broader in time ⇒ narrower in frequency
    - and vice versa
- Symmetry (duality)
  - 2πx(-ω) = ∫X(t) exp(-jωt)dt
    - i.e., Fourier transform 'pairs'

Time limited signal limited has infinite bandwidth;
Signal of finite bandwidth has infinite time support

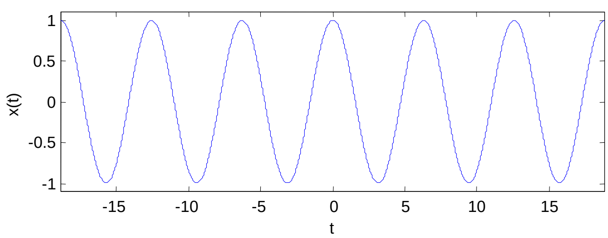
Properties of Fourier Transform

- if…
- x(t) is real
- x(t) is real and even
- x(t) is real and odd

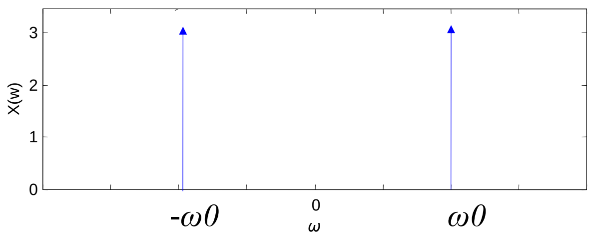
- Then…
- $X(-\omega) = X(\omega)^*$
  - $\Re\{X(\omega)\}$ is even
  - $\Im\{X(\omega)\}$ is odd
  - $|X(\omega)|$ is even
  - $\angle X(\omega)$ is odd
- $X(\omega)$ is real and even
- $X(\omega)$ is imaginary and odd

Fourier Transforms

$$X(\omega) = \delta(\omega - \omega_0) \mid x(t) = F^{-1}\{X(\omega)\} \mid x(t) = \frac{1}{2\pi}\exp(j\omega_0 t)$$



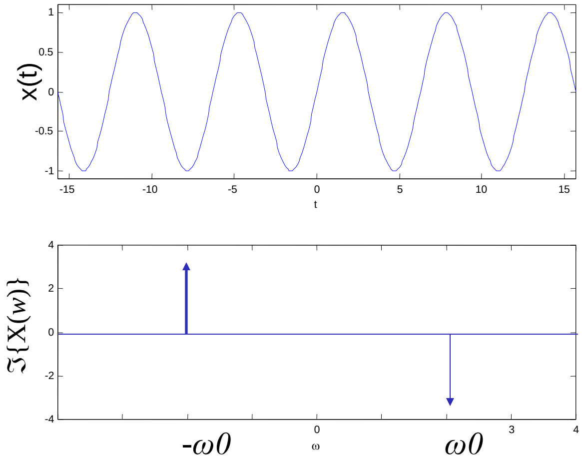
x(t) = cos($\omega_0 t$)
(real & even)



X(w) = $\pi[\delta(\omega-\omega_0)$
$+ \delta(\omega+\omega_0)]$
(real and even)

Note: cos(ω0t) has ∞ energy! But is dual of δ(w – ω0)

Fourier Transforms



$x(t) = \sin(\omega_0 t)$
(real and odd)

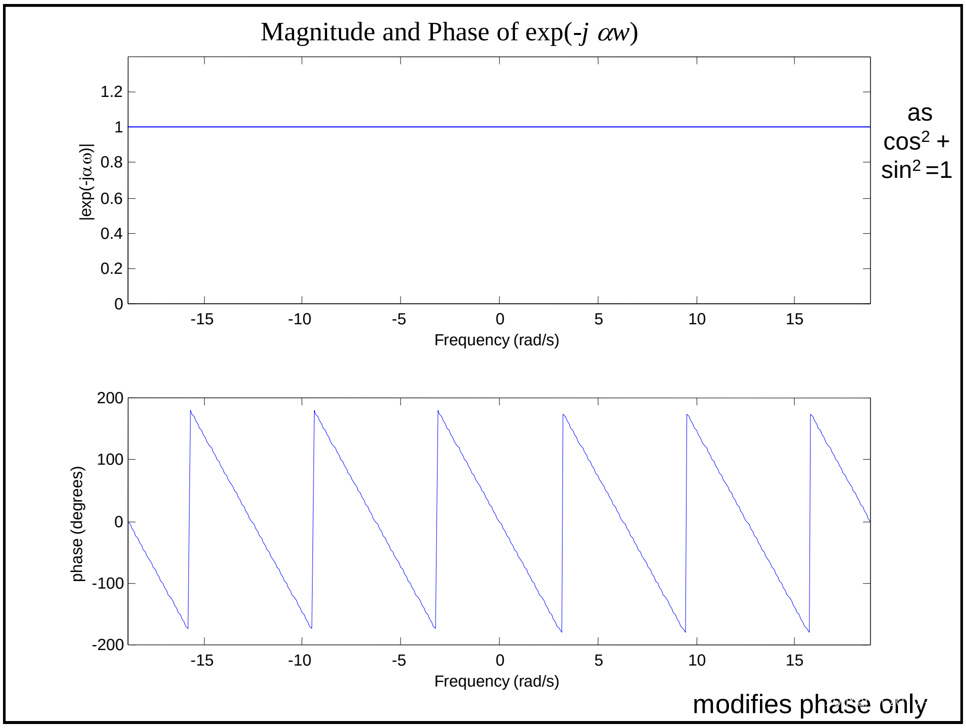
$X(w) = j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$
(imaginary & odd)

