http://elec3004.org

Sampling and
CONVOLUTION

ELEC 3004: **Digital Linear Systems**: Signals & Controls

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Lecture 4

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Lecture Schedule:

Week	Date	Lecture Title
1	2-Mar	Introduction
	3-Mar	Systems Overview
2	9-Mar	Signals as Vectors & Systems as Maps
	10-Mar	[Signals]
3	16-Mar	Sampling & Data Acquisition & Antialiasing Filters
	17-Mar	[Sampling]
4	23-Mar	Sampling & Convolution
	24-Mar	[Convolution & FT]
5	30-Mar	Frequency Response & Filter Analysis
	31-Mar	[Filters]
6	13-Apr	Discrete Systems & Z-Transforms
	14-Apr	[Z-Transforms]
7	20-Apr	Introduction to Digital Control
	21-Apr	[Feedback]
8	27-Apr	Digital Filters
	28-Apr	[Digital Filters]
9	4-May	Digital Control Design
	5-May	[Digitial Control]
10	11-May	Stability of Digital Systems
	12-May	[Stability]
11	18-May	State-Space
	19-May	Controllability & Observability
12	25-May	PID Control & System Identification
	26-May	Digitial Control System Hardware
13	31-May	Applications in Industry & Information Theory & Communications
	2-Jun	Summary and Course Review

Sampling Theorem

- The Nyquist criterion states:

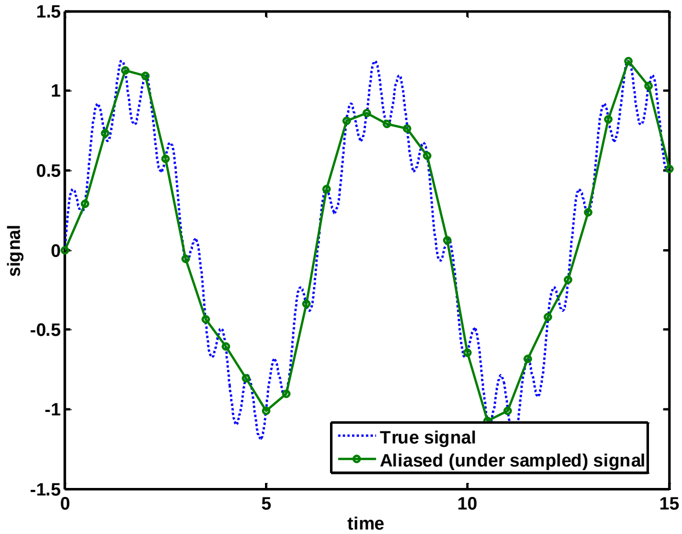
To prevent aliasing, a ***bandlimited*** *signal of bandwidth* w_B *rad/s must be sampled at a rate greater than* $2w_B$ *rad/s*

