Digital Control Systems: Shaping the Dynamic Response

ELEC 3004: **Digital Linear Systems**: Signals & Controls

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Lecture 12

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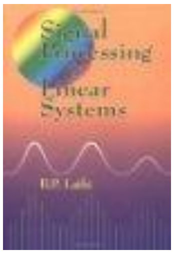
Schedule

Week	Date	Lecture Title
1	2-Mar	Introduction
	3-Mar	Systems Overview
2	9-Mar	Signals as Vectors & Systems as Maps
	10-Mar	[Signals]
3	16-Mar	Sampling & Data Acquisition & Antialiasing Filters
	17-Mar	[Sampling]
4	23-Mar	System Analysis & Convolution
	24-Mar	[Convolution & FT]
5	30-Mar	Discrete Systems & Z-Transforms
	31-Mar	[Z-Transforms]
6	13-Apr	Frequency Response & Filter Analysis
	14-Apr	[Filters]
7	20-Apr	Digital Filters
	21-Apr	[Digital Filters]
8	27-Apr	Discrete Systems Analysis
	28-Apr	[Feedback]
9	4-May	Introduction to (Digital) Control
	5-May	[Digitial Control]
10	11-May	Digital Control Design
	12-May	[Introduction to State-Space]
11	18-May	State-Space - Analysis
	19-May	[Stability]
12	25-May	Digital Control Systems: Shaping the Dynamic Response
	26-May	[Applications in Industry]
13	1-Jun	System Identification & [Summary and Course Review]
	2-Jun	Information Theory + Communications

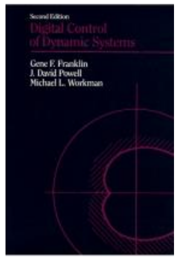
** Additional Reading Materials **



- Chapter 13

  State Space Analysis

  – "s-plane"
  – `tf2ss`

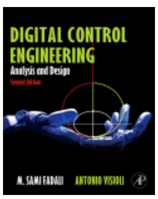


- Chapter 6

  [Design of Digital Control Systems Using State-Space Methods]

  – Extends to "z-plane"
  – `c2d`

Online Reading Materials



M. Sami Fadali and Antonio Visioli
Digital Control Engineering Analysis & Design (2nd Edition), 2012
Online via ScienceDirect

Chapter 7: State Space Representation
Chapter 9: State Feedback Control

Lab 4

Digital State Space Recap

Digital State Space:

- Difference equations in state-space form:

$$x[n+1] = Ax[n] + Bu[n]$$
$$y[n] = Cx[n] + Du[n]$$

- Where:
  - u[n], y[n]: input & output (scalars)
  - x[n]: state vector

Digital Control Law Design

In Chapter 2, we saw that the state-space description of a continuous system is given by (2.43),

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u, \tag{6.1}$$

and (2.44),

$$y = \mathbf{H}\mathbf{x}. \tag{6.2}$$

We assume the control is applied from the computer by a ZOH as shown in Fig. 1.1. Therefore, (6.1) and (6.2) have an exact discrete representation as given by (2.57),

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k),$$
$$y(k) = \mathbf{H}\mathbf{x}(k), \tag{6.3}$$

where

$$\mathbf{\Phi} = e^{\mathbf{F}T}, \tag{6.4a}$$

$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G}, \tag{6.4b}$$

Discretisation FTW!

- We can use the time-domain representation to produce difference equations!

$$\boldsymbol{x}(kT+T) = e^{\mathbf{F}T}\ \boldsymbol{x}(kT) + \int_{kT}^{kT+T} e^{\mathbf{F}(kT+T-\tau)}\mathbf{G}u(\tau)d\tau$$

Notice $\boldsymbol{u}(\tau)$ is not based on a discrete ZOH input, but rather an integrated time-series.

