

# ELEC 3004 / 7312 – Digital Linear Systems: Signals & Controls 2015

## Problem Set 1: Linear Signals SOLUTIONS & RUBRIC

**Total marks:** 25**Due Date:** March 25, 2014 (at 11:59pm, AEST)**Q1.****[10 points]**Consider the following circuit in which  $R = 5\Omega$ ,  $L = 1\text{H}$  and its transfer function  $H(s)$ .

1. Solve a mesh equation to find the time-domain current of this circuit in response to the following  $V_{in}^1$ . [2 points]  $V_{in} = \delta(t)$

The time-domain current is the circuit's *Impulse Response*<sup>2</sup>. How does it relate to the circuit's transfer function? [1 point]

2. What is the time-domain current of this circuit in response to the following  $V_{in}$ ? [1 point]

$$V_{in} = \delta(t-1)$$

How is this related to the current derived in (1) and what property of the system does this demonstrate? [1 point]

3. Without doing any calculations, can you predict the time-domain current of this circuit for the following  $V_{in}$ ? [1 point]

$$V_{in} = 2 \cdot \delta(t) + 5 \cdot \delta(t-2)$$

Justify your answer using principles and properties of the system (the circuit).

[4 points]

**Solution:****Part 1:**

a)

The time-domain current can be derived using the Laplace Transform technique:

$$\delta(t) = 5i + \frac{di}{dt}$$

$$1 = 5I(s) + sI(s)$$

$$1 = I(s)(5 + s)$$

$$\frac{1}{s+5} = I(s)$$

$$L(\delta(t)) \cdot H(s) = H(s)$$

$$L^{-1}\left(\frac{1}{s+5}\right) = u(t) \cdot e^{-5t}$$

The unit step is required. A lone exponential predicts massive voltages at negative time. That does not make sense for our circuit.

b)

The transfer function is the Laplace Transform of the Impulse Response.

<sup>1</sup> This is the [Dirac delta function](#).

<sup>2</sup> This need not always be a current. We have simply defined the circuit's output to be the mesh current.

**Part 1: Points for Parts Done Correctly:**

Correct mesh equation with explicit solution via any method	1 mark
Decaying Exponential $e^{-5t}$	1 mark
Unit Step	1 mark
Recognition that transfer function is Laplace transform of impulse response	1 mark

**Part 2:**

This can be solved a few ways, but by far the easiest is by using Laplace Transforms and the second-shifting theorem, noting that this theorem holds for LTI systems like our RL circuit.

We can quickly solve this in the s-domain.

$$e^{-s} \cdot H(s) = \frac{e^{-s}}{s+5}$$

$$L^{-1}\left(\frac{e^{-s}}{s+5}\right) = u(t-1) \cdot e^{-5(t-1)}$$

The result is exactly the same, shifted in time by 1 second just like the input. While this isn't a proof of time-invariance, this is an demonstration of how a time-invariant system behaves.

**Part 2: Points for Parts Done Correctly:**

Correct time-shifted impulse response from (2), with some mathematical justification; e.g. using second shifting theorem	1 mark
Recognition that the response is identical to (2), just time-shifted and that this demonstrates time invariance	1 mark

**Part 3:**

After calculating the answers to 1, 2, 3 it may seem intuitive that the current is going to be:

$$I_{out} = 2 \cdot u(t) \cdot e^{-5t} + 5 \cdot u(t-2) \cdot e^{-5(t-2)}$$

We have simply added and time-shifted impulse responses. But why does this work?

RLC components are all linear and therefore RLC circuits are linear. Therefore superposition holds for each component and therefore the circuit as a whole. Mathematically:

$$F(ax_1 + bx_2) = aF(x_1) + bF(x_2)$$

where  $F$  is the circuit, its argument is the input voltage and evaluating  $F$  gives the output current. We can apply this to our circuit as follows.

$$F_{sys}(2 \cdot \delta(t) + 5 \cdot \delta(t-1)) = 2 \cdot F_{sys}(\delta(t)) + 5 \cdot F_{sys}(\delta(t-1))$$

$$F_{sys}(\delta(t)) = H(t)$$

$$2 \cdot H(t) + 5 \cdot H(t-2) = 2 \cdot u(t) \cdot e^{-5t} + 5 \cdot u(t-2) \cdot e^{-5(t-2)}$$

Note that the system must be time-invariant for this to work. An impulse later in time must excite the circuit in exactly the same way. If  $R$  or  $L$  changed with time, the second, later impulse would not follow the same exponential function.