$$w_s > 2w_B$$

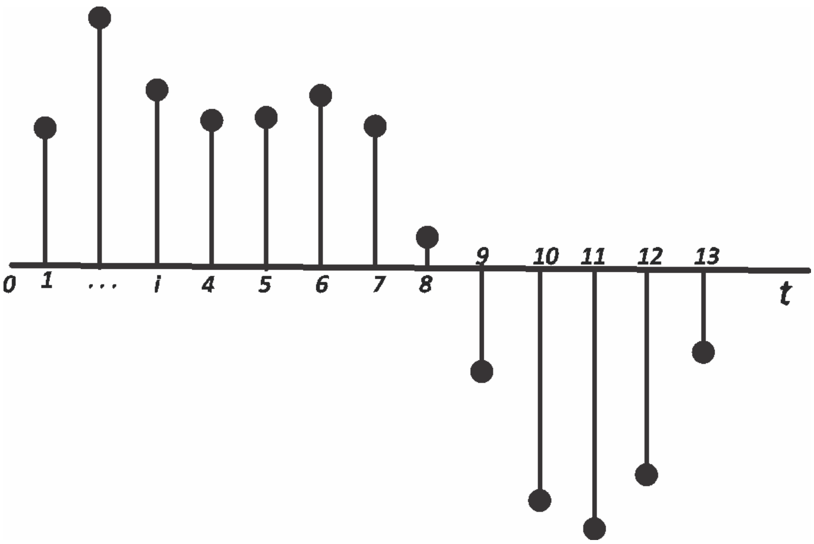
Note: this is a > sign not a ≥

Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

Sampling < Nyquist → Aliasing

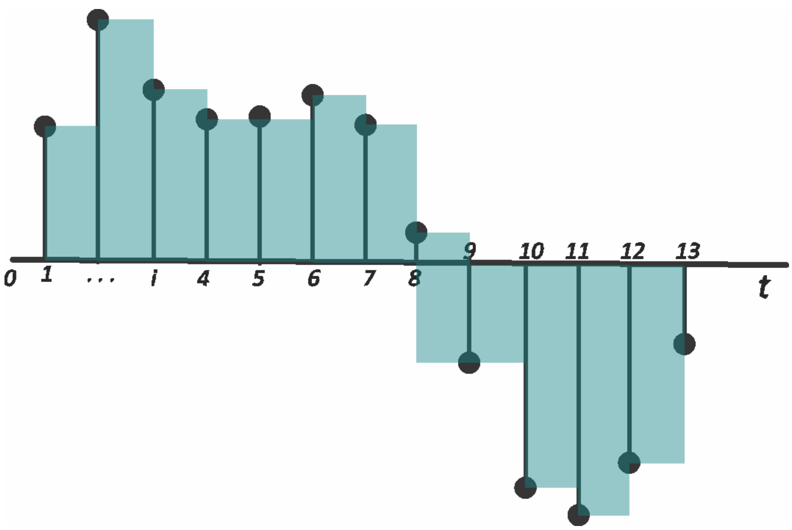


Reconstruction



Reconstruction

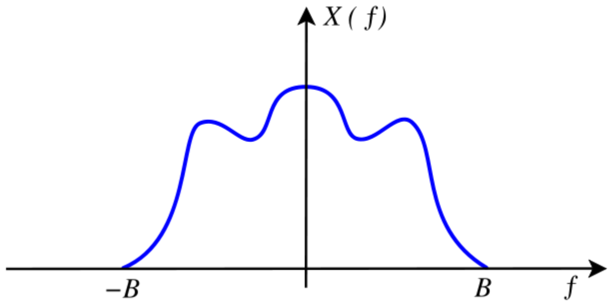
- Zero-Order Hold [ZOH]



