

is known in mathematical literature as the *resolvent* of A . In engineering literature this matrix has been called the *characteristic frequency matrix*[1] or simply the *characteristic matrix*.[4] Regrettably there doesn't appear to be a standard symbol for the resolvent, which we have designated as $\Phi(s)$ in this book.

The fact that the state transition matrix is the inverse Laplace transform of the resolvent matrix facilitates the calculation of the former. It also characterizes the dynamic behavior of the system, the subject of the next chapter. The steps one takes in calculating the state-transition matrix using the resolvent are:

- Calculate $sI - A$.
- Obtain the resolvent by inverting $(sI - A)$.
- Obtain the state-transition matrix by taking the inverse Laplace transform of the resolvent, element by element.

The following examples illustrate the process.

Example 3C DC motor with inertial load In Chap. 2 (Example 2B) we found that the dynamics of a dc motor driving an inertial load are

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\alpha\omega + \beta u\end{aligned}$$

The matrices of the state-space characterization are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

Thus the resolvent is

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 0 & s + \alpha \end{bmatrix}^{-1} = \frac{1}{s(s + \alpha)} \begin{bmatrix} s + \alpha & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s + \alpha)} \\ 0 & \frac{1}{s + \alpha} \end{bmatrix}$$

Finally, taking the inverse Laplace transforms of each term in $\Phi(s)$ we obtain

$$e^{At} = \Phi(t) = \begin{bmatrix} 1 & (1 - e^{-\alpha t})/\alpha \\ 0 & e^{-\alpha t} \end{bmatrix}$$

Example 3D Inverted pendulum The equations of motion of an inverted pendulum were determined to be (approximately)

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \Omega^2\theta + u\end{aligned}$$

Hence the matrices of the state-space characterization are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The resolvent is

$$\Phi(s) = (sI - A)^{-1} = \begin{bmatrix} s & -1 \\ -\Omega^2 & s \end{bmatrix}^{-1} = \frac{1}{s^2 - \Omega^2} \begin{bmatrix} s & 1 \\ \Omega^2 & s \end{bmatrix}$$

and the state-transition matrix is

$$\Phi(t) = e^{At} = \begin{bmatrix} \cosh \Omega t & \sinh \Omega t / \Omega \\ \Omega \sinh \Omega t & \cosh \Omega t \end{bmatrix}$$

For a general k th-order system the matrix $sI - A$ has the following appearance

$$sI - A = \begin{bmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1k} \\ -a_{21} & s - a_{22} & \cdots & -a_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ -a_{k1} & -a_{k2} & \cdots & s - a_{kk} \end{bmatrix} \quad (3.50)$$

We recall (see Appendix) that the inverse of any matrix M can be written as the adjoint matrix, $\text{adj } M$, divided by the determinant $|M|$. Thus

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{|sI - A|}$$

If we imagine calculating the determinant $|sI - A|$ we see that one of the terms will be the product of the diagonal elements of $sI - A$:

$$(s - a_{11})(s - a_{22}) \cdots (s - a_{kk}) = s^k + c_1 s^{k-1} + \cdots + c_k$$

a polynomial of degree k with the leading coefficient of unity. There will also be other terms coming from the off-diagonal elements of $sI - A$ but none will have a degree as high as k . Thus we conclude that

$$|sI - A| = s^k + a_1 s^{k-1} + \cdots + a_k \quad (3.51)$$

This is known as the *characteristic polynomial* of the matrix A . It plays a vital role in the dynamic behavior of the system. The roots of this polynomial are called the *characteristic roots*, or the *eigenvalues*, or the *poles*, of the system and determine the essential features of the unforced dynamic behavior of the system, since they determine the inverse Laplace transform of the resolvent, which is the transition matrix. See Chap. 4.

