http://elec3004.com

Frequency Response & Filter Analysis

ELEC 3004: **Digital Linear Systems**: Signals & Controls

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(with material from Kumaresan, Continuous-Time Fourier Transform, URI)

Lecture 5

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April 1, 2014

Lecture Schedule:

Week	Date	Lecture Title
1	4-Mar	Introduction & Systems Overview
	6-Mar	[Linear Dynamical Systems]
2	11-Mar	Signals as Vectors & Systems as Maps
	13-Mar	[Signals]
3	18-Mar	Sampling & Data Acquisition & Antialiasing Filters
	20-Mar	[Sampling]
4	25-Mar	System Analysis & Convolution
	27-Mar	[Convolution & FT]
5	1-Apr	Frequency Response & Filter Analysis
	3-Apr	[Filters]
6	8-Apr	Discrete Systems & Z-Transforms
	10-Apr	[Z-Transforms]
7	15-Apr	Introduction to Control
	17-Apr	[Feedback]
8	29-Apr	Digital Filters
	1-May	[Digital Filters]
9	6-May	Introduction to Digital Control
	8-May	[Digitial Control]
10	13-May	Stability of Digital Systems
	15-May	[Stability]
11	20-May	State-Space
	22-May	Controllability & Observability
12	27-May	PID Control & System Identification
	29-May	Digitial Control System Hardware
13	3-Jun	Applications in Industry & Information Theory & Communications
	5-Jun	Summary and Course Review

Frequency Response

Fourier Series → Fourier Transforms

Typical Linear Processors

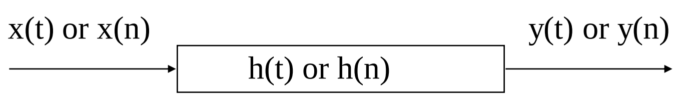
- Convolution h(n,k)=h(n-k)
- Cross Correlation h(n,k)=h(n+k)
- Auto Correlation h(n,k)=x(k-n)
- Cosine Transform h(n,k)= $\cos\left(\frac{2\pi}{N}nk\right)$
- Sine Transform h(n,k)= $\sin\left(\frac{2\pi}{N}nk\right)$
- Fourier Transform h(n,k)= $\exp\left(j\frac{2\pi}{N}nk\right)$

Transform Analysis

- Signal measured (or known) as a function of an independent variable
  - e.g., time: y = f(t)
- However, this independent variable may not be the most appropriate/informative
  - e.g., frequency: Y = f(w)
- Therefore, need to transform from one domain to the other
  - e.g., time ⇔ frequency
  - As used by the human ear (and eye)

Signal processing uses Fourier, Laplace, & *z* transforms etc

Sinusoids and Linear Systems



If $$x(t) = A\cos(\omega_0 t + \theta_0)$$

or $$x(n) = A\cos(\omega_0 nt + \theta_0)$$

then in steady state

$$y(t) = AC(\omega_0)\cos(\omega_0 t + \theta_0 + \theta(\omega_0))$$

$$y(n) = AC(\omega_0 T)\cos(\omega_0 nt + \theta_0 + \theta(\omega_0 T))$$

Sinusoids and Linear Systems

- The pair of numbers C(w0) and q(w0) are the complex gain of the system at the frequency w0 .
- They are respectively, the magnitude response and the phase response at the frequency w0 .

$$y(t) = AC(\omega_0)\cos(\omega_0 t + \theta_0 + \theta(\omega_0))$$
$$y(n) = AC(\omega_0 T)\cos(\omega_0 nt + \theta_0 + \theta(\omega_0 T))$$

Why Use Sinusoids?

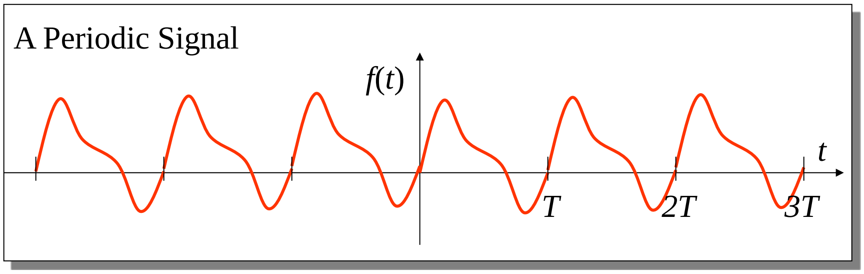
- Why probe system with sinusoids?
- Sinusoids are eigenfunctions of linear systems???
- What the hell does that mean?
- Sinusoid in implies sinusoid out
- Only need to know phase and magnitude (two parameters) to fully describe output rather than whole waveform
  - sine + sine = sine
  - derivative of sine = sine (phase shifted - cos)
  - integral of sine = sine (-cos)
- Sinusoids maintain orthogonality after sampling (not true of most orthogonal sets)

Frequency Response

Fourier Series → Fourier Transforms

Fourier Series

- Deal with continuous-time periodic signals.
- Discrete frequency spectra.



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Two Forms for Fourier Series

Sinusoidal Form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt$$

Complex Form:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt$$

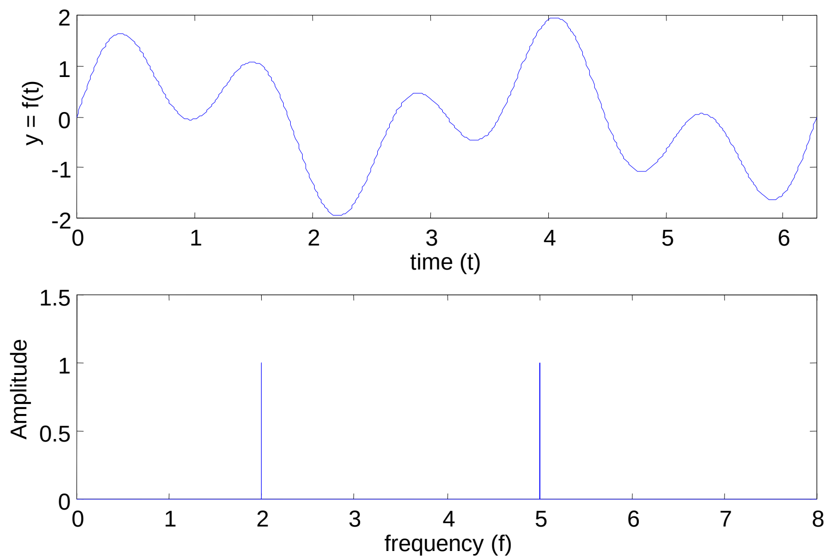
Source: URI ELE436

Fourier Series

- Any finite power, periodic, signal x(t)
  - period T
- can be represented as (∞) summation of
  - sine and cosine waves
- Called: Trigonometrical Fourier Series

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos(nw_0 t) + B_n \sin(nw_0 t)$$

- Fundamental frequency $w_0 = 2\pi/T$ rad/s or $1/T$ Hz
- DC (average) value $A_0/2$



Frequency representation (spectrum) shows signal contains:
• 2Hz and 5Hz components (sinewaves) of equal amplitude