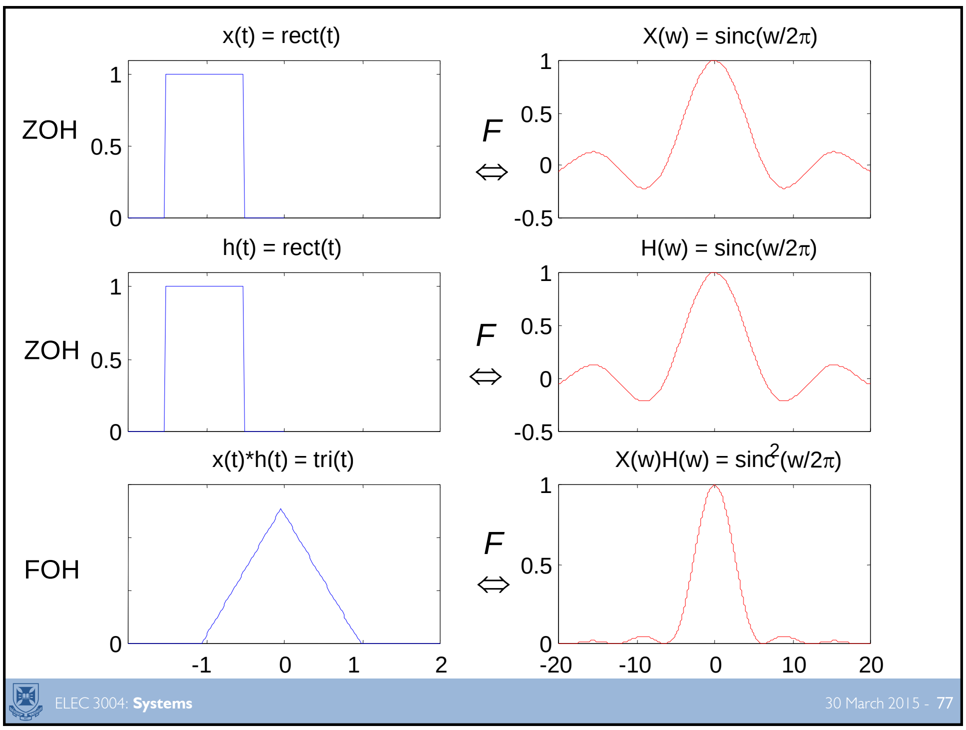
Note: sin & cos have same Mag spectrum, Phase is only difference

Properties of Fourier Transform

- Time Shift
  - $F\{x(t - \alpha)\} = \exp(-j\,\alpha w)X(w)$
    - time shift $\Rightarrow$ phase shift

- Convolution and multiplication
  - $F\{x(t) * y(t)\} = X(\omega) \cdot Y(\omega)$
    - i.e., implement convolution in Fourier domain
  - $F\{x(t) \cdot y(t)\} = 1/2\pi\,(X(\omega) * Y(\omega))$
    - i.e., Fourier interpretation of multiplication (e.g., frequency modulation)


Magnitude and Phase of exp(-j αw)
|exp(-jαω)|
Frequency (rad/s)
as
cos2 +
sin2 =1
phase (degrees)
Frequency (rad/s)
modifies phase only


x(t) = rect(t)
X(w) = sinc(w/2π)
ZOH
F
⇔
h(t) = rect(t)
H(w) = sinc(w/2π)
ZOH
F
⇔
x(t)*h(t) = tri(t)
X(w)H(w) = sinc2(w/2π)
FOH
F
⇔
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More properties of the FT