We can structure this by using the form:

$$u(\tau) = u(kT), \qquad kT \le \tau \le kT + T$$

State-space z-transform

We can apply the z-transform to our system:

$$(z\mathbf{I} - \mathbf{\Phi})\boldsymbol{X}(z) = \mathbf{\Gamma}U(k)$$
$$Y(z) = \mathbf{H}\boldsymbol{X}(z)$$

which yields the transfer function:

$$\frac{Y(z)}{\boldsymbol{X}(z)} = G(z) = \mathbf{H}(z\mathbf{I} - \mathbf{\Phi})^{-1}\mathbf{\Gamma}$$

State-space control design -- Controllability

- Design for discrete state-space systems is just like the continuous case.
  - Apply linear state-variable feedback:

$$u = -\mathbf{K}\boldsymbol{x}$$

such that $\det(z\mathbf{I} - \boldsymbol{\Phi} + \boldsymbol{\Gamma}\mathbf{K}) = \alpha_c(z)$

where $\alpha_c(z)$ is the desired control characteristic equation

Predictably, this requires the system controllability matrix $\mathcal{C} = [\boldsymbol{\Gamma} \quad \boldsymbol{\Phi}\boldsymbol{\Gamma} \quad \boldsymbol{\Phi}^2\boldsymbol{\Gamma} \quad \cdots \quad \boldsymbol{\Phi}^{n-1}\boldsymbol{\Gamma}]$ to be full-rank.

Φ: Solving State Space

- In the conventional, frequency-domain approach the differential equations are converted to transfer functions as soon as possible
  - The dynamics of a system comprising several subsystems is obtained by combining the transfer functions!

- With the state-space methods, on the other hand, the description of the system dynamics in the form of differential equations is retained throughout the analysis and design.

State-transition matrix Φ(t)

- Describes how the state *x(t)* of the system at some time *t* evolves into (or from) the state x(τ) at some other time *T*.

$$x(t) = \Phi(t, \tau)\, x(\tau)$$

- $\mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}$ ➔ $\Phi(t) = e^{\mathbf{A}t}$
- Matrix Exponential:

$$e^{At} = \exp(At) = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$

- Similar idea, but different result, for the control **u** ➔ Γ

Γ: Gamma: Comes from Integrating $\dot{\boldsymbol{x}}$

- $\Gamma = \left(\sum_{k=0}^{\infty} \frac{A^kT^{k+1}}{(k+1)!}\right) TB \approx \left(IT + A\frac{T^2}{2}\right) B$

Why?

- $x(t) = e^{A(t-t_0)}x(t_0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\, d\tau$
- $x(kT+T) = e^{AT}x(kT) + \int_{kT}^{kT+T} e^{A(kt+t-\tau)} Bu(\tau) d\tau$
- u(t) is specified in terms of a continuous time history, though we often assume u(t) is a ZOH:
- $u(\tau) = u(kT) \Rightarrow$ Introduce $\eta = kT + T - \tau$
- $x(kT+T) = e^{AT}x(kT) + \int_{kT}^{kT+T} e^{A\eta} d\eta Bu(kT)$

➔ $\Phi = e^{AT}$, $\Gamma = \int_0^T e^{A\eta} d\eta B$

Solving State Space (optional notes) …

Time-invariant dynamics The simplest form of the general differential equation of the form (3.1) is the "homogeneous," i.e., unforced equation

$$\dot{x} = Ax \tag{3.2}$$

where A is a constant k by k matrix. The solution to (3.2) can be expressed as

$$x(t) = e^{At}c \tag{3.3}$$

where e^{At} is the matrix exponential function

$$e^{At} = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + \cdots \tag{3.4}$$

and c is a suitably chosen constant vector. To verify (3.3) calculate the derivative of $x(t)$

$$\frac{dx(t)}{dt} = \frac{d}{dt}(e^{At})c \tag{3.5}$$

and, from the defining series (3.4),

$$\frac{d}{dt}(e^{At}) = A + A^2t + A^3\frac{t^2}{2!} + \cdots = A\left(I + At + A^2\frac{t^2}{2!} + \cdots\right) = A\,e^{At}$$

Thus (3.5) becomes

$$\frac{dx(t)}{dt} = Ae^{At}c = Ax(t)$$

Solving State Space (optional notes)

which was to be shown. To evaluate the constant c suppose that at some time τ the state $x(\tau)$ is given. Then, from (3.3),

$$x(\tau) = e^{A\tau}c \tag{3.6}$$

Multiplying both sides of (3.6) by the inverse of $e^{A\tau}$ we find that

$$c = (e^{A\tau})^{-1}x(\tau)$$

Thus the general solution to (3.2) for the state $x(t)$ at time t, given the state $x(\tau)$ at time τ, is

$$x(t) = e^{At}(e^{A\tau})^{-1}x(\tau) \tag{3.7}$$

The following property of the matrix exponential can readily be established by a variety of methods—the easiest perhaps being the use of the series definition (3.4)—

$$e^{A(t_1+t_2)} = e^{At_1}e^{At_2} \tag{3.8}$$

for any t_1 and t_2. From this property it follows that

$$(e^{A\tau})^{-1} = e^{-A\tau} \tag{3.9}$$

and hence that (3.7) can be written

$$x(t) = e^{A(t-\tau)}x(\tau) \tag{3.10}$$

Solving State Space (optional notes)