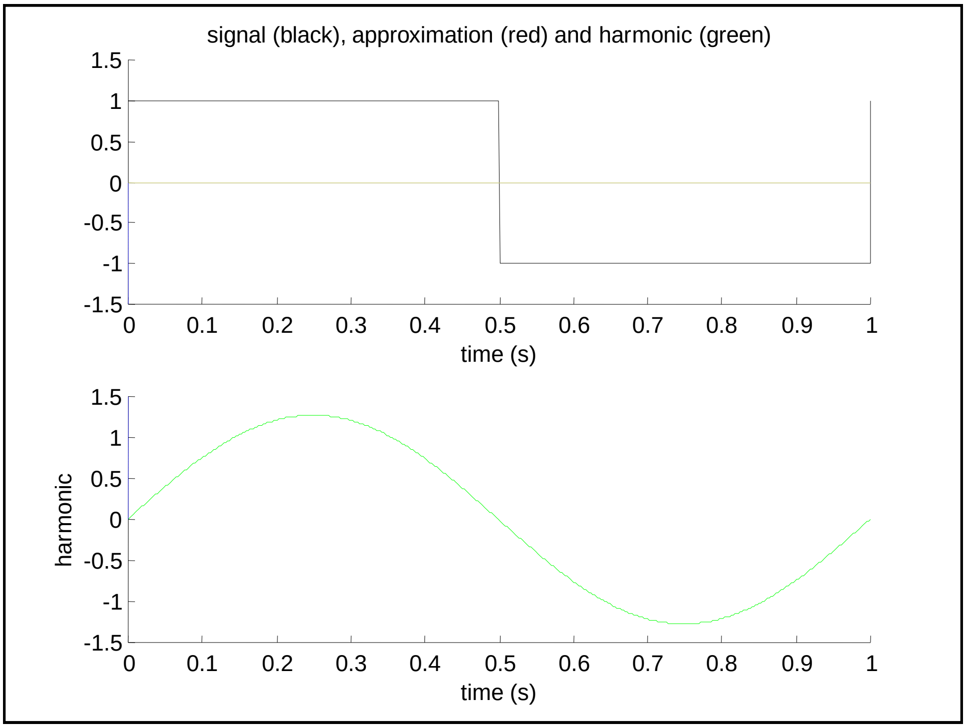
Fourier Series Coefficients

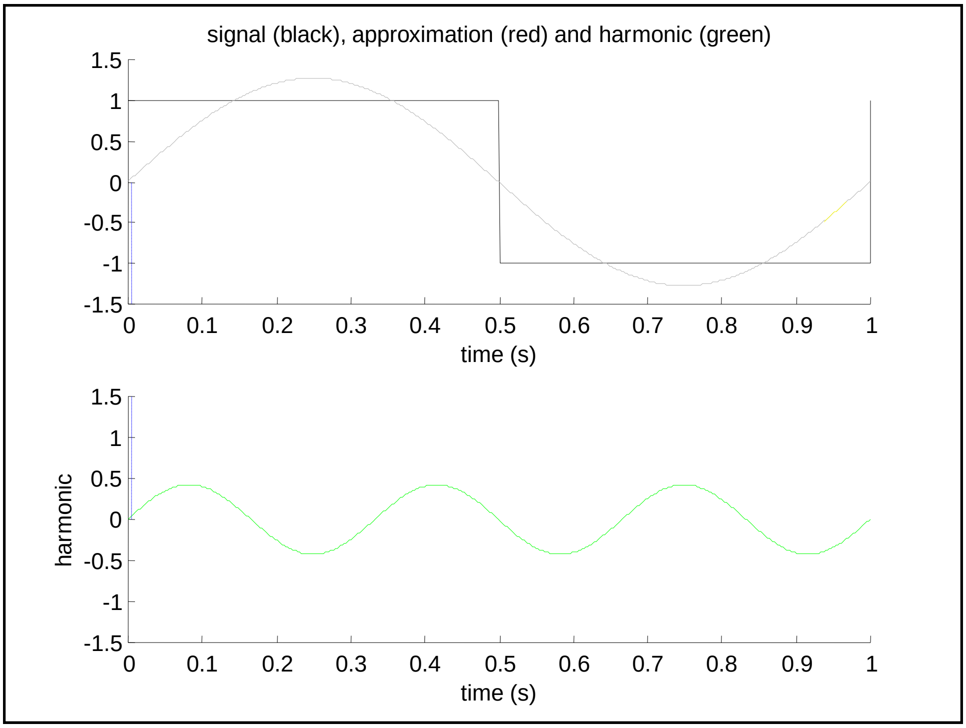
- An & Bn calculated from the signal, x(t)
  - called: Fourier coefficients

$$A_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(nw_0 t)\,dt \qquad n = 0,1,2,\cdots$$

$$B_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(nw_0 t)\,dt \qquad n = 1,2,3,\cdots$$

Note: Limits of integration can vary, provided they cover one period


signal (black), approximation (red) and harmonic (green)
time (s)
harmonic
time (s)


signal (black), approximation (red) and harmonic (green)
time (s)
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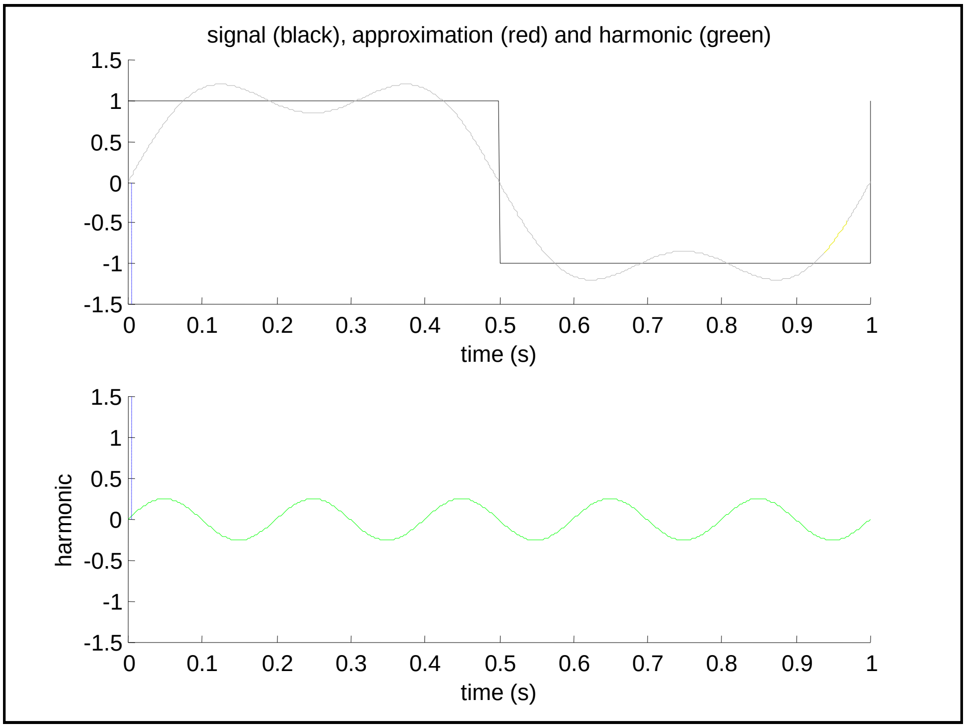
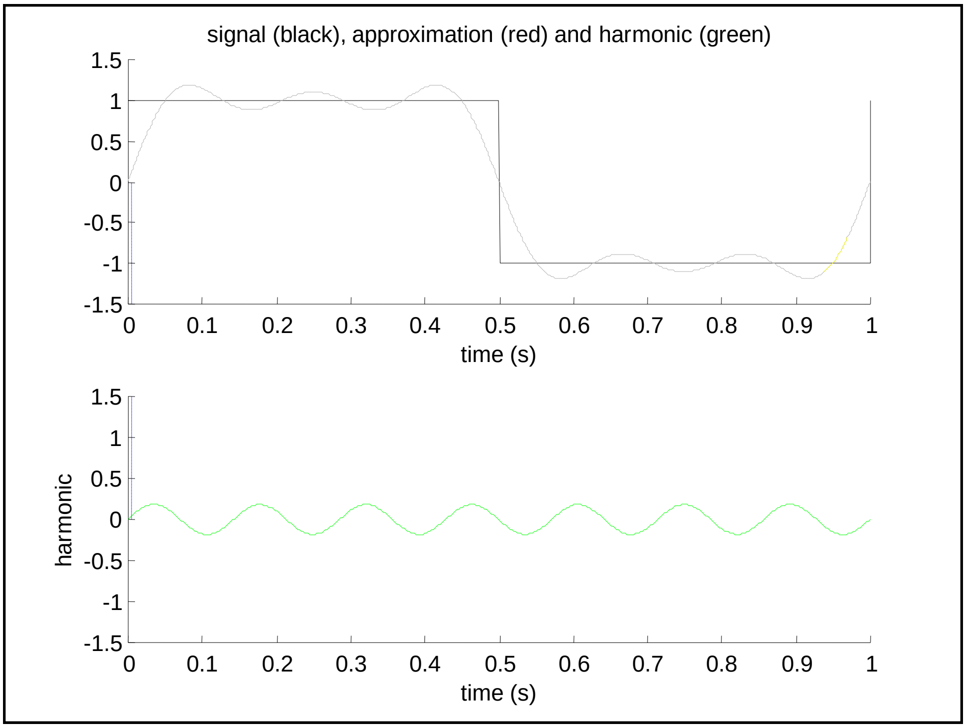


signal (black), approximation (red) and harmonic (green)
time (s)
harmonic
time (s)



signal (black), approximation (red) and harmonic (green)
time (s)
harmonic
time (s)

Fourier Series Coefficients

- Approximation with 1st, 3rd, 5th, & 7th Harmonics added, note:
  - 'Ringing' on edges due to series truncation
  - Often referred to as Gibb's phenomenon
- Fourier series converges to original signal if
  - Dirichlet conditions satisfied
  - Closer approximation with more harmonics