The adjoint of a k by k matrix is itself a k by k matrix whose elements are the cofactors of the original matrix. Each cofactor is obtained by computing the determinant of the matrix that remains when a row and a column of the original matrix are deleted. It thus follows that each element in $\text{adj}(sI - A)$ is a polynomial in s of maximum degree $k - 1$. (The polynomial cannot have degree k when any row and column of $sI - A$ is deleted.) Thus it is seen that the adjoint of $sI - A$ can be written

$$\text{adj}(sI - A) = E_1 s^{k-1} + E_2 s^{k-2} + \cdots + E_k$$

Thus we can express the resolvent in the following form

$$(sI - A)^{-1} = \frac{E_1 s^{k-1} + \cdots + E_k}{s^k + a_1 s^{k-1} + \cdots + a_k} \quad (3.52)$$

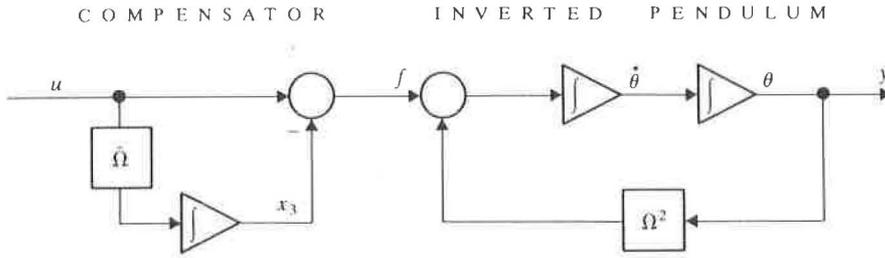


Figure 5.7 Unstabilizable compensation of inverted pendulum.

book. But one way guaranteed to be disastrous is to try to cancel the unstable pole with a zero in the compensator. The reason for the disaster is the subject of this example.

Consider the inverted pendulum of Example 3D with the output being the measured position. The transfer function from the input (force) to the output (position) is

$$H(s) = \frac{y(s)}{f(s)} = \frac{1}{s^2 - \Omega^2} = \frac{1}{(s + \Omega)(s - \Omega)} \quad (5D.1)$$

This is obviously unstable. A much better transfer function would be

$$H(s) = \frac{1}{s(s + \Omega)} \quad (5D.2)$$

which is stable and, because of the pole at the origin, would be a "type-one" system, with zero steady state error. Thus, one might be tempted to "compensate" the unstable transfer function by means of a compensator having the transfer function (Fig. 5.7)

$$G(s) = \frac{s - \bar{\Omega}}{s} = 1 - \frac{\bar{\Omega}}{s} \quad (5D.3)$$

with

$$\bar{\Omega} = \Omega$$

Of course it will not be possible to make $\bar{\Omega}$ precisely equal to Ω so the compensation will not be perfect. But that is not the trouble, as we shall see.

The compensator transfer function (5D.3) represents "proportional plus integral" compensation which is quite customary in practical process control systems. The transfer function of the compensated system is now

$$H_c(s) = G(s)H(s) = \frac{s - \bar{\Omega}}{s(s^2 - \Omega^2)} \rightarrow H(s) \quad \text{as } \bar{\Omega} \rightarrow \Omega \quad (5D.4)$$

A block diagram representation of this system is shown in Fig. 5.7, and the state-space equations corresponding to this representation are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \Omega^2 x_1 - x_3 + u \\ \dot{x}_3 &= \bar{\Omega} u \end{aligned} \quad (5D.5)$$

where x_3 is the state of the integrator in the compensator. The matrices of the process (5D.5) are

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \Omega^2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ \bar{\Omega} \end{bmatrix}$$

INVERT. PEND. STABILITY

LOP

ALTY

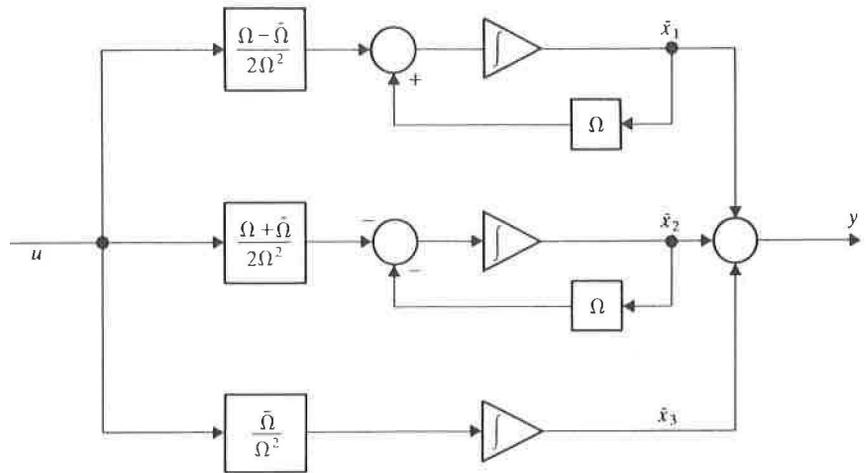


Figure 5.8 Partial fraction representation of Fig. 5.7.