- Differentiation in time
- Integration in time

$$F\left\{\frac{d}{dt}x(t)\right\} = j\omega X(\omega)$$

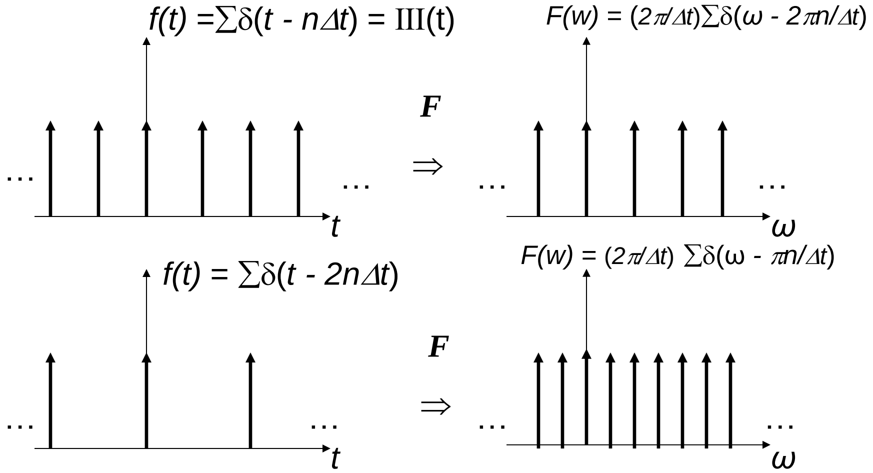
$$F\left\{\frac{d^n}{dt}x(t)\right\} = (j\omega)^n X(\omega)$$

Differentiation ⇒ ×ω
(Note: HPF & DC x zero)

$$F\left\{\int_{-\infty}^{t} x(t)\,dt\right\} = \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$

Integration ⇒ /ω + DC offset
(LPF & opposite of differentiation)

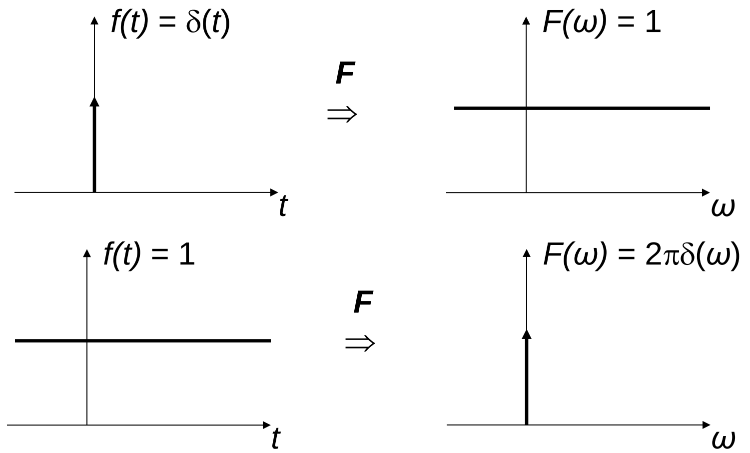
More Fourier Transforms



Impulse train, 'comb' or 'Shah' function

More Fourier Transforms

Limit of previous as $\Delta t \to \infty$ and $\Delta t \to 0$ respectively
Note: f(t) = 1 has ∞ energy! But is dual of $\delta(t)$ ☺



Note: u(t) also has ∞ energy! But F{u(t)} = F{$\int\delta(t)$} i.e., apply integration property

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"Pop Quiz": Questions

- If F{x(t)} = X(w)
  - F{x(2t)} =?
  - F{x(t/4)} =?
- F{δ(t)} = ?
- F{1} = ?

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Pop Quiz: Answers!

… No Worries LectopiaLand ☺ !

- If $F\{x(t)\} = X(\omega)$
  - $F\{x(2t)\} = 1/2X(\omega/2)$
    - narrower in t $\Rightarrow$ broader in freq
  - $F\{x(t/4)\} = 4X(4\omega)$
    - broader in t $\Rightarrow$ narrower in freq (but increased amplitude)
- $F\{\delta(t)\} = 1$
  - i.e. flat spectrum (all frequencies equally)
- $F\{1\} = \delta(\omega)$
  - i.e. impulse at DC only

Interpretation of Fourier Transform

- Represents (usually finite energy) signals
  - as sum of cosine waves
    - at all possible frequencies
    - $|X(\omega)|d\omega/2\pi$ is amplitude of cosine wave
      - i.e., in frequency band ω to $\omega + d\omega$
    - $\angle X(\omega)$ is phase shift of cosine wave
- Also represents finite power, periodic signals
  - Using $\delta(\omega)$
- Distribution with frequency of
  - both magnitude & phase
  - called a Frequency spectrum (continuous)