Example: Square wave

$$x(t) = \begin{cases} 1, & 0 < t < 1; \\ -1, & 1 < t < 2; \\ x(t+2). & \end{cases}$$ ← periodic! i.e., x(t + 2) = x(t)

$$A_n = \int_0^2 x(t)\cos(n\pi t)dt = \int_0^1 \cos(n\pi t)dt - \int_1^2 \cos(n\pi t)dt$$

$$A_n = \left[\frac{-\sin(n\pi t)}{n\pi}\right]_0^1 - \left[\frac{-\sin(n\pi t)}{n\pi}\right]_1^2 = 0$$

No cos terms as sin(nπ) = 0 ∀ n
x(t) has odd symmetry

$$B_n = \int_0^2 x(t)\sin(n\pi t)dt = \int_0^1 \sin(n\pi t)dt - \int_1^2 \sin(n\pi t)dt$$

cos(2nπ) = 1 ∀ n

$$B_n = \left[\frac{-\cos(n\pi t)}{n\pi}\right]_0^1 - \left[\frac{-\cos(n\pi t)}{n\pi}\right]_1^2 = -\frac{\cos(n\pi)}{n\pi} + \frac{1}{n\pi} + \frac{1}{n\pi} - \frac{\cos(n\pi)}{n\pi}$$

$$B_n = \frac{2}{n\pi}(1 - \cos(n\pi))$$

Sin terms only

Example: Square wave

Therefore, Trigonometric Fourier series is,

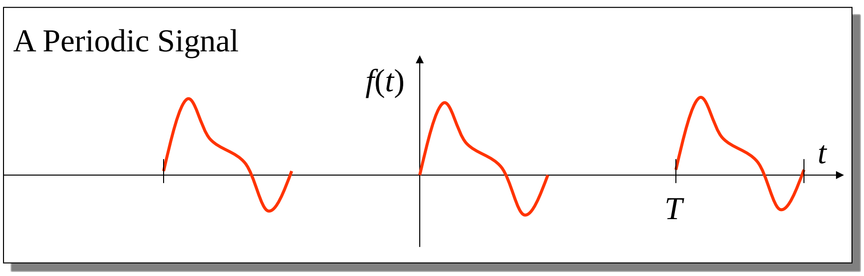
$$x(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1 - \cos(n\pi))\sin(n\pi t)$$

Expanding the terms gives,

$$\begin{aligned} x(t) = \; & \frac{4}{\pi}\sin(\pi t) && \text{(fundamental)} \\ & + 0 && \text{(second harmonic)} \\ & + \frac{4}{3\pi}\sin(3\pi t) && \text{(third harmonic)} \\ & + 0 && \text{(fourth harmonic)} \\ & + \frac{4}{5\pi}\sin(5\pi t) && \text{(fifth harmonic)} \\ & + \text{etc} \end{aligned}$$

- Only odd harmonics;
- In proportion 1,1/3,1/5,1/7,…
- Higher harmonics contribute less;
- Therefore, converges

How to Deal with Aperiodic Signal?



If $T \rightarrow \infty$, what happens?

Source: URI ELE436

Fourier Integral

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau} d\tau\right] e^{jn\omega_0 t} \qquad \omega_0 = \frac{2\pi}{T} \Rightarrow \frac{1}{T} = \frac{\omega_0}{2\pi}$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau} d\tau\right]\omega_0 e^{jn\omega_0 t}$$

Let $\Delta\omega = \omega_0 = \frac{2\pi}{T}$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\left[\int_{-T/2}^{T/2} f_T(\tau) e^{-jn\omega_0 \tau} d\tau\right] e^{jn\omega_0 t}\Delta\omega$$

$T \to \infty \Rightarrow d\omega = \Delta\omega \approx 0$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_T(\tau) e^{-j\omega \tau} d\tau\right] e^{j\omega t} d\omega$$

Source: URI ELE436

Fourier Integral

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\left[\int_{-\infty}^{\infty} f(\tau) e^{-j\omega \tau} d\tau\right]}_{F(j\omega)} e^{j\omega t} d\omega$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega \qquad \text{Synthesis}$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \qquad \text{Analysis}$$

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Fourier Series vs. Fourier Integral

Fourier Series:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$ Period Function

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t) e^{-jn\omega_0 t} dt$$ Discrete Spectra

Fourier Integral:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$ Non-Period Function

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$ Continuous Spectra

Source: URI ELE436

Complex Fourier Series (CFS)

- Also called Exponential Fourier series
  - As it uses Euler's relation

$$A\exp(jw_0 t) = A\cos(w_0 t) + jA\sin(w_0 t)$$

which implies,

$$\cos(nw_0 t) = \frac{\exp(jnw_0 t) + \exp(-jnw_0 t)}{2}$$

$$\sin(nw_0 t) = \frac{\exp(jnw_0 t) - \exp(-jnw_0 t)}{2j}$$

- FS as a Complex phasor summation

$$x(t) = \sum_{n=-\infty}^{+\infty} X_n \exp(jnw_0 t)$$

Where X_n are the CFS coefficients

Complex Fourier Coefficients

- Again, Xn calculated from x(t)
- Only one set of coefficients, Xn
  - but, generally they are complex

$$X_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\exp(-jnw_0 t)dt$$

Remember: fundamental $w_0 = 2\pi/T$!

Relationships

- There is a simple relationship between
  - trigonometrical and
  - complex Fourier coefficients,

$$X_0 = \frac{A_0}{2}$$

$$X_n = \begin{cases} \frac{A_n - jB_n}{2}, & n > 0; \\ \frac{A_n + jB_n}{2}, & n < 0. \end{cases}$$

Constrained to be symmetrical, i.e., complex conjugate

$$X_{-n} = X_n^*$$

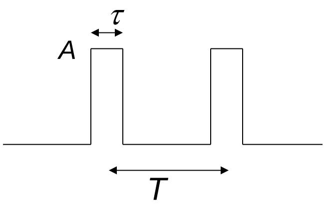
Therefore, can calculate simplest form and convert

Example: Complex FS

- Consider the pulse train signal

$$x(t) = \begin{cases} A, & 0 \le |t| \le \frac{\tau}{2}; \\ 0, & \frac{\tau}{2} < |t| \le T; \\ x(t+T). & \end{cases}$$

- Has complex Fourier series:



$$X_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\exp(-jn\omega_0 t)dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} A\exp(-jn\omega_0 t)dt$$

$$= \frac{-A\tau}{jn\omega_0 T\tau}\left[\exp\left(\frac{-jn\omega_0\tau}{2}\right) - \exp\left(\frac{jn\omega_0\tau}{2}\right)\right]$$

Note: × by τ/τ …

Note: n is the ind. variable

Example: Complex FS

- Which using Euler's identity reduces to:

$$X_n = \frac{A\tau}{T}\frac{\sin(nw_0\,\tau/2)}{nw_0\,\tau/2} = \frac{A\tau}{T}\mathrm{sa}(nw_0\,\tau/2)$$

$$w_0 = \frac{2\pi}{T}$$

Note: letting $\theta = \frac{n\omega_0\tau}{2}$

$$\exp(-j\theta) - \exp(j\theta)$$
$$= \cos(-\theta) + j\sin(-\theta) - (\cos(\theta) + j\sin(\theta))$$
$$= \cos(\theta) - j\sin(\theta) - \cos(\theta) - j\sin(\theta) = -2j\sin(\theta)$$

Note:
cos(-θ) = cos(θ): even
sin(-θ) = -sin(θ): odd

Dirichlet Conditions

For Fourier series to converge,
f(t) must be:

$$\int_{-\infty}^{\infty} | f(t) | \, dt < \infty$$

- defined & single valued
- continuous and have a finite number of finite discontinuities within a periodic interval, and
- piecewise continuous in periodic interval, as must f'(t) be absolutely integrable; i.e.,
  - i.e., have finite energy
- have a finite number of finite discontinuities within a finite interval, and
- have a finite number of maxima and minima within a finite interval

Note: Periodic signals have FT, if we use impulse functions, δ(w)

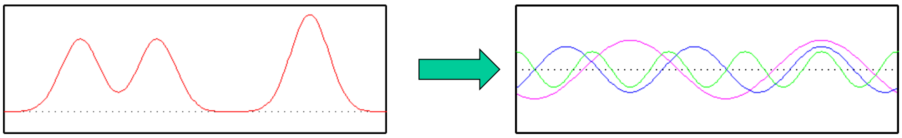
Frequency Response

Fourier Series → **Fourier Transforms**

Fourier Transform

- A Fourier Transform is an integral transform that re-expresses a function in terms of different sine waves of varying amplitudes, wavelengths, and phases.