The matrix $e^{A(t-\tau)}$ is a special form of the *state-transition matrix* to be discussed subsequently.

We now turn to the problem of finding a "particular" solution to the nonhomogeneous, or "forced," differential equation (3.1) with A and B being constant matrices. Using the "method of the variation of the constant,"[1] we seek a solution to (3.1) of the form

$$x(t) = e^{At}c(t) \tag{3.11}$$

where $c(t)$ is a function of time to be determined. Take the time derivative of $x(t)$ given by (3.11) and substitute it into (3.1) to obtain:

$$Ae^{At}c(t) + e^{At}\dot{c}(t) = Ae^{At}c(t) + Bu(t)$$

or, upon cancelling the terms $A\,e^{At}c(t)$ and premultiplying the remainder by e^{-At},

$$\dot{c}(t) = e^{-At}Bu(t) \tag{3.12}$$

Thus the desired function $c(t)$ can be obtained by simple integration (the mathematician would say "by a quadrature")

$$c(t) = \int_T^t e^{-A\lambda}Bu(\lambda)\,d\lambda$$

The lower limit T on this integral cannot as yet be specified, because we will need to put the particular solution together with the solution to the

Solving State Space (optional notes)

homogeneous equation to obtain the complete (general) solution. For the present, let T be undefined. Then the particular solution, by (3.11), is

$$x(t) = e^{At}\int_T^t e^{-A\lambda}Bu(\lambda)\,d\lambda = \int_T^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.13}$$

In obtaining the second integral in (3.13), the exponential e^{At}, which does not depend on the variable of integration λ, was moved under the integral, and property (3.8) was invoked to write $e^{At}e^{-A\lambda} = e^{A(t-\lambda)}$.

The complete solution to (3.1) is obtained by adding the "complementary solution" (3.10) to the particular solution (3.13). The result is

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_T^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.14}$$

We can now determine the proper value for lower limit T on the integral. At $t = \tau$ (3.14) becomes

$$x(\tau) = x(\tau) + \int_T^\tau e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.15}$$

Thus, the integral in (3.15) must be zero for any $u(t)$, and this is possible only if $T = \tau$. Thus, finally we have the complete solution to (3.1) when A and B are constant matrices

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_\tau^t e^{A(t-\lambda)}Bu(\lambda)\,d\lambda \tag{3.16}$$

Solving State Space (optional notes)

This important relation will be used many times in the remainder of the book. It is worthwhile dwelling upon it. We note, first of all, that the solution is the sum of two terms: the first is due to the "initial" state $x(\tau)$ and the second—the integral—is due to the input $u(\tau)$ in the time interval $\tau \leqq \lambda \leqq t$ between the "initial" time τ and the "present" time t. The terms initial and present are enclosed in quotes to denote the fact that these are simply convenient definitions. There is no requirement that $t \geqq \tau$. The relationship is perfectly valid even when $t \leqq \tau$.

Another fact worth noting is that the integral term, due to the input, is a "convolution integral": the contribution to the state $x(t)$ due to the input u is the convolution of u with $e^{At}B$. Thus the function $e^{At}B$ has the role of the impulse response[1] of the system whose output is $x(t)$ and whose input is $u(t)$.

If the output y of the system is not the state x itself but is defined by the observation equation

$$y = Cx$$

then this output is expressed by

$$y(t) = Ce^{A(t-\tau)}x(t) + \int_{\tau}^{t} Ce^{A(t-\lambda)}Bu(\lambda)\,d\lambda \qquad (3.17)$$

Solving State Space (optional notes)

and the impulse response of the system with y regarded as the output is $Ce^{A(t-\lambda)}B$.