Mathematically, impulses must map to the same function, regardless of time.

$$\delta(t) \rightarrow H(t)$$

$$\delta(t - \Delta) \rightarrow H(t - \Delta)$$

These two properties of linearity and time-invariance combined allow us to simply add and shift impulse responses together without recalculating everything from the beginning.

### Part 3: Points for Parts Done Correctly:

Correct output current	1 mark
Statement and application of the superposition principle	1 mark
Statement and application of time invariance	1 mark
Insightful presentation of results. Does the answer convey genuine understanding in your opinion?	1 mark

**Q2.****[5 points]**

In Question 1, part 3 you observed a circuit's response to a sum of time-shifted impulses. In fact, **every signal is a sum of time-shifted impulses**. The following integral states this fact mathematically:

$$\int_{-\infty}^{\infty} \delta(t-T)f(T)dT = f(t)$$

1. Given this fact, construct an integral that could be used to evaluate a circuit's response to an **input f(t)**, given the circuit's **impulse response H(t)** [1 point].
2. Discuss why the integral produces the system output. [4 points]

**Solution:**

This "mystery" integral is the convolution integral.

$$\int_{-\infty}^{\infty} H(t-T)f(T)dT = F(t)$$

The lower bound is generally zero by convention in engineering.

A signal entering an LTI system is simply a sum of scaled, time-delayed impulses. By the properties of linearity and time-invariance, the LTI system's response will be a sum of impulse responses with the same scale and time-delay as the input impulses.

In Question 1, part 4, the sum was discrete with only two elements. Here, the sum is "continuous"; i.e. it's an integral. However, we can still see that the expression being integrated (summed) over time is an impulse response that is:

- Scaled to the value of the input function at the present point in time
- Time-shifted to the current point in time
- Mirrored about the y-axis (time reversed) by the negative time argument

The only gotcha is the time-reversal. The reason for this is clearer when you view convolution graphically. In short, time reversal makes sure that the signal's voltage at  $t = 0$  enters the system first, not last.

**Points for Parts Done Correctly:**

Insightful explanation of the convolution integral. This is mostly your opinion on the accuracy and insight of the discussion.	4 marks
Correct statement of integral	1 Mark