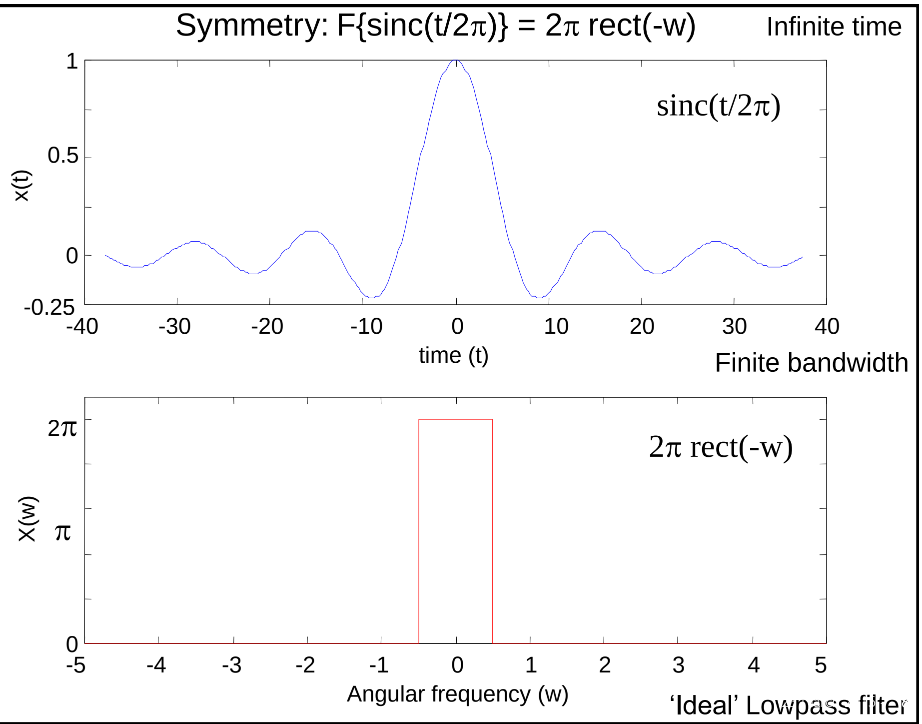
Reconstruction

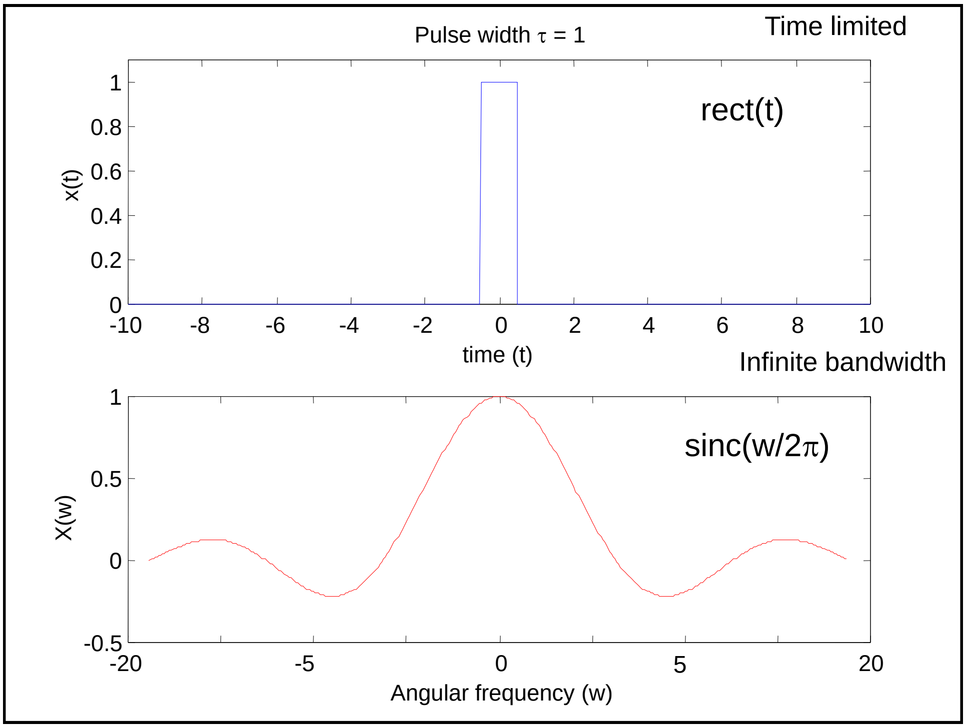
- Whittaker–Shannon interpolation formula

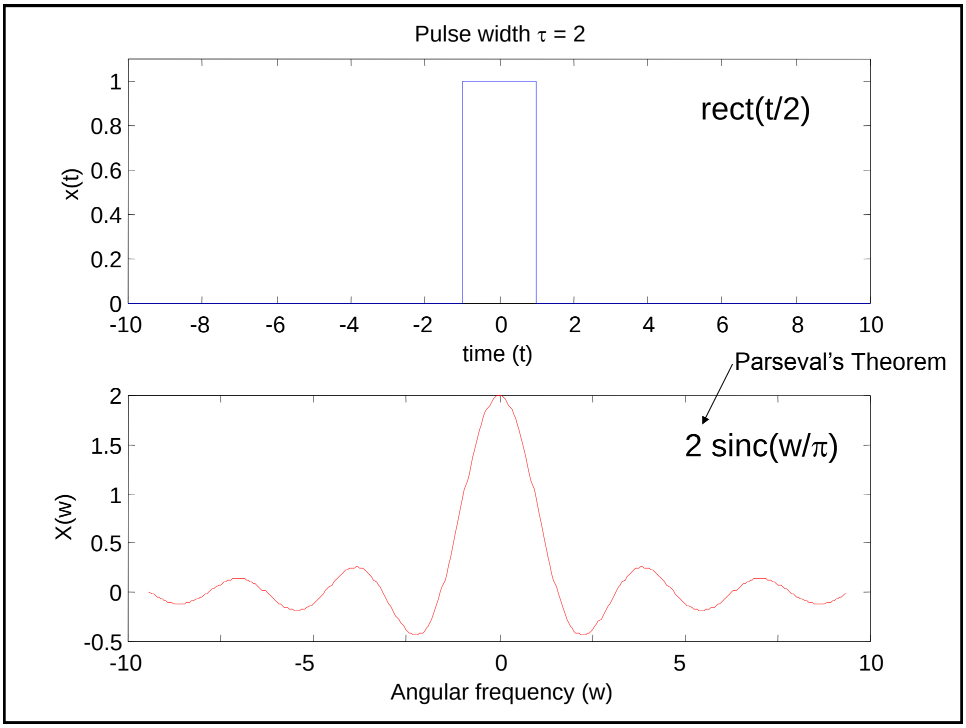
$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$