1-D Example:



- When you let these three waves interfere with each other you get your original wave function!

Source: Tufts Uni Sykes Group

Fourier Series

- What we have produced is a processor to calculate one coefficient of the complex Fourier Series
- Fourier Series Coefficients = Heterodyne and average over observation interval T

$$C_k = \frac{1}{T}\int_0^T h(t)e^{-j\frac{2\pi}{T}kt}\,dt$$

Fourier Transform

- If we change the limits of integration to the entire real line, remove the division by T, and make the frequency variable continuous, we get the Fourier Transform

$$C(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$

Fourier Transform (is not the Fourier Series per se)

	Continuous Time	Discrete Time
Periodic	Fourier Series	Discrete Fourier Transform
Aperiodic	Continuous Fourier Transform	Fourier Transform

Source: URI ELE436

Fourier Transform

- Fourier series
  - Only applicable to periodic signals

- Real world signals are rarely periodic

- Develop Fourier transform by
  - Examining a periodic signal
  - Extending the period to infinity

Fourier Transform

- Problem: as $T \rightarrow \infty$, $Xn \rightarrow 0$
  - i.e., Fourier coefficients vanish!
- Solution: re-define coefficients
  - $Xn' = T \times Xn$
- As $T \rightarrow \infty$
  - (harmonic frequency) nw0 $\rightarrow$ w (continuous freq.)
  - (discrete spectrum) Xn' $\rightarrow$ X(w) (continuous spect.)
  - w0 (fundamental freq.) reduces $\rightarrow$ dw (differential)
    - Summation becomes integration

Fourier Transform Pair

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}\,d\omega$$ Synthesis

Fourier Transform:

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$ Analysis

Source: URI ELE436

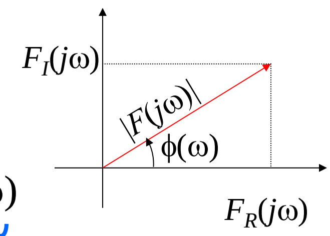
Continuous Spectra

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$$

$$F(j\omega) = F_R(j\omega) + jF_I(j\omega)$$

$$= \underbrace{|F(j\omega)|}_{\text{Magnitude}}\, e^{\overbrace{j\phi(\omega)}^{}}$$

Phase



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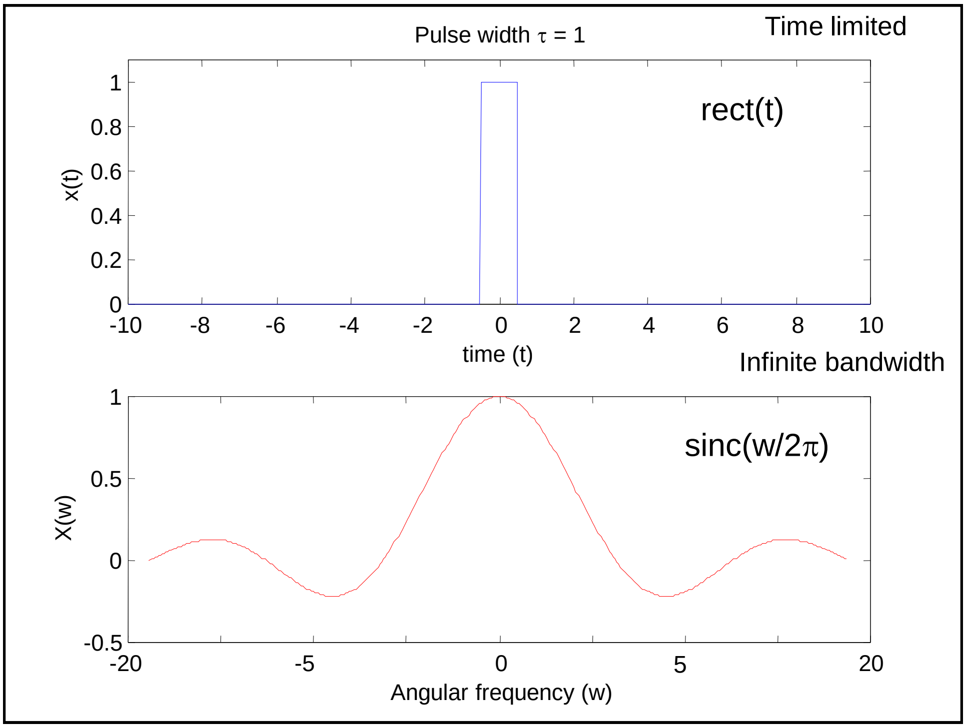
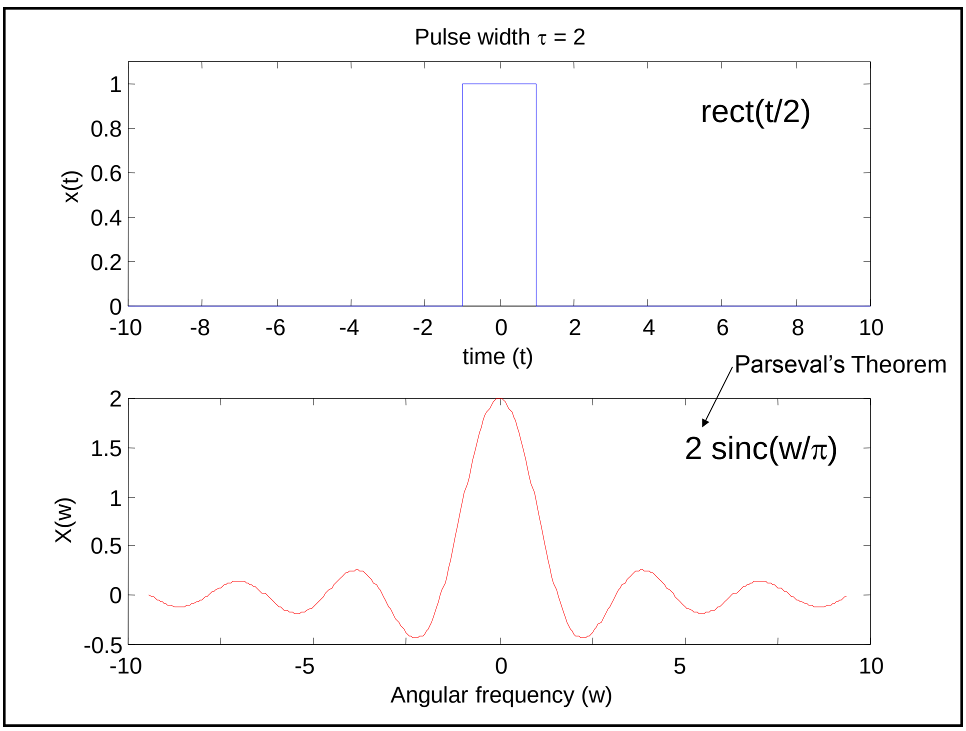
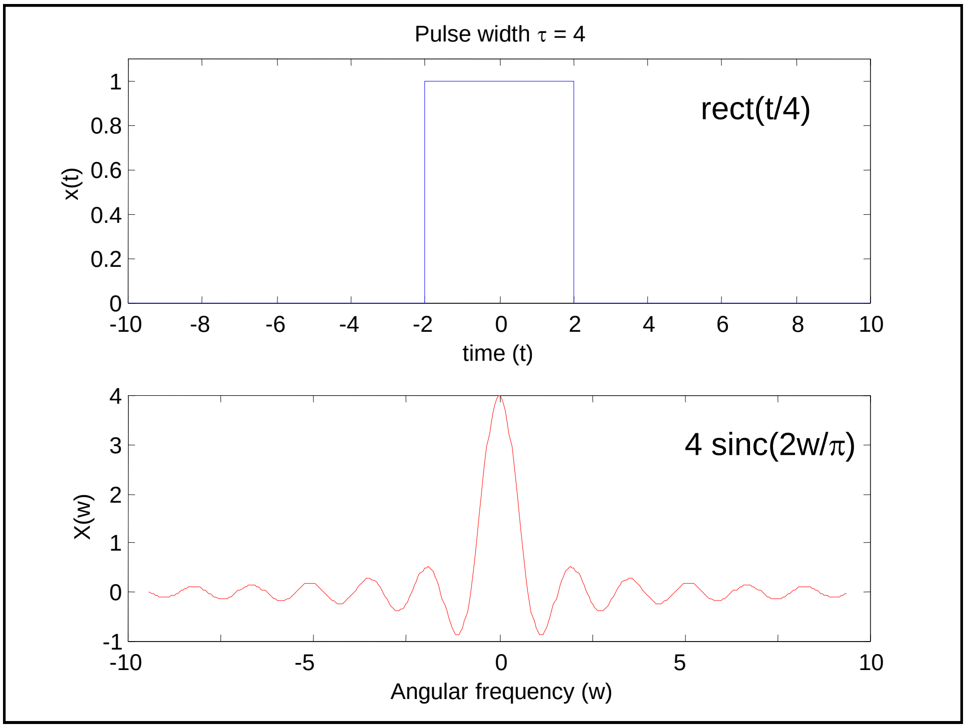
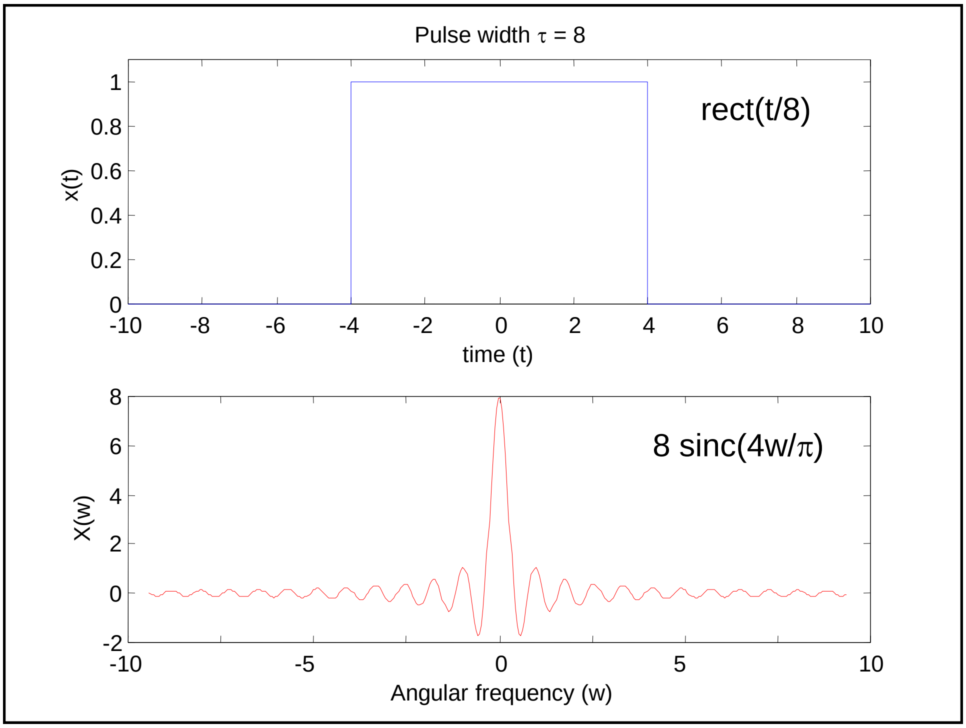
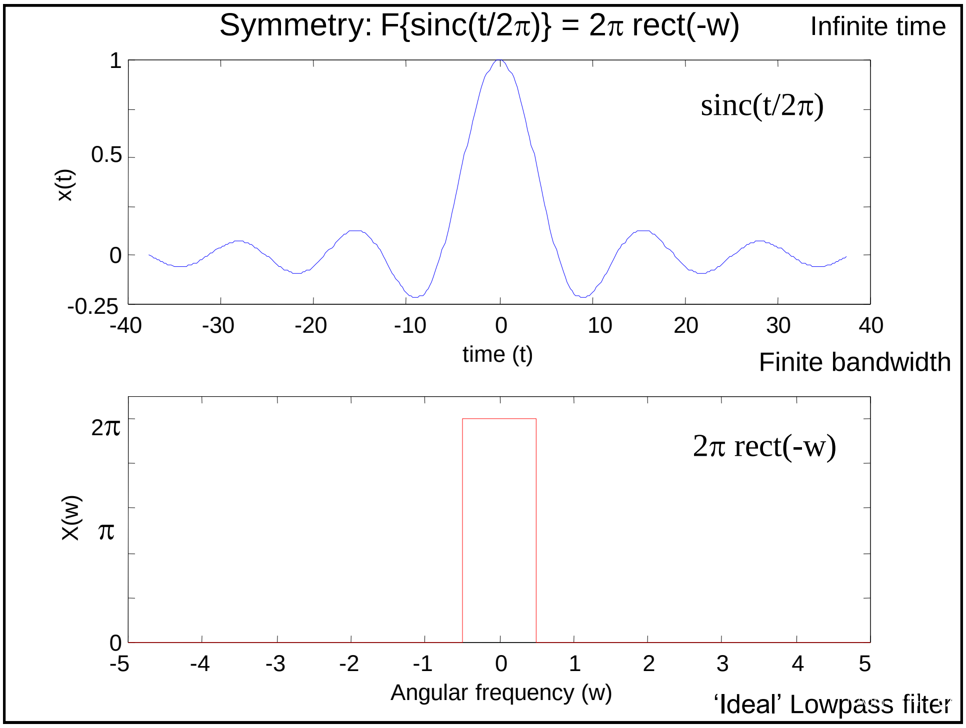