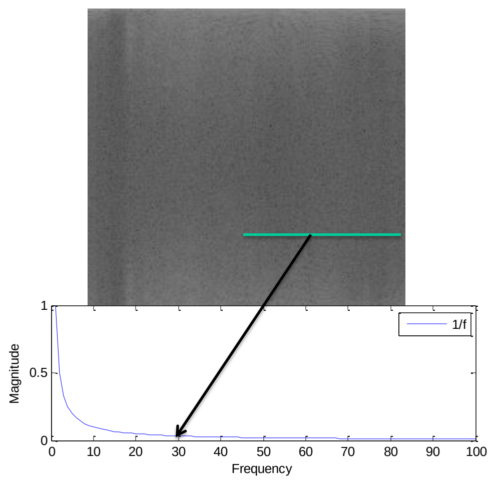
Fourier Image Examples

Lena

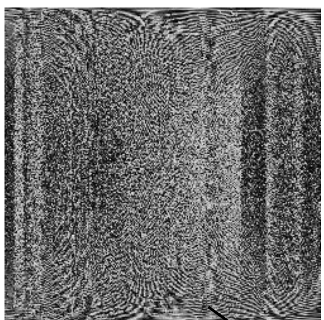
Bridge

Fourier Magnitude and Phase

20*log10(abs(fft(Lena)))



angle(fft(Lena))

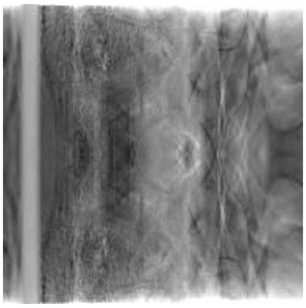


‘random’
range($\pm\pi$)

Bridge spectra look similar

Magnitude and Phase Only

ifft(abs(fft(Lena)) + angle(0))



Lena magnitude only

ifft(abs(fft(Bridge)) + angle(fft(Lena)))

Lena phase + bridge magnitude

Note: titles are illustrative only and are not the actual Matlab commands used!

FFT Fourier Transform
(**not** just yet – we'll come back to this)

Fourier Analysis of Discrete Time Signals

- For a discrete time sequence we define two classes of Fourier Transforms:

- the *<u>DTFT</u> (Discrete Time FT) for sequences having **<u>infinite</u>** duration,*

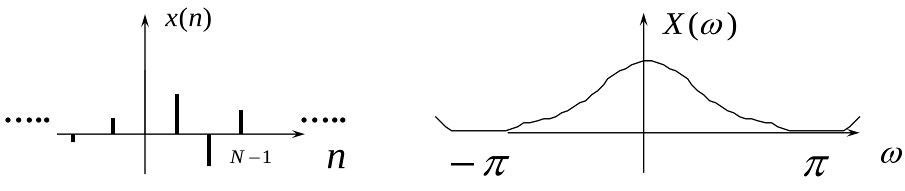
- the *<u>DFT</u> (Discrete FT) for sequences having **<u>finite</u>** duration.*

The *Discrete Time Fourier Transform (DTFT)*

- *Given a sequence x(n) having infinite duration, we define the DTFT as follows:*

$$X(\omega) = DTFT\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$$

$$x(n) = IDTFT\{X(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(\omega)e^{j\omega n}d\omega$$



<u>discrete time</u>

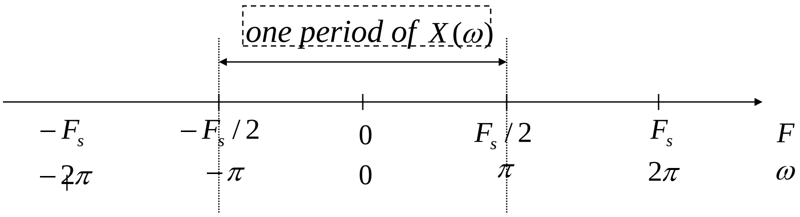
<u>continuous frequency</u>

Observations:

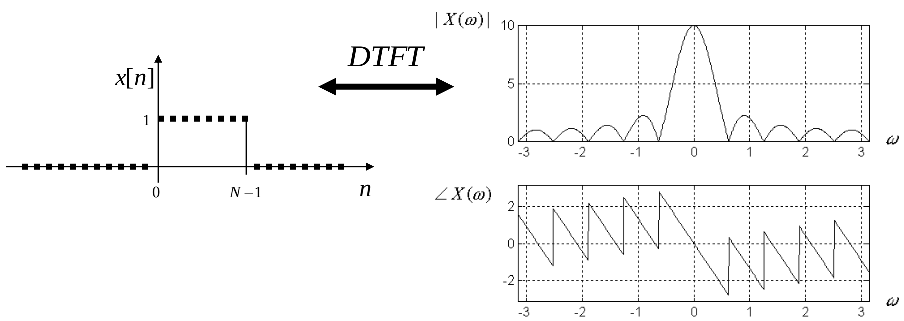
• *The DTFT* $X(\omega)$ *is periodic with period* 2π *;*