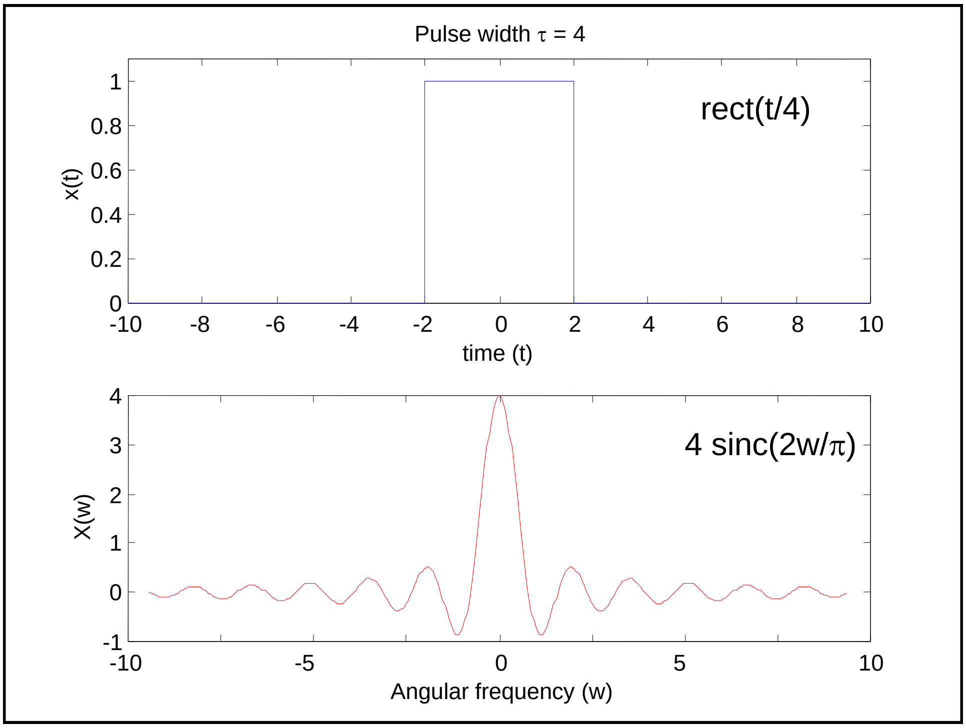


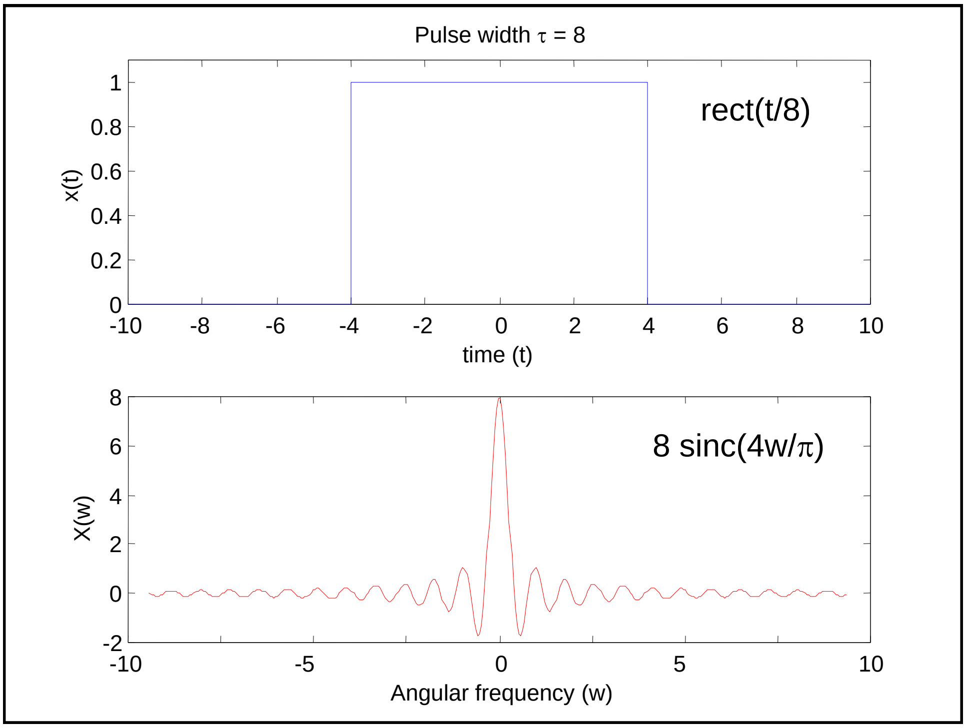
Symmetry: F{sinc(t/2π)} = 2π rect(-w)




Pulse width τ = 1
Time limited
rect(t)
x(t)
time (t)
Infinite bandwidth
sinc(w/2π)
X(w)
Angular frequency (w)

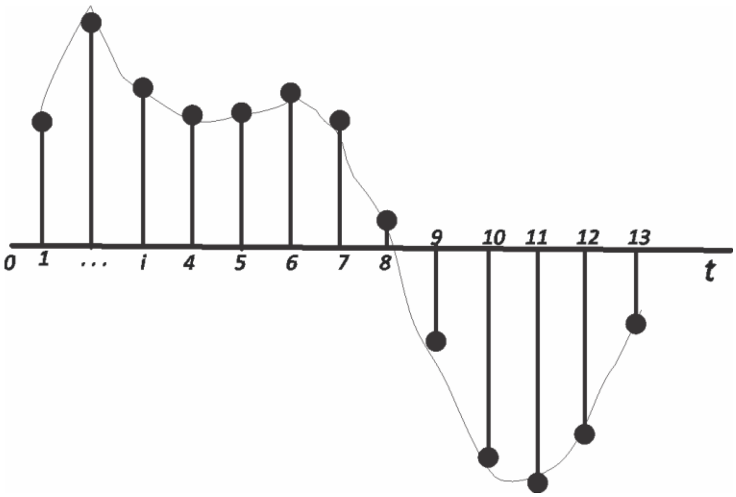

Pulse width τ = 2
rect(t/2)
x(t)
time (t)
Parseval's Theorem
2 sinc(w/π)
X(w)
Angular frequency (w)