The A matrix can be transformed to diagonal form by the transformation matrix

$$T = \frac{1}{2\Omega^2} \begin{bmatrix} \Omega^2 & \Omega & -1 \\ \Omega^2 & -\Omega & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad T^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ \Omega & -\Omega & 0 \\ 0 & 0 & \Omega^2 \end{bmatrix}$$

We find that

$$\bar{A} = TAT^{-1} = \begin{bmatrix} \Omega & 0 & 0 \\ 0 & -\Omega & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{B} = TB = \frac{1}{2\Omega^2} \begin{bmatrix} \Omega - \bar{\Omega} \\ -(\Omega + \bar{\Omega}) \\ 2\bar{\Omega} \end{bmatrix}$$

The state-space representation of the transformed system is as shown in Fig. 5.8. This block-diagram corresponds directly to the partial-fraction expansion of (5D.4):

$$H_c(s) = \frac{\bar{\Omega}/\Omega^2}{s} + \frac{(\Omega - \bar{\Omega})/2\Omega^2}{s - \Omega} + \frac{-(\Omega + \bar{\Omega})/2\Omega^2}{s + \Omega} \quad (5D.6)$$

Note carefully what happens when $\bar{\Omega} \rightarrow \Omega$. In the block-diagram the connection between the control input u and the *unstable* state x_1 is broken, rendering the system uncontrollable and *unstabilizable*. In (5D.6) the residue at the unstable pole vanishes. But now we understand that the vanishing of a residue at a pole of a transfer function does not imply that the subsystem giving rise to the pole disappears, but rather that it becomes “invisible.”

If the original inverted pendulum could have arbitrary initial conditions, the transformed system (5D.5) could also have arbitrary initial conditions and hence the inverted pendulum would most assuredly not remain upright, regardless of how the loop were closed between the measurement y and the control input u .

~~**More reasons for unobservability** The foregoing examples were instances of uncontrollable systems. Instances of unobservable systems are even more abundant. An unobservable system results any time a state variable is not measured~~

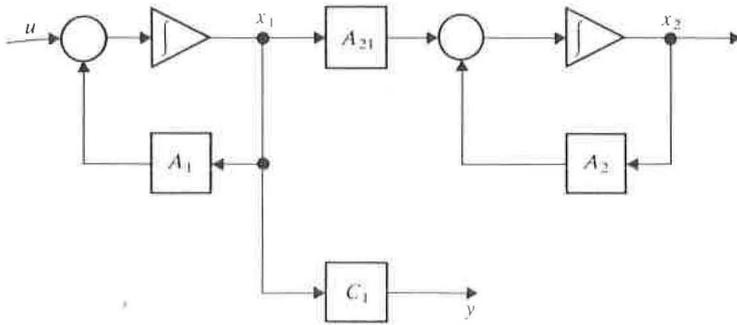


Figure 5.9 Systems in tandem that are unobservable.

directly and is not fed back to those state variables that are measured. Thus, any system comprising two subsystems in tandem (as shown in Fig. 5.9, in which none of the states of the right-hand subsystem can be measured) is unobservable. The transfer function from the inputs to the outputs obviously depends only on the left-hand subsystem.

Physical processes which have the structure shown in Fig. 5.9 are not uncommon. A mass m acted upon by a control force f is unobservable if only its velocity, and not its position, can be measured. This means that no method of velocity feedback can serve as a means of controlling position. In this regard it is noted that the integral of the measured velocity is not the same as the actual position. A control system shown in Fig. 5.10 will not be effective in controlling the position x of the mass, no matter how well it controls the velocity \dot{x} ; any initial position error will remain in the system indefinitely.

In addition to the obvious reasons for unobservability there are also some of the more subtle reasons such as symmetry, as was illustrated by Example 5C.