• *The frequency* ω *is the digital frequency and therefore it is limited to the interval* $-\pi < \omega < +\pi$

Recall that the digital frequency ω is a *normalized frequency* relative to the sampling frequency, defined as $\omega = 2\pi \frac{F}{F_s}$



Example



since

$$X(\omega) = \sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1-e^{-j\omega N}}{1-e^{-j\omega}}$$

$$= e^{-j\omega(N-1)/2} \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Fast Fourier Transform Algorithms

- Consider DTFT

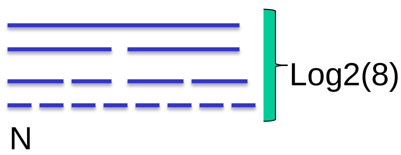
$$X[k] = \sum_{n=0}^{N-1} x[n] W^{kn}$$

There are approximately N^2 complex multiplications and additions required to implement this (N for each value of $X[k]$).

If $N = 2^{10} = 1024$, then $N^2 = 2^{20} = 10^6$, a very large number!

However, the FFT would only require about 5000, a substantial savings in complexity (the actual calculation is $\frac{N}{2} \log_2 N$).

- Basic idea is to split the sum into 2 subsequences of length N/2 and continue all the way down until you have N/2 subsequences of length 2



Radix-2 FFT Algorithms - Two point FFT

- We assume N=2^m
  - This is called **Radix-2 FFT Algorithms**
- Let's take a simple example where only two points are given n=0, n=1; N=2

$$\boxed{Y[k] = \sum_{n=0}^{1} y[n] W_2^{kn}} = y[0] + W_2^k y[1]$$

$$W_2 = e^{-\frac{j2\pi}{2}} = e^{-j\pi} = -1$$

So we get,

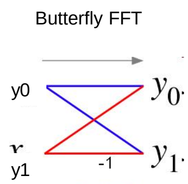
$$Y[k] = y[0] + (-1)^k y[1]$$

and:

$$Y[0] = y[0] + y[1]$$

$$Y[1] = y[0] - y[1]$$

Advantage: Less computationally intensive: **O**(N/2*log(N))



Source: http://www.cmlab.csie.ntu.edu.tw/cml/dsp/training/coding/transform/fft.html

General FFT Algorithm

- First break x[n] into even and odd

$$X[k] = \sum_{n=0}^{N-1} x[n] W^{kn}$$

- Let n=2m for even and n=2m+1 for odd
- Even and odd parts are both DFT of a N/2 point sequence

$$X[k] = \sum_{n\,\text{even}} x[n] W^{kn} + \sum_{n\,\text{odd}} x[n] W^{kn}$$

$$X[k] = \sum_{m=0}^{\frac{N}{2}-1} x[2m] W^{2mk} + \sum_{m=0}^{\frac{N}{2}-1} x[2m+1] W^{k(2m+1)} =$$

- Break up the size N/2 subsequent in half by letting 2m→m
- The first subsequence here is the term x[0], x[4], …
- The second subsequent is x[2], x[6], …

$$\boxed{\sum_{m=0}^{N/2-1} W_{N/2}{}^{mk} x[2m] + W_N{}^{k} \left(\sum_{m=0}^{N/2-1} W_{N/2}{}^{mk} x[2m+1] \right)}$$

$$W_N{}^{2mk} = W_{N/2}{}^{mk}$$

$$W_{N/2}{}^{m+N/2} = W_{N/2}{}^{m} W_{N/2}{}^{N/2} = W_{N/2}{}^{m}$$

$$W_N{}^{N} = e^{-2\pi j} = \cos(-2\pi) - j\sin(-2\pi) = 1$$

$$W_N{}^{N/2} = -1$$

Example

Let's take a simple example where only two points are given n=0, n=1; N=2

$$\boxed{X[k] = \sum_{m=0}^{N/2-1} W_{N/2}{}^{mk} x[2m] + W_N{}^{k} \left(\sum_{m=0}^{N/2-1} W_{N/2}{}^{mk} x[2m+1] \right)}$$

$$W_N{}^{2mk} = W_{N/2}{}^{mk}$$

$$W_{N/2}{}^{m+N/2} = W_{N/2}{}^{m} W_{N/2}{}^{N/2} = W_{N/2}{}^{m}$$

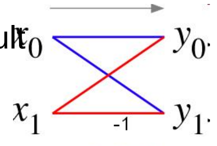
$$W_N{}^{N} = e^{-2\pi j} = \cos(-2\pi) - j\sin(-2\pi) = 1$$

$$W_N{}^{N/2} = -1$$

$$X[k=0] = \sum_{m=0}^{0} W_1{}^{0.0} x[0] + W_1{}^{0} \left(\sum_{m=0}^{0} W_1{}^{0.0} x[1] \right) = x[0] + x[1]$$

