and the covariance matrix,

$$\mathcal{E}\{(\mathbf{x} - \mu_{\mathbf{x}})(\mathbf{x} - \boldsymbol{\mu}_{\mathbf{x}})^T\} = \mathbf{R}_{\mathbf{x}}.$$

Like the scalar normal density, the multivariable law is described entirely by the two parameters $\boldsymbol{\mu}$ and $\mathbf{R}$, the difference being that the multivariable case is described by matrix parameters rather than scalar parameters. In (D.19) we require the inverse of $\mathbf{R}_{\mathbf{x}}$ and have thus implicitly assumed that this covariance matrix is nonsingular. [See Parzen (1962) for a discussion of the case when $\mathbf{R}_{\mathbf{x}}$ is singular.]

D.4 STOCHASTIC PROCESSES

In a study of dynamic systems, it is natural to have random variables that evolve in time much as the states and control inputs evolve. However, with random time variables it is not possible to compute z-transforms in the usual way; and furthermore, because specific values of the variables have little value, we need formulas to describe how the means and covariances evolve in time. A random variable that evolves in time is called a *stochastic process*, and here we consider only discrete time.

Suppose we deal first with a stochastic process $w(n)$, where w is a scalar distributed according to the density $f_w(\xi; N)$. Note that the density function depends on the time of occurrence of the random variable. If a variable has statistical properties (such as f_w) that are independent of the origin of time, then we say the process is *stationary*. Considering values of the process at distinct times, we have separate random variables, and we define the covariance of the process w as

$$R_w(j,k) = \mathcal{E}(w(j) - \overline{w}(j))(w(k) - \overline{w}(k)). \tag{D.20}$$

If the process is stationary, then the covariance in (D.20) depends only on the magnitude of the difference in observation times, $k-j$, and we often will write $R_w(j,k) = R_w(k - j)$ and drop the second argument. Because a stochastic process is both random and time dependent, we can imagine averages that are computed over the time variable as well as by the expectation. For example, for a stationary process $w(n)$ we can define the mean as

$$\widetilde{w}(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} w(n+k), \tag{D.21}$$

and the second-order mean or autocorrelation

$$\begin{aligned}
&\widetilde{(w(j) - \tilde{w})(w(k) - \tilde{w}(k))} \\
&= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \{(w(n+j) - \tilde{w}(j))(w(n+k - \tilde{w}(k)))\}.
\end{aligned} \tag{D.22}$$

For a stationary process, the time average in (D.21) is usually equal to the distribution average, and likewise the second-order average in (D.22) is the same as the covariance in (D.20). Processes for which time averages give the same limits as distribution averages are called *ergodic*.

A very useful aid to understanding the properties of stationary stochastic processes is found by considering the response of a linear stationary system to a stationary input process. Suppose we let the input be w, a stationary scalar process with zero mean and covariance $R_w(j)$, and suppose we take the output to be $y(k)$. We let the unit-pulse response from w to y be $h(j)$. Thus from standard analysis (see Chapter 2), we have

$$y(j) = \sum_{k=-\infty}^{\infty} h(k)w(j-k), \tag{D.23}$$

and the covariance of $y(j)$ with $y(j+\ell)$ is

$$\begin{aligned}
R_y(\ell) &= \mathcal{E}y(j+\ell)y(j) \\
&= \mathcal{E}\left\{\sum_{k=-\infty}^{\infty} h(k)w(j+\ell-k)\right\}\left\{\sum_{n=-\infty}^{\infty} h(n)w(j-n)\right\}.
\end{aligned} \tag{D.24}$$

Because the system unit-pulse response, $h(k)$, is not random, both $h(k)$ and $h(n)$ can be removed from the integral implied by the $\mathcal{E}$ operation, with the result

$$R_y(\ell) = \sum_{k=-\infty}^{\infty} h(k) \sum_{n=-\infty}^{\infty} h(n)\mathcal{E}w(j+\ell-k)w(j-n). \tag{D.25}$$

The expectation in (D.25) is now recognized as $R_w(\ell - k + n)$, and substituting this expression in (D.25), we find

$$R_y(\ell) = \sum_{k=-\infty}^{\infty} h(k) \sum_{n=-\infty}^{\infty} h(n)R_w(\ell - k + n). \tag{D.26}$$