The development leading to (3.16) and (3.17) did not really require that B and C be constant matrices. By retracing the steps in the development it is readily seen that when B and C are time-varying, (3.16) and (3.17) generalize to

$$x(t) = e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} e^{A(t-\lambda)}B(\lambda)u(\lambda)\,d\lambda \qquad (3.18)$$

and

$$y(t) = C(t)\,e^{A(t-\tau)}x(\tau) + \int_{\tau}^{t} C(t)\,e^{A(t-\lambda)}B(\lambda)u(\lambda)\,d\lambda \qquad (3.19)$$

Pole Placement (Following FPW – Chapter 6)

- FPW has a slightly different notation:

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u,$$
$$y = \mathbf{H}\mathbf{x}.$$

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}u(k),$$
$$y(k) = \mathbf{H}\mathbf{x}(k),$$

$$\mathbf{\Phi} = e^{\mathbf{F}T},$$

$$\mathbf{\Gamma} = \int_0^T e^{\mathbf{F}\eta} d\eta \mathbf{G},$$

Pole Placement

- Start with a simple feedback control law ("controller")

$$u = -\mathbf{K}\mathbf{x} = -[K_1 K_2 \ldots] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$$

- It's actually a regulator
  ∵ it does not allow for a reference input to the system.
  (there is no "reference" **r** (r = 0))

- Substitute in the difference equation

$$x(k+1) = \Phi x(k) - \Gamma K x(k)$$

- $\mathcal{Z}$ Transform:

$$(zI - \Phi + \Gamma K)X(z) = 0$$

➔Characteristic Eqn:

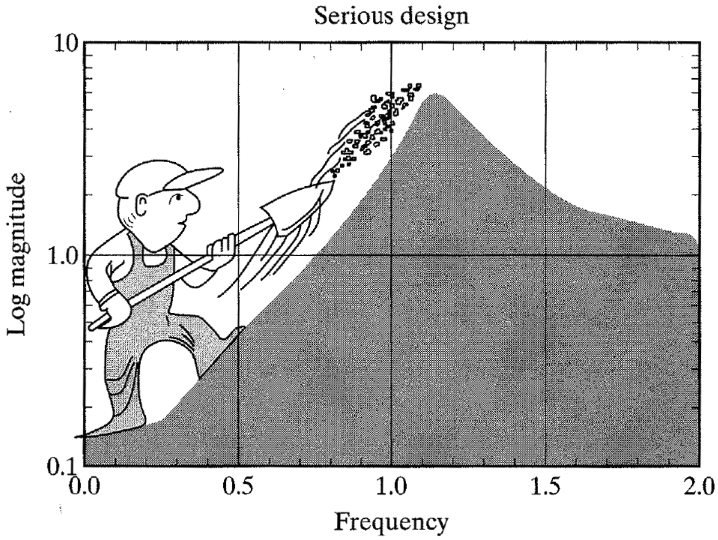
$$\det|zI - \Phi + \Gamma K| = 0$$

Pole Placement

Pole placement: Big idea:

- Arbitrarily select the desired root locations of the closed-loop system and see if the approach will work.
- AKA: full state feedback
  ∵ enough parameters to influence all the closed-loop poles
- Finding the elements of K so that the roots are in the desired locations. Unlike classical design, where we iterated on parameters in the compensator (hoping) to find acceptable root locations, the full state feedback, pole-placement approach guarantees success and allows us to arbitrarily pick any root locations, providing that *n* roots are specified for an n^{th}-order system.

Meaning…



Back to Pole Placement

- Given:

$$z_i = \beta_1, \beta_2, \beta_3, \ldots$$

- This gives the desired control-characteristic equation as:

$$a_c(z) = (z - \beta_1)(z - \beta_2)(z - \beta_3) \ldots =$$

- Now we "just solve" for **K** and "bingo"

Pole Placement Example (FPW p. 241)

Example 6.1: Suppose we want to design a control law for the satellite attitude-control system described by (2.45) with $\boldsymbol{x} = [\boldsymbol{x}_1 \;\; \boldsymbol{x}_2]$. Example 2.13 showed that the discrete model for this system is

$$\boldsymbol{\Phi} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Gamma} = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}.$$