Pulse width τ = 1
Time limited
rect(t)
x(t)
time (t)
Infinite bandwidth
sinc(w/2π)
X(w)
Angular frequency (w)


Pulse width τ = 2
rect(t/2)
x(t)
time (t)
Parseval's Theorem
2 sinc(w/π)
X(w)
Angular frequency (w)


Pulse width τ = 4
rect(t/4)
x(t)
time (t)
4 sinc(2w/π)
X(w)
Angular frequency (w)


Pulse width τ = 8
rect(t/8)
x(t)
time (t)
8 sinc(4w/π)
X(w)
Angular frequency (w)



Properties of Fourier Transform

- Linearity
  - F {a x(t) + b y(t)} = a X(w) + b Y(w)
- Time and frequency scaling
  - F {x(at)} = 1/a X(w/a)
  - broader in time ⇒ narrower in frequency
    - and vice versa
- Symmetry (duality)
  - $2\pi x(-w) = \int X(t) \exp(-jwt)dt$
    - i.e., Fourier transform 'pairs'

Time limited signal limited has infinite bandwidth;
Signal of finite bandwidth has infinite time support

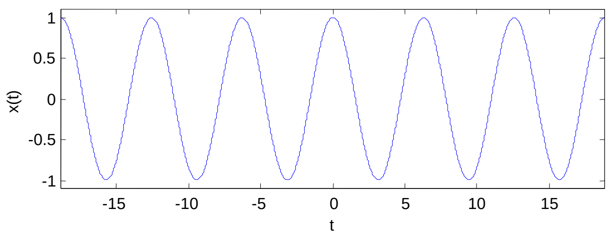
Properties of Fourier Transform

if…	Then…
x(t) is real	X(-w) = X(w)* – $\Re\{X(w)\}$ is even – $\Im\{X(w)\}$ is odd – |X(w)| is even – $\angle X(w)$ is odd
x(t) is real and even	X(w) is real and even
x(t) is real and odd	X(w) is imaginary and odd

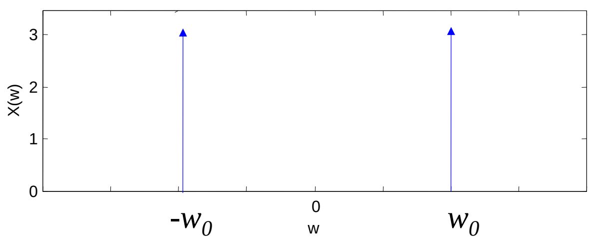
Fourier Transforms

$$X(\omega) = \delta(\omega - \omega_0)$$
$$x(t) = F^{-1}\{X(\omega)\}$$
$$x(t) = \frac{1}{2\pi}\exp(j\omega_0 t)$$

Note: $\cos(w_0 t)$ has ∞ energy! But is dual of $\delta(w - w_0)$



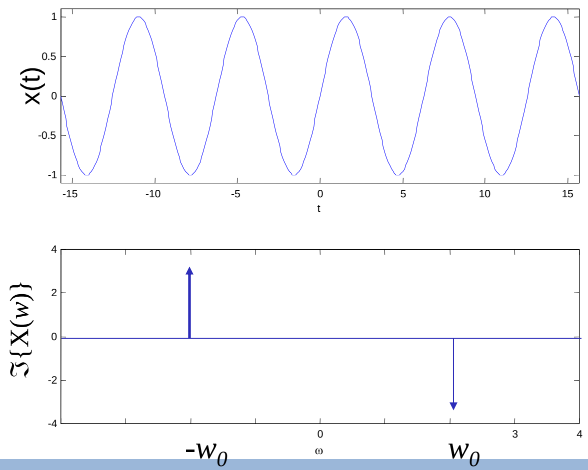
$x(t) = \cos(w_0 t)$ (real & even)