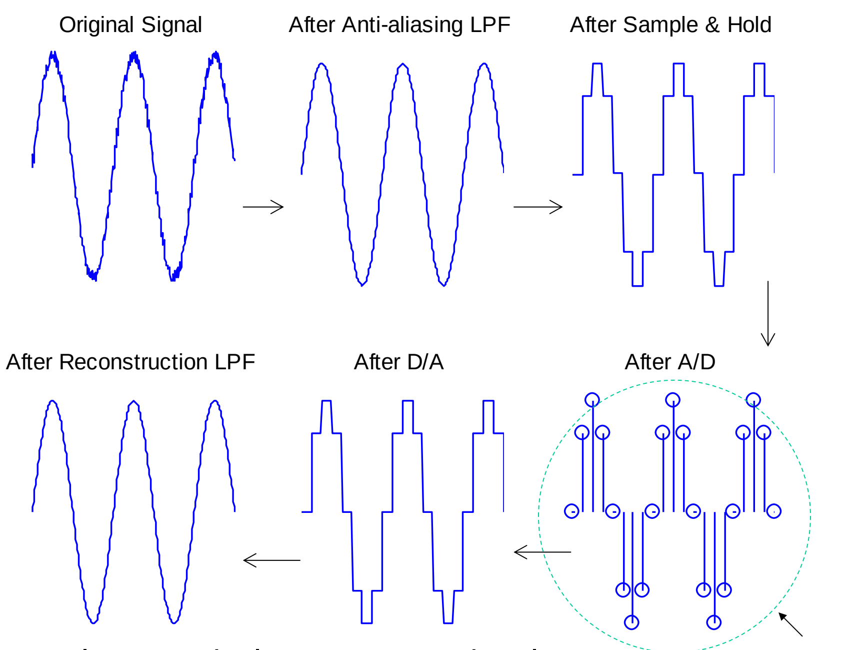

Pulse width τ = 4
rect(t/4)
x(t)
time (t)
4 sinc(2w/π)
X(w)
Angular frequency (w)


Pulse width τ = 8
rect(t/8)
x(t)
time (t)
8 sinc(4w/π)
X(w)
Angular frequency (w)

Reconstruction

- Whittaker–Shannon interpolation formula





Complete practical DSP system signals

Convolution

Convolution Definition

The convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as:

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

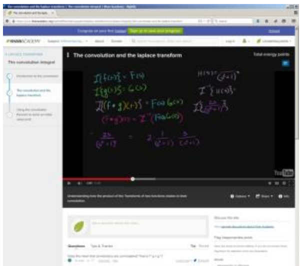
$$= f_1(t) * f_2(t)$$

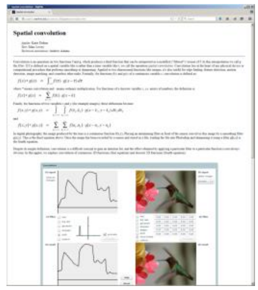
Source: URI ELE436

Convolution: Concepts

- My goal is to give you a feel for the Convolution in Systems

- For the mechanics of Convolution:
  - Many good Convolution reviews online
  - EG: Khan Academy…
    https://www.khanacademy.org/math/differential-equations/laplace-transform/convolution-integral/v/the-convolution-and-the-laplace-transform
  - & More… http://graphics.stanford.edu/courses/cs178/applets/convolution.html



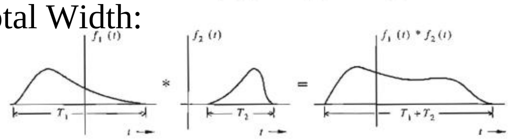


Convolution & Properties

$$f_1(t) * f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

Properties:

- Commutative: $f_1(t)*f_2(t) = f_2(t)*f_1(t)$
- Distributive: $f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$
- Associative: $f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t)$
- Shift:
  if $f_1(t)*f_2(t)=c(t)$, then $f_1(t-\mathbf{T})*f_2(t)= f_1(t)*f_2(t-\mathbf{T})=c(t-\mathbf{T})$
- Identity (Convolution with an Impulse):
  $f(t) * \delta(t) = f(t)$
- Total Width:



Based on Lathi, *SPLS*, Sec 2.4-1

Convolution & Properties [II]

- Convolution systems are **linear**:

$$h * (\alpha u_1 + \beta u_2) = \alpha(h * u_1) + \beta(h * u_2)$$

- Convolution systems are **causal**: the output y(t) at time t depends only on past inputs