We want to pick z-plane roots of the closed-loop characteristic equation so that the equivalent s-plane roots have a damping ratio of $\zeta = 0.5$ and real part of $s = -1.8$ rad/sec (i.e., $s = -1.8 \pm j3.12$ rad/sec). Using $z = e^{sT}$ with a sample period of $T = 0.1$ sec, we find that $z = 0.8 \pm j0.25$, as shown in Fig. 6.1. The desired characteristic equation is then

$$z^2 - 1.6z + 0.70 = 0, \qquad (6.9)$$

and the evaluation of (6.7) for any control law $\mathbf{K}$ leads to

$$\det \left| z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} [K_1 \quad K_2] \right| = 0$$

or

$$z^2 + (TK_2 + (T^2/2)K_1 - 2)z + (T^2/2)K_1 - TK_2 + 1 = 0. \qquad (6.10)$$

Pole Placement Example (FPW p. 241)

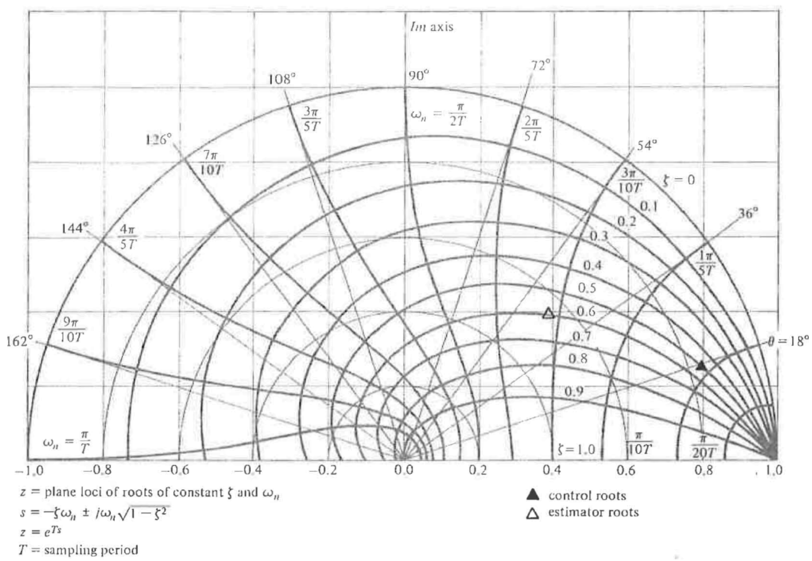
Equating coefficients in (6.9) and (6.10) with like powers of z, we obtain two simultaneous equations in the two unknown elements of $\mathbf{K}$:

$$TK_2 + (T^2/2)K_1 - 2 = -1.6,$$
$$(T^2/2)K_1 - TK_2 + 1 = 0.70,$$

which are easily solved for the coefficients and evaluated for $T = 0.1$ sec:

$$K_1 = \frac{0.10}{T^2} = 10, \qquad K_2 = \frac{0.35}{T} = 3.5.$$

Pole Placement Example (FPW p. 241)



Ackermann's Formula (FPW p. 245)

- Gains maybe approximated with:

$$\mathbf{K} = [0 \ldots 0 \quad 1][\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \quad \mathbf{\Phi}^2\mathbf{\Gamma} \ldots \mathbf{\Phi}^{n-1}\mathbf{\Gamma}]^{-1}\alpha_c(\mathbf{\Phi}),$$

- Where: C = controllability matrix, *n* is the order of the system (or number of state elements) and α_c:

$$\mathcal{C} = [\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma} \ldots]$$

$$\alpha_c(\mathbf{\Phi}) = \mathbf{\Phi}^n + \alpha_1\mathbf{\Phi}^{n-1} + \alpha_2\mathbf{\Phi}^{n-2} + \cdots + \alpha_n\mathbf{I},$$

  - α_i: coefficients of the desired characteristic equation

$$\alpha_c(z) = |\, z\mathbf{I} - \mathbf{\Phi} + \mathbf{\Gamma K} \,| = z^n + \alpha_1 z^{n-1} + \cdots + \alpha_n.$$

Ackermann's Formula Example (FPW p.246)

Example 6.2: Applying Ackermann's formula to the satellite attitude-control system of Example 6.1, we find from (6.9) that

$$\alpha_1 = -1.6, \qquad \alpha_2 = +0.70,$$

and therefore

$$\alpha_c(\mathbf{\Phi}) = \begin{bmatrix} 1 & 2T \\ 0 & 1 \end{bmatrix} - 1.6\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} + 0.70\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix}.$$