5.3 DEFINITIONS AND CONDITIONS FOR CONTROLLABILITY AND OBSERVABILITY

In Secs. 5.1 and 5.2 we found that uncontrollable and/or unobservable systems were characterized by the property that the transfer function from the input to

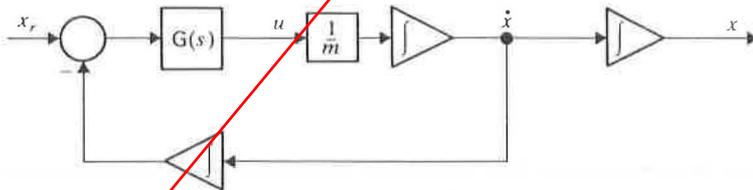


Figure 5.10 Position of mass cannot be observed and cannot be controlled using only velocity feedback.

or

$$g_1 = \bar{a}_2/\beta \quad g_2 = (\bar{a}_1 - \alpha)/\beta$$

which is the same as (6A.5).

Note that the position and velocity gains g_1 and g_2 , respectively, are proportional to the amounts we wish to move the coefficients from their open-loop positions. The position gain g_1 is necessary to produce a stable system: $\bar{a}_2 > 0$. But if the designer is willing to settle for $a_1 = \alpha$, i.e., to accept the open-loop damping, then the gain g_2 can be zero. This of course eliminates the need for a tachometer and reduces the hardware cost of the system. It is also possible to alter the system damping without the use of a tachometer, by using an estimate $\hat{\omega}$ of the angular velocity ω . This estimate is obtained by means of an observer as discussed in Chap. 7.

Example 6B Stabilization of an inverted pendulum An inverted pendulum can readily be stabilized by a closed-loop feedback system, just as a person of moderate dexterity can do it.

A possible control system implementation is shown in Fig. 6.3, for a pendulum constrained to rotate about a shaft at its bottom point. The actuator is a dc motor. The angular position of the pendulum, being equal to the position of the shaft to which it is attached, is measured by means of a potentiometer. The angular velocity in this case can be measured by a "velocity pick-off" at the top of the pendulum. Such a device could consist of a coil of wire

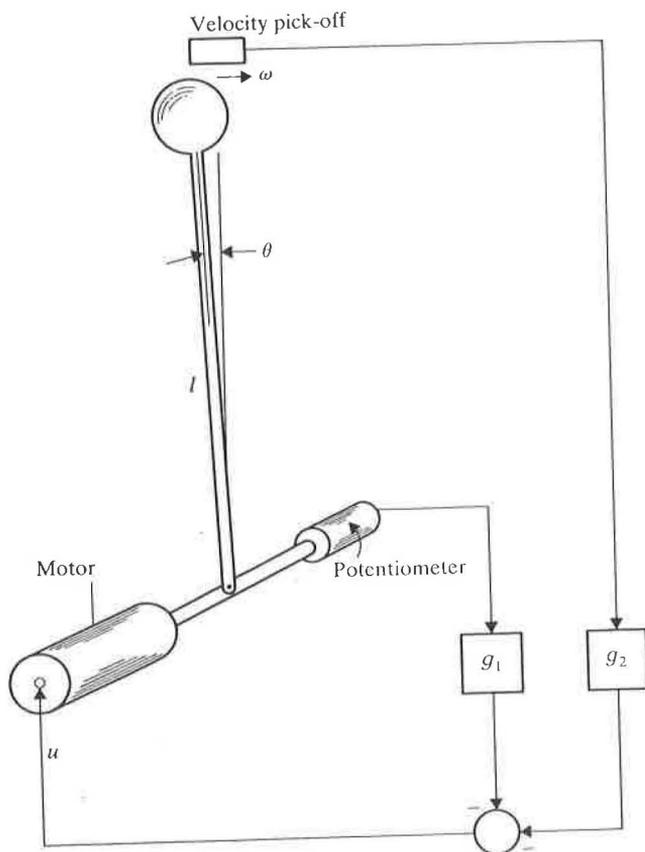


Figure 6.3 Implementation of system to stabilize inverted pendulum.

in a magnetic field created by a small permanent magnet in the pendulum bob. The induced voltage in the coil is proportional to the linear velocity of the bob as it passes the coil. And since the bob is at a fixed distance from the pivot point the linear velocity is proportional to the angular velocity. The angular velocity could of course also be measured by means of a tachometer on the dc motor shaft.

As determined in Prob. 2.2, the dynamic equations governing the inverted pendulum in which the point of attachment does not translate is given by

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= \Omega^2 \theta - \alpha \omega + \beta u \end{aligned} \tag{6B.1}$$

where α and β are given in Example 6A, with the inertia J being the total reflected inertia:

$$J = J_m + ml^2$$

where m is the pendulum bob mass and l is the distance of the bob from the pivot. The natural frequency Ω is given by

$$\Omega^2 = \frac{mgl}{J + ml^2} = \frac{g}{l + J/ml}$$

(Note that the motor inertia J_m affects the natural frequency.)