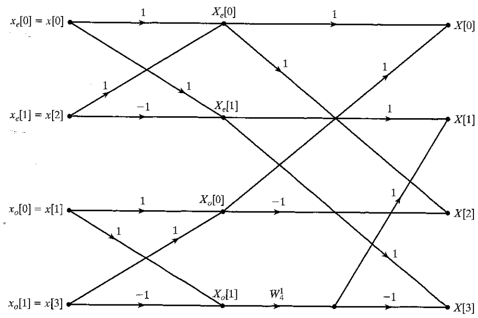
$$X[k=1] = \sum_{m=0}^{0} W_1{}^{0.1} x[0] + W_1{}^{1} \left(\sum_{m=0}^{0} W_1{}^{0.1} x[1] \right) = x[0] + W_1{}^{1} x[1] = x[0] - x[1]$$

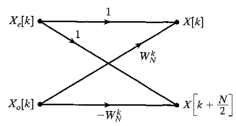
Same result



FFT Algorithms - Four point FFT

First find even and odd parts and then combine them:

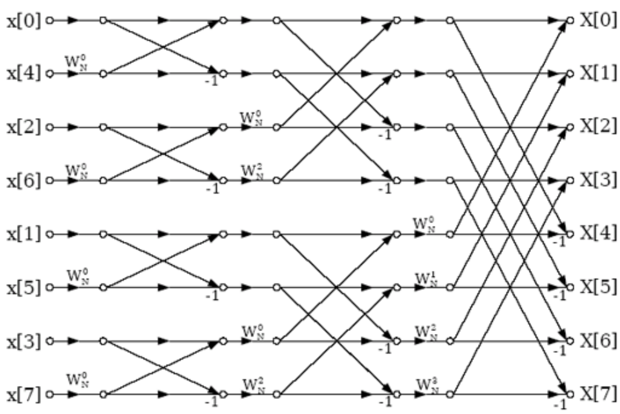




The general form:

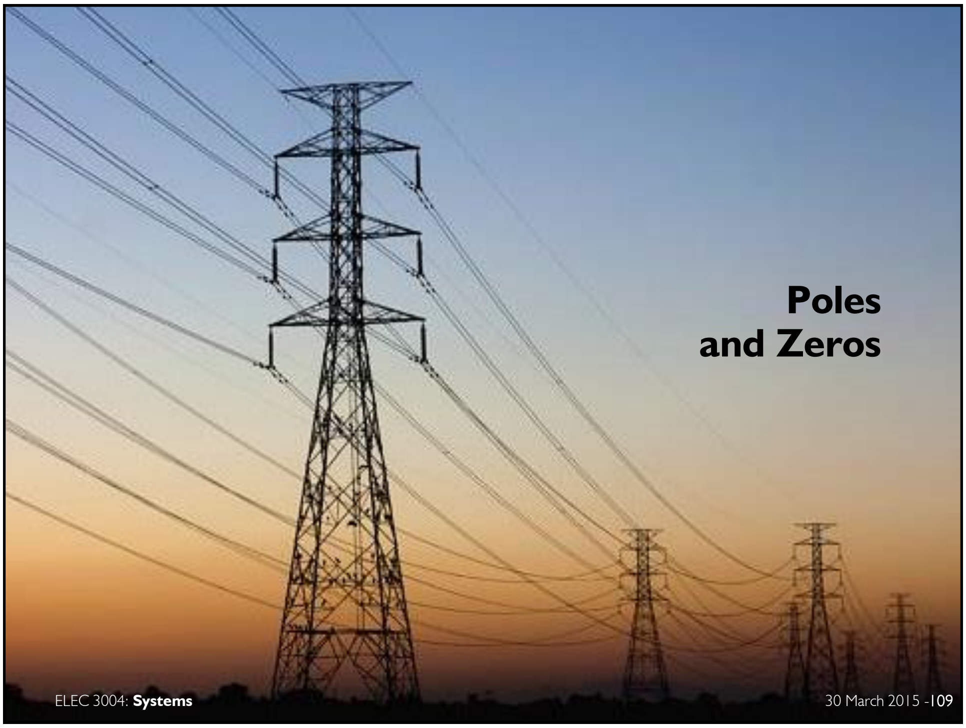
$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ F(3) \end{bmatrix} = \begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -j & -1 & +j \\ +1 & -1 & +1 & -1 \\ +1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

FFT Algorithms - 8 point FFT



Applet:
http://www.falstad.com/fourier/directions.html

http://www.engineeringproductivitytools.com/stuff/T0001/PT07.HTM



Poles and Zeros

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n},$$

assume b and a have no common factors (cancel them out if they do . . .)