Furthermore, we find that

$$[\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma}] = \begin{bmatrix} T^2/2 & 3T^2/2 \\ T & T \end{bmatrix}$$

and

$$[\mathbf{\Gamma} \quad \mathbf{\Phi\Gamma}]^{-1} = 1/T^2 \begin{bmatrix} -1 & +3T/2 \\ 1 & -T/2 \end{bmatrix},$$

and finally

$$\mathbf{K} = [K_1 \quad K_2] = (1/T^2)[0 \quad 1]\begin{bmatrix} -1 & 3T/2 \\ 1 & -T/2 \end{bmatrix}\begin{bmatrix} 0.1 & 0.4T \\ 0 & 0.1 \end{bmatrix};$$

therefore

$$[K_1 \quad K_2] = \frac{1}{T^2}[0.1 \quad 0.35T]$$
$$= [10 \quad 3.5],$$

which is the same result as that obtained earlier.

Stability

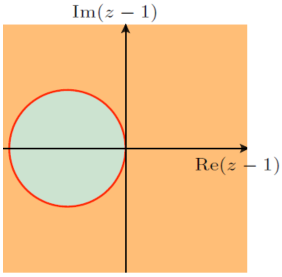
Fast sampling revisited

- For small T:

$$z = e^{sT} = 1 + sT + \frac{(sT)^2}{2} + \ldots \approx 1 + sT$$

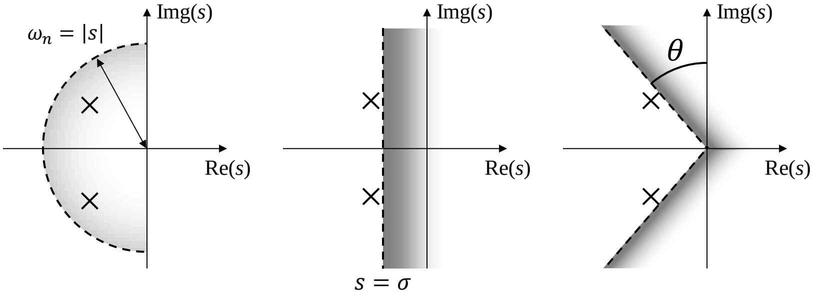
$$\rightarrow z \approx 1 + sT \rightarrow s = \frac{z-1}{T}$$

- Hence, the unit circle under the map from z to s-plane becomes:



Specification bounds

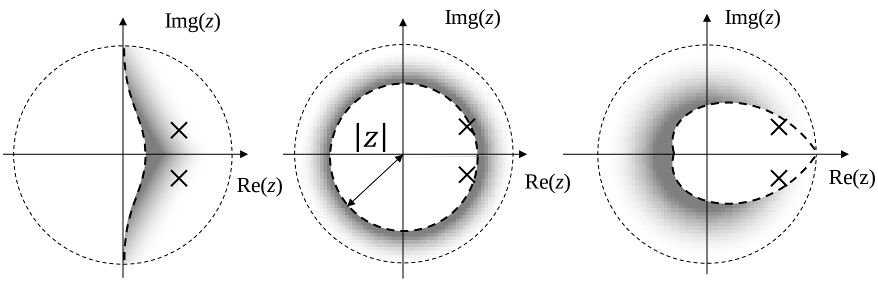
- Recall in the continuous domain, response performance metrics map to the s-plane:



$|s| = \frac{1.8}{t_r}$ $\quad s = \frac{4.6}{t_s}$ $\quad \theta = \sin^{-1}\zeta$

Discrete bounds

- These map to the discrete domain:



$|z| = e^{-t_s T}$

In practice, you'd use Matlab to plot these, and check that the spec is satisfied

Example Code:

```
%% Input System Model G
numg=5; deng=[1 20 0]; sysg=tf(numg, deng);

%% Approximate the ZOH (1-e^-sT)/(s)
[nd, dd]=pade(1,2);  %pade gives us the "hold" or -e^-sT of a ZOH
sysp=tf(nd, dd); sysi=tf([1],[1,0]); %Now we need the "1/s" portion
sys1=series(1-sysp, sysi); % Approximation as a series

%% Open loop response
syso=series(sys1, sysg); % computer the open loop G with the ZOH
sys=feedback(syso,1); % Computer the unity feedback response
step(sys) % Display the step response
```

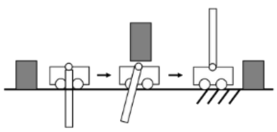

</assessable>

- Nothing beyond this point is on the exam.
  (except for the exam review ☺)
- Do not pay attention.
- Do not attempt to learn.