$X(w) = \pi[\delta(w-w_0) + \delta(w+w_0)]$ (real and even)

Fourier Transforms

Note: sin & cos have same Mag spectrum
Phase is only difference



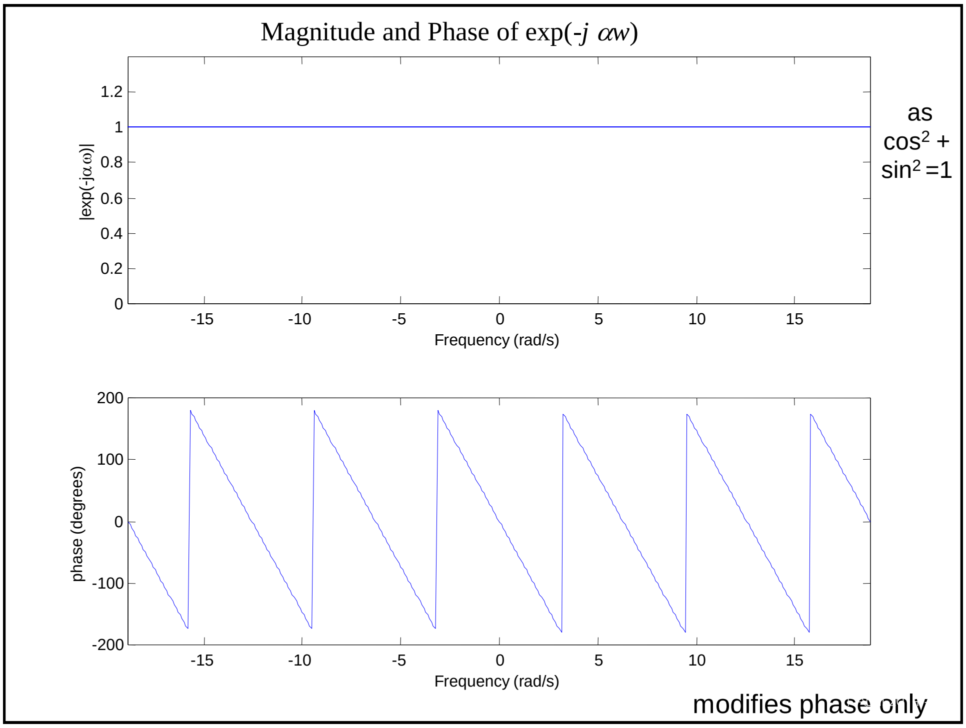
$x(t) = \sin(w_0 t)$
(real and odd)

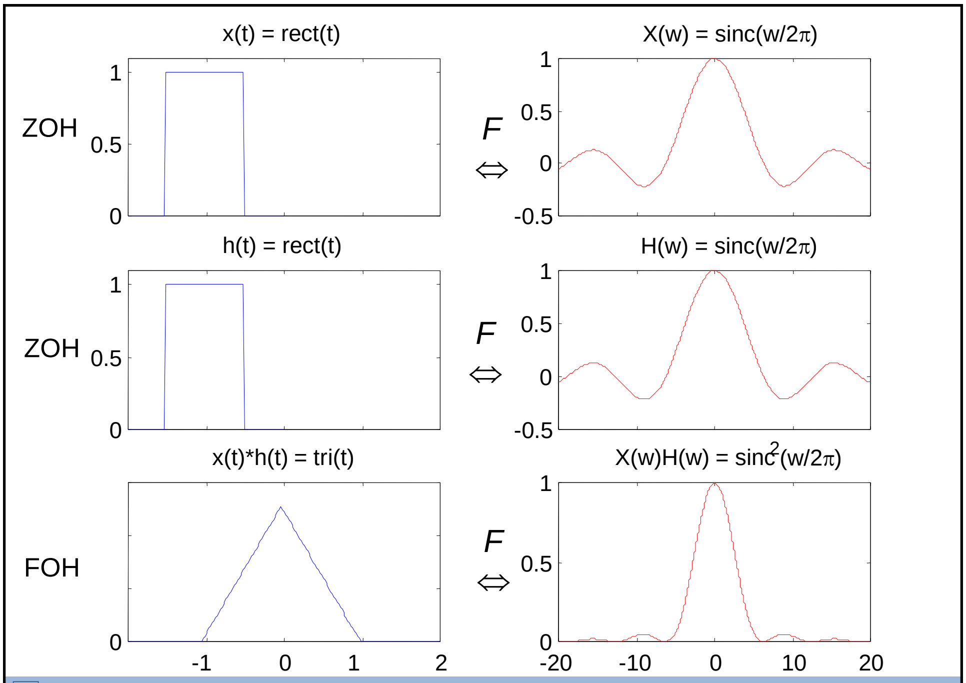
$X(w) = j\pi[\delta(w+w_0) - \delta(w-w_0)]$
(imaginary & odd)

Properties of Fourier Transform

- Time Shift
  - $F\ \{x(t - \alpha)\} = \exp(-j\ \alpha w)X(w)$
    - time shift $\Rightarrow$ phase shift

- Convolution and multiplication
  - $F\ \{x(t) * y(t)\} = X(w) \cdot Y(w)$
    - i.e., implement convolution in Fourier domain
  - $F\ \{x(t) \cdot y(t)\} = 1/2\pi\ (X(w) * Y(w))$
    - i.e., Fourier interpretation of multiplication (e.g., frequency modulation)


Magnitude and Phase of exp(-j αw)
|exp(-jαω)|
1.2
1
0.8
0.6
0.4
0.2
0
-15
-10
-5
0
5
10
15
Frequency (rad/s)
as cos² + sin² =1
200
100
0
-100
-200
phase (degrees)
-15
-10
-5
0
5
10
15
Frequency (rad/s)
modifies phase only


x(t) = rect(t)
X(w) = sinc(w/2π)
ZOH
F ⇔
h(t) = rect(t)
H(w) = sinc(w/2π)
ZOH
F ⇔
x(t)*h(t) = tri(t)
X(w)H(w) = sinc²(w/2π)
FOH
F ⇔
1
0.5
0
-0.5
-1
0
1
2
-20
-10
0
10
20

More properties of the FT

- Differentiation in time
- Integration in time

$$F\left\{\frac{d}{dt}x(t)\right\} = j\omega X(\omega)$$

$$F\left\{\frac{d^n}{dt}x(t)\right\} = (j\omega)^n X(\omega)$$

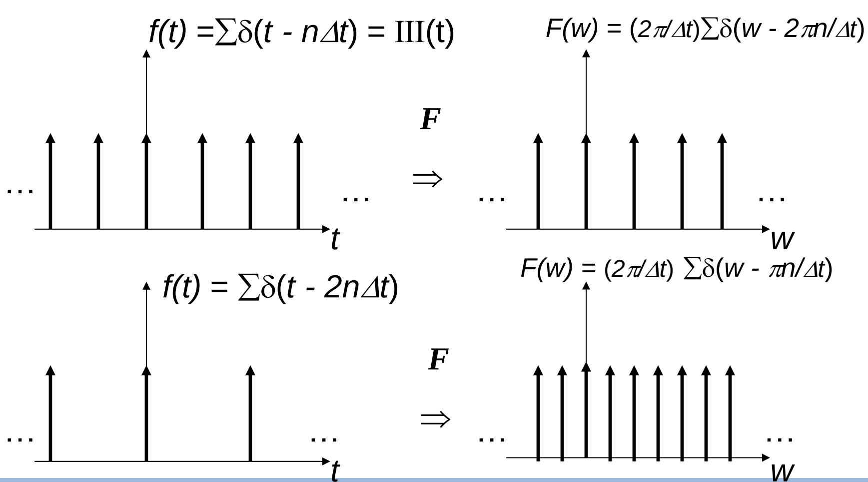
Differentiation ⇒ ×ω
(Note: HPF & DC x zero)

$$F\left\{\int_{-\infty}^{t} x(t)\,dt\right\} = \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$

Integration ⇒ /ω + DC offset (LPF & opposite of differentiation)