- the m roots of b are called the *zeros* of F; λ is a zero of F if $F(\lambda) = 0$
- the n roots of a are called the *poles* of F; λ is a pole of F if $\lim_{s \to \lambda} |F(s)| = \infty$

the *multiplicity* of a zero (or pole) λ of F is the multiplicity of the root λ of b (or a)

example: $\dfrac{6s + 12}{s^2 + 2s + 1}$ has one zero at $s = -2$, two poles at $s = -1$

Source: Boyd, EE102,5-12

Poles and Zeros

factored or *pole-zero* form of F:

$$F(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n} = k \frac{(s - z_1)\cdots(s - z_m)}{(s - p_1)\cdots(s - p_n)}$$

where

- $k = b_m / a_n$
- $z_1, \ldots, z_m$ are the zeros of F ($i.e.$, roots of b)
- $p_1, \ldots, p_n$ are the poles of F ($i.e.$, roots of a)

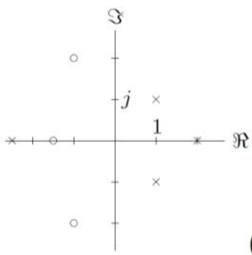
(assuming the coefficients of a and b are real) complex poles or zeros come in complex conjugate pairs

can also have *real factored form* . . .

Source: Boyd, EE102,5-13

Pole Zero Plot

poles & zeros of a rational functions are often shown in a *pole-zero plot*



($\times$ denotes a pole; $\circ$ denotes a zero)

this example is for

$$\begin{aligned} F(s) &= k\frac{(s+1.5)(s+1+2j)(s+1-2j)}{(s+2.5)(s-2)(s-1-j)(s-1+j)} \\ &= k\frac{(s+1.5)(s^2+2s+5)}{(s+2.5)(s-2)(s^2-2s+2)} \end{aligned}$$

(the plot doesn't tell us k)

Source: Boyd, EE102,5-14

Partial Fraction Expansion

$$F(s) = \frac{b(s)}{a(s)} = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots + a_n s^n}$$

let's assume (for now)

- no poles are repeated, *i.e.*, all roots of a have multiplicity one
- $m < n$

then we can write F in the form

$$F(s) = \frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}$$

called **partial fraction expansion** of F

- $\lambda_1, \ldots, \lambda_n$ are the poles of F
- the numbers $r_1, \ldots, r_n$ are called the **residues**
- when $\lambda_k = \overline{\lambda_l}$, $r_k = \overline{r_l}$

Source: Boyd, EE102,5-15

Partial Fraction Expansion Example

example:

$$\frac{s^2 - 2}{s^3 + 3s^2 + 2s} = \frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2}$$

let's check:

$$\frac{-1}{s} + \frac{1}{s+1} + \frac{1}{s+2} = \frac{-1(s+1)(s+2) + s(s+2) + s(s+1)}{s(s+1)(s+2)}$$

in partial fraction form, **inverse Laplace transform** is **easy:**

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{r_1}{s - \lambda_1} + \cdots + \frac{r_n}{s - \lambda_n}\right) \\ &= r_1 e^{\lambda_1 t} + \cdots + r_n e^{\lambda_n t} \end{aligned}$$

(this is real since whenever the poles are conjugates, the corresponding residues are also)

Source: Boyd, EE102,5-16

How to Handle the Digitization?

(z-Transforms)

The z-transform

- The discrete equivalent is the z-Transform†:

$$\mathcal{Z}\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k} = F(z)$$

and

$$\mathcal{Z}\{f(k-1)\} = z^{-1} F(z)$$

$$x(k) \longrightarrow \boxed{F(z)} \longrightarrow y(k)$$

Convenient!