- Convolution systems are **time-invariant** (if we shift the signal, the output similarly shifts)

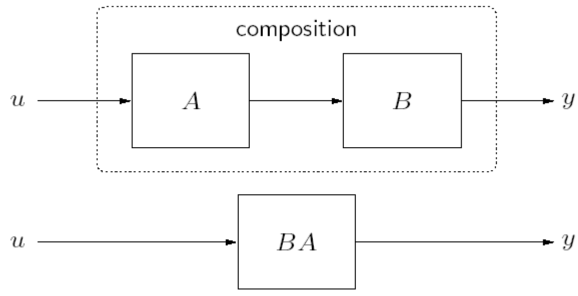
➔

$$\tilde{u}(t) = \begin{cases} 0 & t < T \\ u(t-T) & t \geq 0 \end{cases}$$

$$\tilde{y}(t) = \begin{cases} 0 & t < T \\ y(t-T) & t \geq 0 \end{cases}$$

Convolution & Properties [III]

- Composition of convolution systems corresponds to:
  - multiplication of transfer functions
  - convolution of impulse responses



- Thus:
  - We can manipulate block diagrams with transfer functions as if they were simple gains
  - convolution systems commute with each other

Properties of Convolution

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=-\infty}^{\tau=\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$= \int_{t-\tau=-\infty}^{t-\tau=\infty} f_1(t-\tau) f_2[t-(t-\tau)] d(t-\tau)$$

$$= -\int_{\tau=\infty}^{\tau=-\infty} f_1(t-\tau) f_2(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau \;\; = f_2(t) * f_1(t)$$

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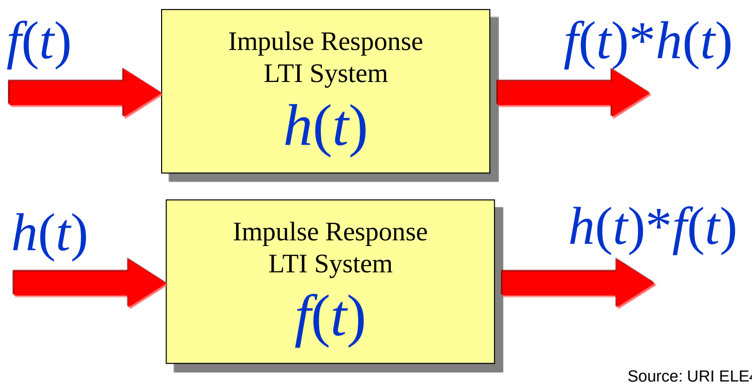
Properties of Convolution

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$



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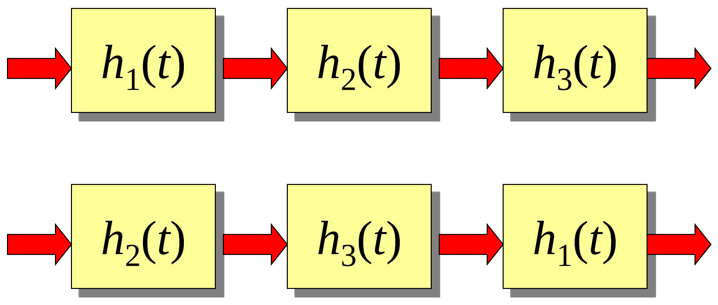
Properties of Convolution

$$[f_1(t) * f_2(t)] * f_3(t) = f_1(t) * [f_2(t) * f_3(t)]$$



- The two systems are identical!

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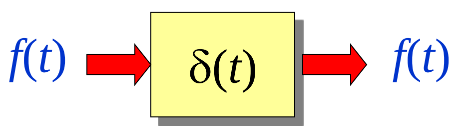
Properties of Convolution

$$f(t) * \delta(t) = f(t)$$



$$\begin{aligned} f(t) * \delta(t) &= \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau \\ &= \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau \\ &= f(t) \end{aligned}$$

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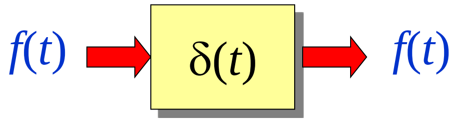
Properties of Convolution

$$f(t) * \delta(t) = f(t)$$



$$f(t) * \delta(t-T) = f(t-T)$$

$$f(t) * \delta(t-T) = \int_{-\infty}^{\infty} f(\tau)\delta(t-T-\tau)d\tau$$
$$= \int_{-\infty}^{\infty} f(t-T-\tau)\delta(\tau)d\tau$$
$$= f(t-T)$$