Since the linearization is valid only when the pendulum is nearly vertical, we shall assume that the control objective is to maintain $\theta = 0$. Thus we have a simple regulator problem.

The matrices A and b for this problem are

$$A = \begin{bmatrix} 0 & 1 \\ \Omega^2 & -\alpha \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$$

The open-loop characteristic polynomial is

$$|sI - A| = \begin{vmatrix} s & -1 \\ -\Omega^2 & s + \alpha \end{vmatrix} = s^2 + \alpha s - \Omega^2$$

Thus

$$\begin{aligned} a_1 &= \alpha \\ a_2 &= -\Omega^2 \end{aligned}$$

The open-loop system is unstable, of course.

The controllability test matrix and the W matrix are given respectively by

$$Q = \begin{bmatrix} 0 & \beta \\ \beta & -\alpha\beta \end{bmatrix} \quad W = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

(which are the same as they were for the instrument servo). And

$$[(QW)]^{-1} = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix}$$

Thus the gain matrix required for pole placement using (6.34), is

$$g = \begin{bmatrix} 0 & 1/\beta \\ 1/\beta & 0 \end{bmatrix} \begin{bmatrix} (\bar{a}_1 - \alpha) \\ \bar{a}_2 + \Omega^2 \end{bmatrix} = \begin{bmatrix} (\bar{a}_2 + \Omega^2)/\beta \\ (\bar{a}_1 - \alpha)/\beta \end{bmatrix}$$

Example 6C Control of spring-coupled masses The dynamics of a pair of spring-coupled masses, shown in Fig. 3.7(a), were shown in Example 3I to have the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -K/\bar{M} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The system has the characteristic polynomial

$$D(s) = s^4 + (K/\bar{M})s^2$$

Hence

$$a_1 = a_3 = a_4 = 0, \quad a_2 = K/\bar{M}.$$

The controllability test and W matrices are given, respectively, by

$$Q = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -K/\bar{M} \\ 1 & 0 & -K/\bar{M} & 0 \end{bmatrix} \quad W = \begin{bmatrix} 1 & 0 & K/\bar{M} & 0 \\ 0 & 1 & 0 & K/\bar{M} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (6C.1)$$

Multiplying we find that

$$QW = (QW)' = (QW)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (6C.2)$$

(This rather simple result is not really as surprising as it may at first seem. Note that A is in the first companion form but using the right-to-left numbering convention. If the left-to-right numbering convention were used the A matrix would already be in the companion form of (6.11) and would not require transformation. The transformation matrix T given by (6C.2) has the effect of changing the state variable numbering order from left-to-right to right-to-left, and vice versa.)

The gain matrix g is thus given by

$$g = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{a}_1 \\ \bar{a}_2 - K/\bar{M} \\ \bar{a}_3 \\ \bar{a}_4 \end{bmatrix} = \begin{bmatrix} \bar{a}_4 \\ \bar{a}_3 \\ \bar{a}_2 - K/\bar{M} \\ \bar{a}_1 \end{bmatrix}$$

A suitable pole "constellation" for the closed-loop process might be a Butterworth pattern as discussed in Sec. 6.5. To achieve this pattern the characteristic polynomial should be of the form

$$\bar{D}(s) = s^4 + (1 + \sqrt{3})\Omega s^3 + (2 + \sqrt{3})\Omega^2 s^2 + (1 + \sqrt{3})\Omega^3 s + \Omega^4$$

Thus

$$\bar{a}_1 = (1 + \sqrt{3})\Omega$$

$$\bar{a}_2 = (2 + \sqrt{3})\Omega^2$$

$$\bar{a}_3 = (1 + \sqrt{3})\Omega^3$$

$$\bar{a}_4 = \Omega^4$$

Thus the gain matrix g is given by

$$g = \begin{bmatrix} \Omega^4 \\ (1 + \sqrt{3})\Omega^3 \\ (2 + \sqrt{3})\Omega^2 - K/\bar{M} \\ (1 + \sqrt{3})\Omega \end{bmatrix}$$

6.3 MULTIPLE-INPUT SYSTEMS

If the dynamic system under consideration

$$\dot{x} = Ax + Bu$$