†This is not an approximation, but approximations are easier to derive

The z-Transform

- It is defined by:

$$z = re^{j\omega}$$

Or in the Laplace domain:

$$z = e^{sT}$$

- Thus: $Y(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$ or $y[n] \overset{\mathcal{Z}}{\longleftrightarrow} Y(z)$

- I.E., It's a discrete version of the Laplace:

$$f(kT) = e^{-akT} \Rightarrow \mathcal{Z}\{f(k)\} = \frac{z}{z - e^{-aT}}$$

The *z*-transform

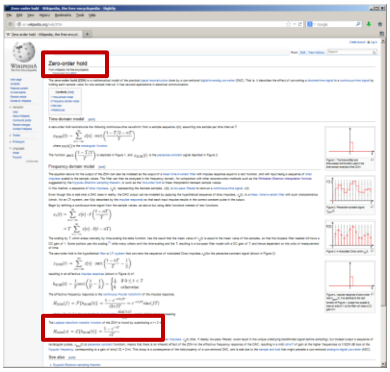
- In practice, you'll use look-up tables or computer tools (ie. Matlab) to find the *z*-transform of your functions

$\boldsymbol{F(s)}$	$\boldsymbol{F(kt)}$	$\boldsymbol{F(z)}$
$\frac{1}{s}$	1	$\frac{z}{z-1}$
$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$
$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$
$\frac{1}{(s+a)^2}$	kTe^{-akT}	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\frac{1}{s^2+a^2}$	$\sin(akT)$	$\frac{z \sin aT}{z^2 - (2\cos aT)z + 1}$

$\mathcal{L}$(ZOH)=??? : What is it?

$$\frac{1-e^{-Ts}}{Ts}$$

- Wikipedia

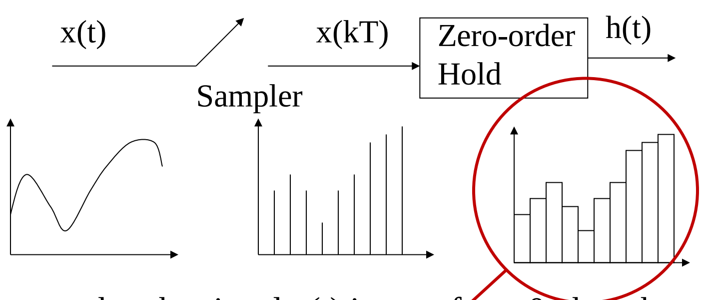


$$\frac{1-e^{-Ts}}{s}$$

- Lathi
- Franklin, Powell, Workman
- Franklin, Powell, Emani-Naeini
- Dorf & Bishop
- Oxford Discrete Systems: (Mark Cannon)
- MIT 6.002 (Russ Tedrake)
- Matlab

Proof!

Zero-order-hold (ZOH)



- Assume that the signal x(t) is zero for t<0, then the output h(t) is related to x(t) as follows:

$$h(t) = x(0)[1(t) - 1(t-T)] + x(T)[1(t-T) - 1(t-2T)] + \cdots$$

$$= \sum_{k=0}^{\infty} x(kT)[1(t-kT) - 1(t-(k+1)T)]$$

Transfer function of Zero-order-hold (ZOH)

- Recall the Laplace Transforms ($\mathcal{L}$) of:

$$\mathcal{L}[\delta(t)] = 1 \qquad \mathcal{L}[f(t-kT)] = F(s)e^{-kTs}$$

$$\mathcal{L}\left[\delta(t-kT)\right] = e^{-kTs} \qquad \mathcal{L}[1(t-kT)] = \frac{e^{-kTs}}{s}$$

- Thus the $\mathcal{L}$ of h(t) becomes:

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

Transfer function of Zero-order-hold (ZOH)

… Continuing the $\mathcal{L}$ of h(t) …

$$\mathcal{L}[h(t)] = \mathcal{L}[\sum_{k=0}^{\infty} x(kT)[1(t - kT) - 1(t - (k+1)T)]]$$

$$= \sum_{k=0}^{\infty} x(kT)L[1(t - kT) - 1(t - (k+1)T)] = \sum_{k=0}^{\infty} x(kT)[\frac{e^{-kTs}}{s} - \frac{e^{-(k+1)Ts}}{s}]$$

$$= \sum_{k=0}^{\infty} x(kT)\frac{e^{-kTs} - e^{-(k+1)Ts}}{s} = \sum_{k=0}^{\infty} x(kT)\frac{1-e^{-Ts}}{s}e^{-kTs} = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\rightarrow X(s) = \mathcal{L}\left[\sum_{k=0}^{\infty} x(kT)\delta(t-kT)\right] = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$\therefore H(s) = \mathcal{L}[h(t)] = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs} = \frac{1-e^{-Ts}}{s}X(s)$$

➔ Thus, giving the transfer function as:

$$\boxed{G_{\mathrm{ZOH}}(s) = \frac{H(s)}{X(s)} = \frac{1-e^{-Ts}}{s}} \xrightarrow{\mathcal{Z}} \boxed{G_{ZOH}(z) = \frac{\left(1-e^{-aT}\right)}{z-e^{-aT}}}$$