Inverted Pendulm

Digital Control

Wikipedia,
Cart and pole



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$

$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

and the equations of motion are:

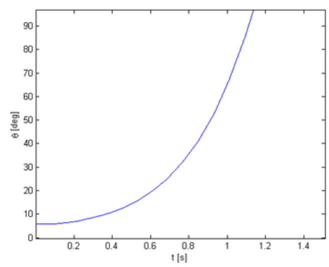
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$

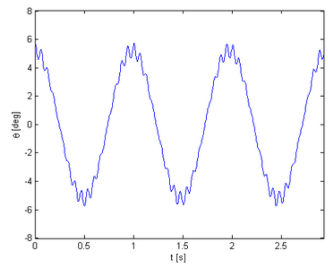
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

substituting L in these equations and simplifying leads to the equations that describe the motion o

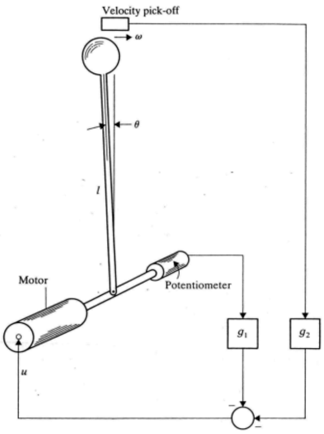
$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$

$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$





Inverted Pendulum



$$L = \frac{1}{2}Mv_1^2 + \frac{1}{2}mv_2^2 - mg\ell\cos\theta$$

where v_1 is the velocity of the cart and v_2 is the velocity of the point mass m. v_1 and v_2 can be expressed in terms of x and θ by writing the velocity as the first derivative of the position;

$$v_1^2 = \dot{x}^2$$
$$v_2^2 = \left(\frac{d}{dt}(x - \ell\sin\theta)\right)^2 + \left(\frac{d}{dt}(\ell\cos\theta)\right)^2$$

Simplifying the expression for v_2 leads to:

$$v_2^2 = \dot{x}^2 - 2\ell\dot{x}\dot{\theta}\cos\theta + \ell^2\dot{\theta}^2$$

The Lagrangian is now given by:

$$L = \frac{1}{2}(M+m)\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\cos\theta$$

and the equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

substituting L in these equations and simplifying leads to the equations that describe the motion o

$$(M+m)\ddot{x} - m\ell\ddot{\theta}\cos\theta + m\ell\dot{\theta}^2\sin\theta = F$$
$$\ell\ddot{\theta} - g\sin\theta = \ddot{x}\cos\theta$$

Inverted Pendulum – Equations of Motion

- The equations of motion of an inverted pendulum (under a small angle approximation) may be linearized as:

$$\dot{\theta} = \omega$$
$$\dot{\omega} = \ddot{\theta} = Q^2\theta + Pu$$

Where:

$$Q^2 = \left(\frac{M+m}{Ml}\right)g$$
$$P = \frac{1}{Ml}.$$

If we further assume unity ***Ml*** ($Ml \approx 1$), then $P \approx 1$

Inverted Pendulum –State Space

- We then select a state-vector as:

$$\boldsymbol{x} = \begin{bmatrix} \theta \\ \omega \end{bmatrix}, \text{ hence } \dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix}$$

- Hence giving a state-space model as:

$$A = \begin{bmatrix} 0 & 1 \\ Q^2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- The resolvent of which is:

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ -Q^2 & s \end{bmatrix}^{-1} = \frac{1}{s^2 - Q^2} \begin{bmatrix} s & 1 \\ Q^2 & s \end{bmatrix}$$

- And a state-transition matrix as:

$$\Phi(t) = \begin{bmatrix} \cosh Qt & \dfrac{\sinh Qt}{Q} \\ Q\, sinh\, Qt & \cosh Qt \end{bmatrix}$$