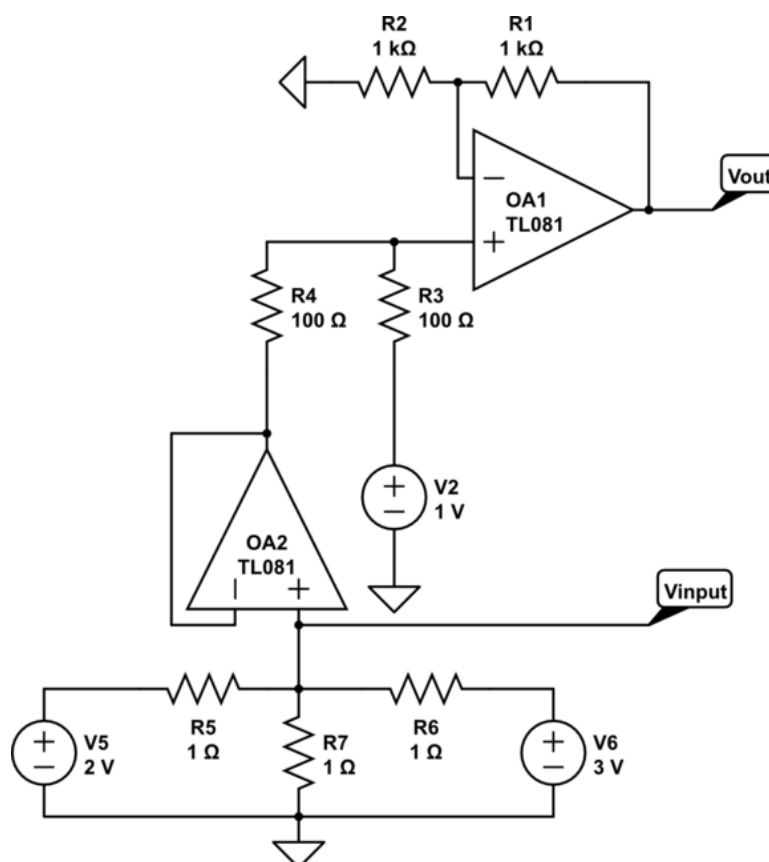
**Q3. [5 points]**

Try solving the next circuit using superposition **on the inputs** (do not consider V2 in your superposition since it's not part of the input to this entire circuit. It's part of a stage in this circuit).

**Clarification:**

As a mode of clarification perhaps the diagram below helps you visualise what the “input” is and to see this as less of a circuit analysis problem and more of a signals and systems problem comprised of 2 system blocks (op amp stages) each with a system function. What are the 2 op amp topologies and their equations?

1. Present your superposition calculations briefly. Pages of equations should not be necessary. [1 point]
2. Briefly calculate the output voltage without using superposition and compare it to the voltage from (1). [1 point]
3. Explain your results, citing and applying principles you deem relevant. Does superposition give the correct result? Why? Based on your superposition result, do you think the above system is linear or nonlinear? [3 points]

**Solution:**

Realising that each source V5, V6 sees a voltage divider

$$1\Omega \parallel 1\Omega = 0.5\Omega$$

$$\frac{0.5\Omega}{1.5\Omega} = \frac{1}{3}$$

$$V_{input} = \frac{1}{3}(2V + 3V) = \frac{5}{3}$$

The op-amp circuit simply adds 1 volt to its input.

$$O(V_{input}) = V_{input} + 1$$

If we solve via superposition the following happens.

$$O(\frac{2}{3}) = \frac{2}{3} + 1 = \frac{5}{3}$$

$$O(1) = 1 + 1 = 2$$

$$O(\frac{2}{3}) + O(1) = \frac{5}{3} + 2 = \frac{11}{3}$$

Yet, we get a different answer when calculating (correctly) as follows.

$$O(\frac{2}{3} + 1) = \frac{5}{3} + 1 = \frac{8}{3}$$

Of course, the relevant principle is the superposition principle. Does our op amp circuit uphold superposition?

$$O(V_{input}) = V_{input} + 1$$

$$O(a + b) = a + b + 1$$

$$O(a) + O(b) = a + 1 + b + 1$$

$$O(a + b) \neq O(a) + O(b)$$

This simple result is often unintuitive. A system  $F(x) = x + 1$  is actually **nonlinear**. The superposition principle defines linearity. If a system does not uphold superposition, it is not linear. Yet, in calculus if we see the function  $y = x + 1$ , we are told that is linear. In the context of calculus, that is the correct terminology. However, in LTI system theory we define linearity by the superposition principle. A constant term in a function does not uphold superposition and therefore renders a system nonlinear.

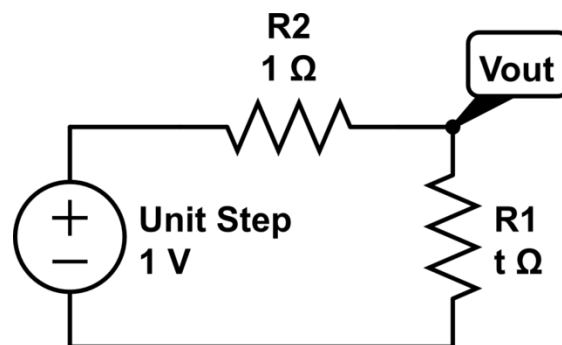
**Points for Parts Done Correctly:**

Calculation of voltage without superposition as 1 volt less than with superposition	1 mark
Usage of black box model $f(x) = x + 1$ or sufficiently short analysis of op amp circuit	1 mark
Explanation citing the superposition principle with application to the black box model $f(x) = x + 1$ demonstrating the failure of superposition. Add up to 3 marks depending on quality of explanation in your opinion.	3 marks

**Q4. [5 points]**

Suppose the resistance of a copper motor winding increases by  $1\Omega$  per second during periods of extremely high current. We'd like to see how hard this is to model.