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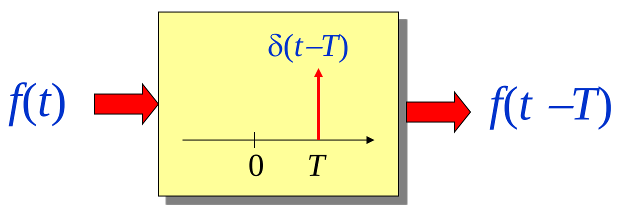
Properties of Convolution

$$f(t) * \delta(t-T) = f(t-T)$$



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Properties of Convolution

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$

$$F[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)e^{-j\omega t}dt\right]d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau)F_2(j\omega)e^{-j\omega\tau}d\tau$$

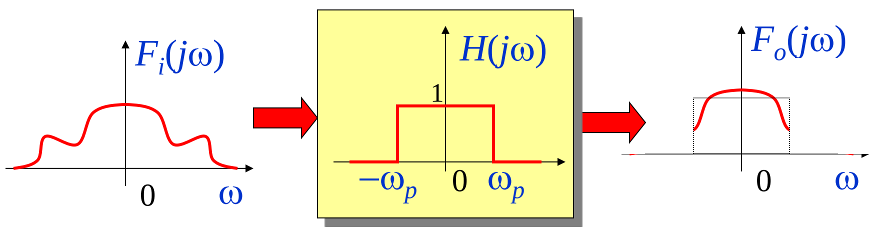
$$= F_2(j\omega)\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}d\tau \quad = F_1(j\omega)F_2(j\omega)$$

Time Domain	Frequency Domain
convolution	multiplication

LE436

Properties of Convolution

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$



An Ideal Low-Pass Filter

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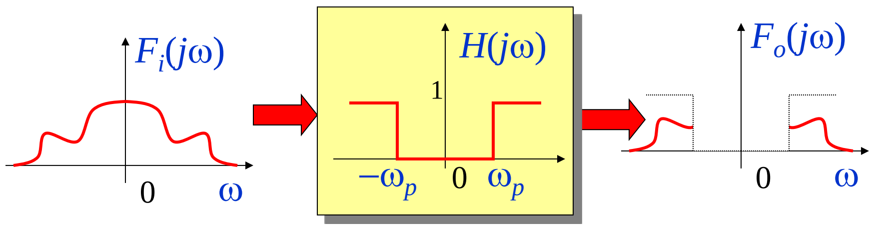
Properties of Convolution

$$f_1(t) * f_2(t) \overset{\mathcal{F}}{\longleftrightarrow} F_1(j\omega)F_2(j\omega)$$



An Ideal High-Pass Filter

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Convolution & Systems

- Convolution system with input u (u(t) = 0, t < 0) and output y:

$$y(t) = \int_0^t h(\tau)u(t-\tau)\, d\tau = \int_0^t h(t-\tau)u(\tau)\, d\tau$$

- abbreviated:

$$y = h * u$$

- in the frequency domain:

$$Y(s) = H(s)U(s)$$

Systems Interpretation

Transfer function

take Laplace transform of $\dot{x} = Ax + Bu$:

$$sX(s) - x(0) = AX(s) + BU(s)$$

hence

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s)$$

so

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-\tau)A}Bu(\tau)\, d\tau$$

- $e^{tA}x(0)$ is the unforced or autonomous response
- $e^{tA}B$ is called the input-to-state impulse response or impulse matrix
- $(sI - A)^{-1}B$ is called the *input-to-state transfer function* or *transfer matrix*

Source: Lecture Notes for EE263, Stephen Boyd, Stanford 2012., Slide: 13-6

Systems Interpretation

with $y = Cx + Du$ we have:

$$Y(s) = C(sI - A)^{-1}x(0) + (C(sI - A)^{-1}B + D)U(s)$$

so

$$y(t) = Ce^{tA}x(0) + \int_0^t Ce^{(t-\tau)A}Bu(\tau)\, d\tau + Du(t)$$

- output term $Ce^{tA}x(0)$ due to initial condition
- $H(s) = C(sI - A)^{-1}B + D$ is called the *transfer function* or *transfer matrix*
- $h(t) = Ce^{tA}B + D\delta(t)$ is called the *impulse response* or *impulse matrix* (δ is the Dirac delta function)

Source: Lecture Notes for EE263, Stephen Boyd, Stanford 2012., Slide: 13-7

Systems Interpretation

with zero initial condition we have:

$$Y(s) = H(s)U(s), \qquad y = h * u$$

where $*$ is convolution (of matrix valued functions)

intepretation:

- H_{ij} is transfer function from input u_j to output y_i

Source: Lecture Notes for EE263, Stephen Boyd, Stanford 2012., Slide: 13-8

Systems Interpretation

Impulse response

impulse response $h(t) = Ce^{tA}B + D\delta(t)$

with $x(0) = 0$, $y = h * u$, *i.e.*,

$$y_i(t) = \sum_{j=1}^{m} \int_0^t h_{ij}(t-\tau)u_j(\tau)\, d\tau$$

interpretations:

- $h_{ij}(t)$ is impulse response from jth input to ith output
- $h_{ij}(t)$ gives y_i when $u(t) = e_j\delta$
- $h_{ij}(\tau)$ shows how dependent output i is, on what input j was, τ seconds ago
- i indexes output; j indexes input; τ indexes time lag

Source: Lecture Notes for EE263, Stephen Boyd, Stanford 2012., Slide: 13-9

Systems Interpretation

Step response

the *step response* or *step matrix* is given by

$$s(t) = \int_0^t h(\tau)\, d\tau$$

interpretations:

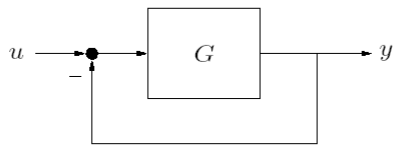
- $s_{ij}(t)$ is step response from jth input to ith output
- $s_{ij}(t)$ gives y_i when $u = e_j$ for $t \geq 0$

for invertible A, we have

$$s(t) = CA^{-1}\left(e^{tA} - I\right)B + D$$

Source: Lecture Notes for EE263, Stephen Boyd, Stanford 2012., Slide: 13-10

Convolution & Feedback



- In the time domain:

$$y(t) = \int_0^t g(t-\tau)(u(\tau) - y(\tau))\, d\tau$$

- In the frequency domain:
  - Y=G(U-Y)
  - ➔Y(s) = H(s)U(s)

$$H(s) = \frac{G(s)}{1 + G(s)}$$

Graphical Understanding of Convolution

➔ For c(τ)= $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau)\, d\tau$:

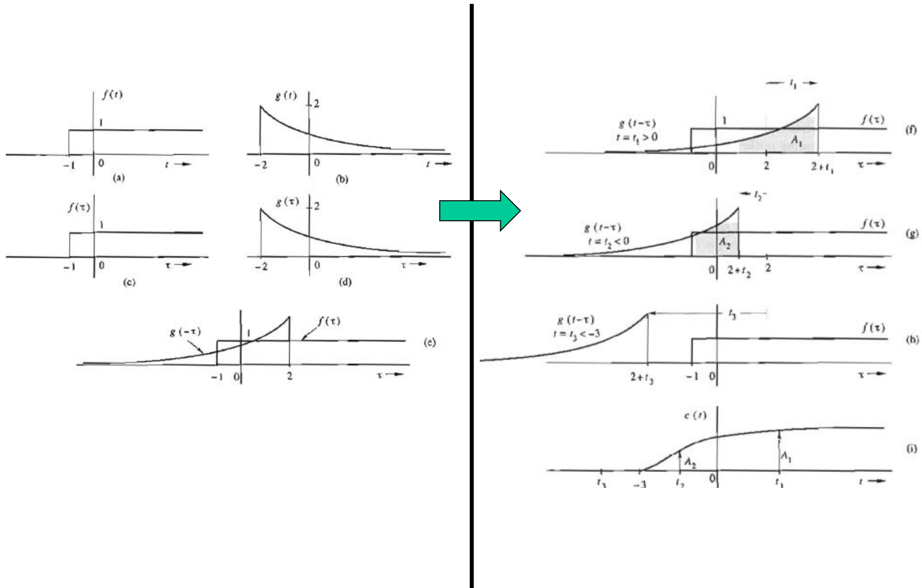
1. Keep the function ***f* (*τ*)** fixed
2. **Flip** (invert) the function ***g*(*τ*)** about the vertical axis (τ=0)
   = this is ***g*(-*τ*)**
3. **Shift** this frame (***g*(-*τ*)**) along τ (horizontal axis) by $\mathbf{t_0}$.
   = this is $\boldsymbol{g(t_0 - \tau)}$

➔ For c($\mathbf{t_0}$):

4. $\mathbf{c(t_0)}$ = the area under the product of ***f* (*τ*) and** $\boldsymbol{g(t_0 - \tau)}$

5. Repeat this procedure, shifting the frame by different values (positive and negative) to obtain **c(t) for all values of t**.

Graphical Understanding of Convolution (Ex)



Another View

e.g. convolution

x(n) = 1 2 3 4 5

h(n) = 3 2 1

x(k)	0 0 1 | 2 3 4 5	0 0 1 2 | 3 4 5	0 0 1 2 3 | 4 5	
h(n,k)	1 2 3 | 0 0 0 0	0 1 2 3 | 0 0 0	0 0 1 2 3 | 0 0	h(n-k)
y(n,k)	3	2 6	1 4 9	
y(n)	3	8	14	

Sum over all k

Notice the gain

Matrix Formulation of Convolution

$$\mathbf{y} = \mathbf{H}\mathbf{x}$$

$$\begin{bmatrix} 3 \\ 8 \\ 14 \\ 20 \\ 26 \\ 14 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

Toeplitz Matrix