1. Derive a time-domain expression for  $V_{out} / V_{in}$  as a function of time from  $t = 0$  onwards. Do not go into the s-domain. [1 point]
2. Find the transfer function  $H(s)$  of this circuit. Now solve in the s-domain for  $V_{out}$  with a unit step input. [1 point]
3. Does your s-domain solution agree with your solution from (1)? Why do you think this is the case? [3 points]

**Solution:**

This is a voltage divider with a unit step input. Therefore:.

$$V_{out}(t) = \frac{t}{1+t}$$

Taking the resistors into the s-domain we might be tempted to write the following.

$$R_2(s) = 1; R_1(s) = \frac{1}{s^2}$$

If we follow this reasoning, we get the following.

$$H(s) = \frac{\frac{1}{s^2}}{1 + \frac{1}{s^2}} = \frac{1}{s^2 + 1}$$

The input is a unit step, so we also multiply our transfer function by  $1/s$ .

$$V_{out}(s) = \frac{1}{s} \cdot H(s) = \frac{1}{s} \cdot \frac{1}{s^2 + 1}$$

$$L^{-1}(V_{out}(s)) = 1 - \cos(t)$$



**In any solution using this method, the s-domain and time-domain solutions will not agree.** This is because multiplication in the s-domain is equivalent to convolution, by the [Convolution Theorem](#). However, convolution of the input and impulse response is only equivalent the solution of a differential equation if the system is LTI. However, this system has a resistance dependent on time. The system time-varying. Therefore, the response to a time-delayed impulse is not the time-delayed impulse response as usual. Therefore, the convolution will not provide a valid solution.

**Points for Parts Done Correctly:**

Calculation of time domain expression using voltage divider	1 mark
Derivation of some transfer function and solution via s-domain	1 mark
Discussion of the reasons behind any observed differences in the solutions and the cause of the difference. Why were the solutions not the same? Give marks based on the insight demonstrated by the discussion in your opinion.	3 marks

**Q5. Sampling: In Sync With Nyquist****[5 points]****True or False:**

1. If a system is linear, it can be written as  $f(x)=Ax$
2. Given the canonical form ( $\dot{x} = Ax + Bu$ , etc.), if  $B=0$ , then the system is autonomous.
3. A Linear Time Invariant (LTI) system must have constant system, control, etc. matrices (**A**, **B**, **C**, and **D** in canonical form).
4. The two requirements for a system to be linear are (1) Additivity and (2) Homogeneity
5. A capacitor is an example of an invertible system
6. All real-time systems must be causal
7. The inverse of a first-difference operation ( $y[n] = x[n] - x[n-1]$ ) is the accumulator  

$$(y[n] = \sum_{k=-\infty}^n x[k]).$$
8. If systems **A** and **B** are LTI, then so is the overall system
9. If systems **A** and **B** are non-linear, then so is the overall system
10. If systems **A** and **B** are stable, then so is the overall system

**Solution:****Points for Parts Done Correctly: +½ (ALL) or 0 (NOTHING).****Answers MUST have full & well-justified explanations.**

1. **TRUE** -- If a system is linear, it can be written as  $f(x)=Ax$

A linear system is one that admits superposition. Matrix multiplication is a linear function for which the converse is true. Any linear function  $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$  can be written as  $f(x)=Ax$  for some  $A \in \mathbb{R}^{m \times n}$ .

[The “canonical” form may be written this way by augmenting this way by augmenting the state vector  $x_{\text{new}} \equiv [x \mid u]$ ]

2. **FALSE** -- Given the canonical form ( $\dot{x} = Ax + Bu$ , etc.), if  $B=0$ , then the system is autonomous.

If  $u=0$ , then the system is autonomous. (if  $B=0$ , it just means that there is no input effect on the state, but there could be **feedthrough** ( $D \neq 0$ )).

[While such a configuration is not common, it is possible]

3. **TRUE** -- A Linear Time Invariant (LTI) system must have constant system, control, etc. matrices (**A**, **B**, **C**, and **D** in canonical form).

If the system is Time-Invariant, then the system (etc) matrices  $A(t)$ , etc. are not a function of time and, thus, are constant. What about the case, where the matrices are a function of something else (say temperature,  $T$ ) such that  $A(T)$ ? In this class we are concerned with Linear Dynamical Systems of the form  $\dot{x} = Ax + Bu$  which are first-order vector differential equations. Such a system ( $A(T,t)$  in this case) would be better represented as  $A(t)$  with the Temperature ( $T$ ) as part of the state vector  $x$ .

4. **TRUE** -- The two requirements for a system to be linear are (1) Additivity and (2) Homogeneity

By definition. The two requirements for a system to be linear are (1) Additivity and (2) Homogeneity. These can be combined into a single condition, termed "superposition"

5. **TRUE** -- A capacitor is an example of an invertible system

A system is invertible if we can determine its input signal ( $x(t)$ ) uniquely by observing its output signal ( $y(t)$ ).

6. **TRUE** -- All real-time systems must be causal

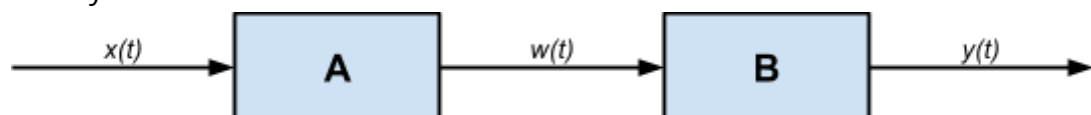
A causal system is one that for an arbitrary time  $t=t_a$ , the output only depends on the input for  $t \leq t_a$ . For a system to be real-time it has to have fixed time slices and be up to the present (implying that it can not depend on or know about future time). To implement a non-causal system in real-time, one would have to add a delay (or clairvoyance :-)).

7. **TRUE** -- The inverse of a first-difference operation ( $y[n] = x[n] - x[n-1]$ ) is the accumulator

$$(y[n] = \sum_{k=-\infty}^n x[k]).$$

By definition. Another way to see this is to consider that the accumulator has a unit-step impulse response of:  $y[n]=u[n]$ . The inverse to such a system is the first-difference operator  $y_{\text{inverse}}[n]=\delta[n]-\delta[n-1]$  and that  $y[n]*y_{\text{inverse}}[n]=u[n]-u[n-1]=\delta[n]$

Imagine two systems in cascade:



For 8-10, imagine these two systems have impulse responses  $h_A(t)$  and  $h_B(t)$ . When we cascade the two systems, the outputs may be seen as two convolutions:

$$w(t) = x(t) * h_A(t),$$

$$y(t) = w(t) * h_B(t) = (x(t) * h_A(t)) * h_B(t) = (h_A(t) * h_B(t)) * x(t)$$

8. **TRUE** -- If systems **A** and **B** are LTI, then so is the overall system

Convolution is a linear operator. Thus  $(h_A(t) * h_B(t))$  is linear.

9. **FALSE** -- If systems **A** and **B** are non-linear, then so is the overall system

It is possible, but not probable, that the systems A and B multiply to give a linear result. For example if  $B=A^{-1}$ , then even if A is non-linear, their cascade would be identity.

10. **TRUE** -- If systems **A** and **B** are stable, then so is the overall system

We assume that  $x(t)$  is bounded. Then,  $w(t)$  would be bounded because A is BIBO. If  $w(t)$  is bounded, then the BIBO stability of B implies that  $y(t)$  should be bounded.