More Fourier Transforms

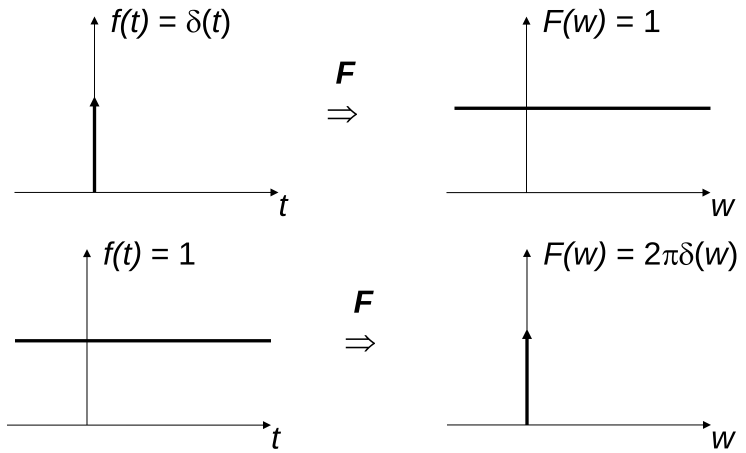
See Tutorial 2 for proof…



Impulse train, 'comb' or 'Shah' function

More Fourier Transforms

Limit of previous as $\Delta t \to \infty$ and $\Delta t \to 0$ respectively
Note: f(t) = 1 has ∞ energy! But is dual of $\delta(t)$ ☺



Note: u(t) also has ∞ energy! But F{u(t)} = F{$\int\delta(t)$} i.e., apply integration property

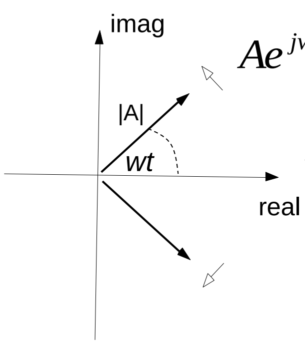
Interpretation of Fourier Transform

- Represents (usually finite energy) signals
  - as sum of cosine waves
    - at all possible frequencies
    - $|X(w)|dw/2\pi$ is amplitude of cosine wave
      - i.e., in frequency band w to w + dw
    - $\angle X(w)$ is phase shift of cosine wave
- Also represents finite power, periodic signals
  - Using $\delta(w)$
- Distribution with frequency of
  - both magnitude & phase
  - called a Frequency spectrum (continuous)

Negative Frequency

- Q: What is negative frequency?
- A: A mathematical convenience
- Trigonometrical FS
  - periodic signal is made up from
  - sum 0 to ∞ of sine and cosines ‘harmonics’
- Complex FS and the FT
  - use exp(±jwt) instead of cos(wt) and sin(wt)
  - signal is sum from 0 to ∞ of exp(±jwt)
  - same as sum -∞ to ∞ of exp(-jwt)
  - which is more compact (i.e., less chalk!)

Negative Frequency



$$Ae^{jwt} = A(\cos(wt) + j\sin(wt))$$

+ve frequency

$$\begin{aligned} Ae^{-jwt} &= A\cos(-wt) + jA\sin(-wt) \\ &= A(\cos(wt) - j\sin(wt)) \end{aligned}$$

-ve frequency

Fourier Image Examples

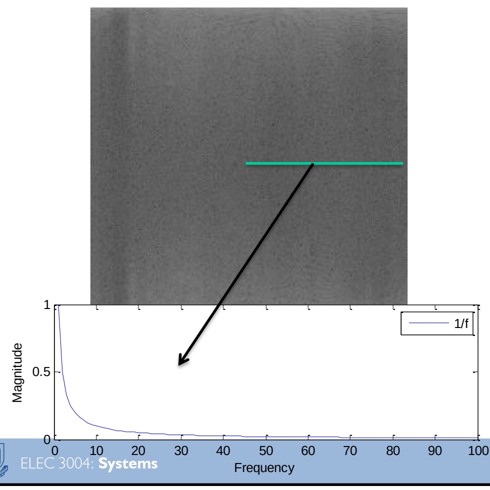
Lena

Bridge

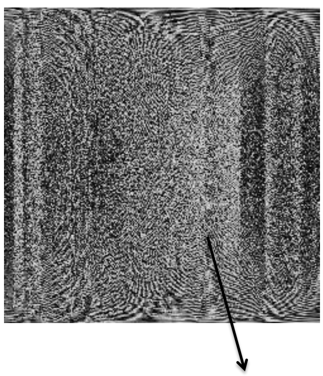
Fourier Magnitude and Phase

Bridge spectra look similar

20*log10(abs(fft(Lena)))



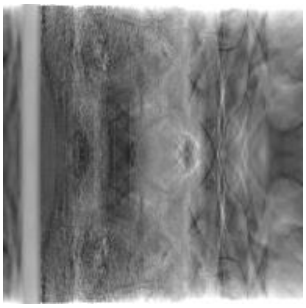
angle(fft(Lena))



'random'
range($\pm\pi$)

Magnitude and Phase Only

ifft(abs(fft(Lena)) + angle(0))



Lena magnitude only

ifft(abs(fft(Bridge)) + angle(fft(Lena)))

Lena phase + bridge magnitude

Note: titles are illustrative only and are not the actual Matlab commands used!

Questions

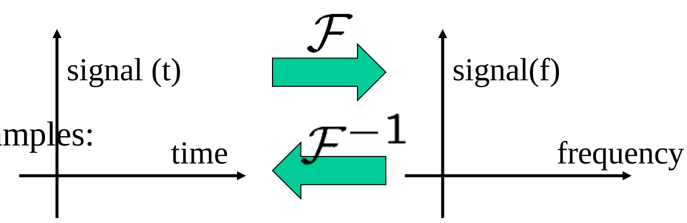
- If F{x(t)} = X(w)
  - F{x(2t)} =?
  - F{x(t/4)} =?
- F{δ(t)} = ?
- F{1} = ?

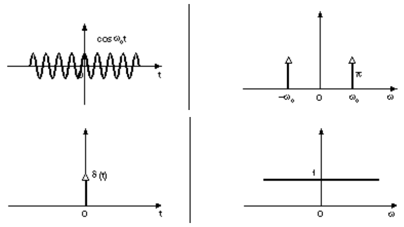
Questions

- If F{x(t)} = X(w)
  - F{x(2t)} = 1/2X(w/2)
    - narrower in t ⇒ broader in freq
  - F{x(t/4)} = 4X(4w)
    - broader in t ⇒ narrower in freq (but increased amplitude)
- F{δ(t)} = 1
  - i.e. flat spectrum (all frequencies equally)
- F{1} = δ(w)
  - i.e. impulse at DC only

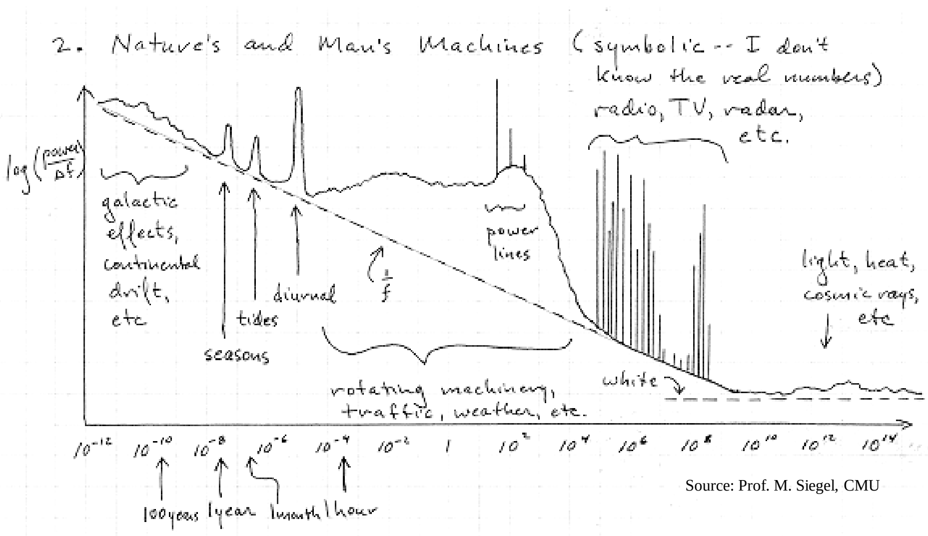
Frequency

- How often the signal repeats
- Can be analyzed through Fourier Transform
- Examples:





Noise



Source: Prof. M. Siegel, CMU

Note: this picture illustrates the concepts but it is not quantitatively precise

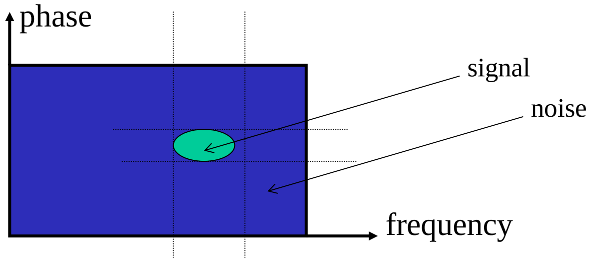
Noise [2]

Various Types:

- Thermal (white):
  - Johnson noise, from thermal energy inherent in mass.
- Flicker or 1/f noise:
  - Pink noise
  - More noise at lower frequency
- Shot noise:
  - Noise from quantum effects as current flows across a semiconductor barrier
- Avalanche noise:
  - Noise from junction at breakdown (circuit at discharge)

How to beat the noise

- Filtering (Narrow-banding): Only look at particular portion of **frequency space**
- Multiple measurements …
- Other (modulation, etc.) …



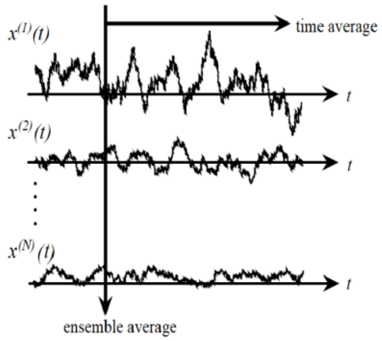
Noise ⊆ Uncertainty

- **Uncertainty**:
  All measurement has some approximation
  A. Statistical uncertainty: quantified by mean & variance
  B. Systematic uncertainty: non-random error sources

- **Law of Propagation of Uncertainty**
  – Combined uncertainty is root squared

$$u_c = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$$

Treating Uncertainty with Multiple Measurements



1. **Over time:** multiple readings of a quantity over time
   - "stationary" or "ergodic" system
   - Sometimes called "integrating"

2. **Over space:** single measurement (summed) from multiple sensors each distributed in space

3. **Same Measurand**: multiple measurements take of the **same observable quantity** by multiple, related instruments

   e.g., measure position & velocity simultaneously

→ Basic "sensor fusion"

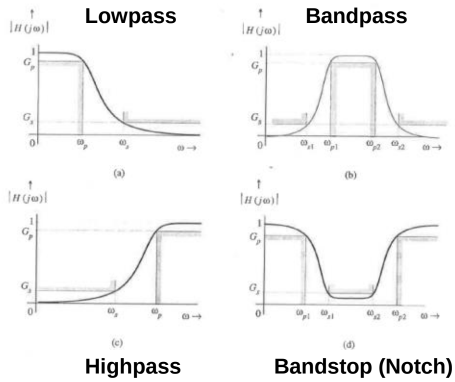
.

$$\sigma_{\text{final}} = \left[\sigma_1^{-1} + \sigma_2^{-1} + \cdots + \sigma_n^{-1}\right]^{-1}$$

Multiple Measurements Example

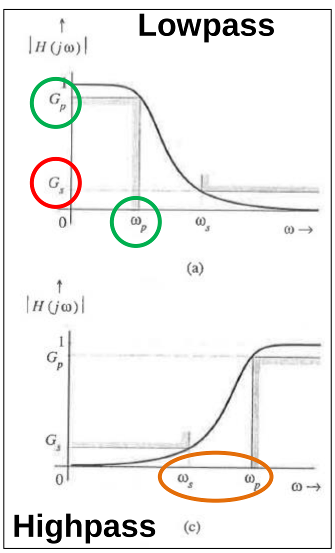
- What time was it when this picture was taken?
- What was the temperature in the room?

Filters



- *Frequency-shaping filters*: LTI systems that change the shape of the spectrum
- *Frequency-selective filters:* Systems that pass some frequencies undistorted and attenuate others

Filters



Specified Values:

- Gp = minimum passband gain

Typically:

$$G_p = \frac{1}{\sqrt{2}} = -3dB$$

- Gs = maximum stopband gain
  - **Low**, not zero (sorry!)
  - For realizable filters, the gain cannot be zero over a finite band (Paley-Wiener condition)
- Transition Band:

  transition from the passband to the stopband ➔ $\omega p \neq \omega s$

Filter Design & z-Transform

Filter Type	Mapping	Design Parameters
Low-pass	$z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$	$\alpha = \frac{\sin[(\omega_c - \omega'_c)/2]}{\sin[(\omega_c + \omega'_c)/2]}$ $\omega'_c =$ desired cutoff frequency
High-pass	$z^{-1} \rightarrow -\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}$	$\alpha = -\frac{\cos[(\omega_c + \omega'_c)/2]}{\cos[(\omega_c - \omega'_c)/2]}$ $\omega'_c =$ desired cutoff frequency
Bandpass	$z^{-1} \rightarrow -\frac{z^{-2} - [2\alpha\beta/(\beta+1)]z^{-1} + [(\beta-1)/(\beta+1)]}{[(\beta-1)/(\beta+1)]z^{-2} - [2\alpha\beta/(\beta+1)]z^{-1} + 1}$	$\alpha = \frac{\cos[(\omega_{c2} + \omega_{c1})/2]}{\cos[(\omega_{c2} - \omega_{c1})/2]}$ $\beta = \cot[(\omega_{c2} - \omega_{c1})/2] \tan(\omega_c/2)$ $\omega_{c1} =$ desired lower cutoff frequency $\omega_{c2} =$ desired upper cutoff frequency
Bandstop	$z^{-1} \rightarrow \frac{z^{-2} - [2\alpha/(\beta+1)]z^{-1} + [(1-\beta)/(1+\beta)]}{[(1-\beta)/(1+\beta)]z^{-2} - [2\alpha/(\beta+1)]z^{-1} + 1}$	$\alpha = \frac{\cos[(\omega_{c1} + \omega_{c2})/2]}{\cos[(\omega_{c1} - \omega_{c2})/2]}$ $\beta = \tan[(\omega_{c2} - \omega_{c1})/2] \tan(\omega_c/2)$ $\omega_{c1} =$ desired lower cutoff frequency $\omega_{c2} =$ desired upper cutoff frequency

Butterworth Filters

- Butterworth: Smooth in the pass-band
- The amplitude response $|H(j\omega)|$ of an n[th] order Butterworth low pass filter is given by:

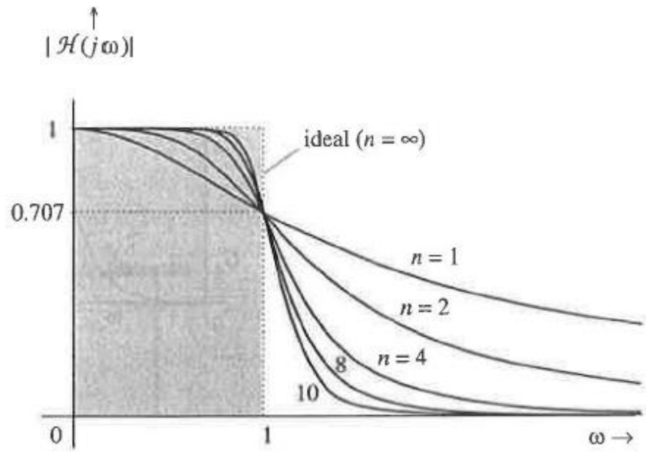
$$|H(j\omega)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}}$$

- The normalized case ($\omega_c = 1$)

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}} \quad \Rightarrow \quad \mathcal{H}(j\omega)\mathcal{H}(-j\omega) = |\mathcal{H}(j\omega)|^2 = \frac{1}{1 + \omega^{2n}}$$

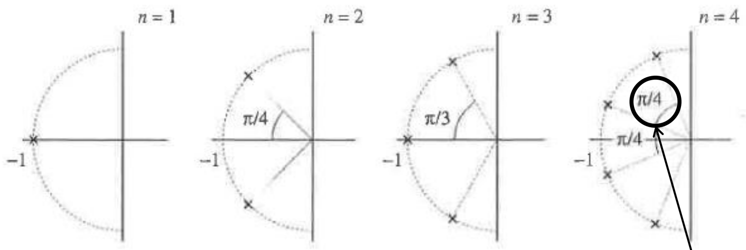
Recall that: $|H(j\omega)|^2 = H(j\omega)\, H(-j\omega)$

Butterworth Filters



Butterworth Filters of Increasing Order: Seeing this Using a Pole-Zero Diagram

- Increasing the order, increases the number of poles:



➔Odd orders (n=1,3,5…):

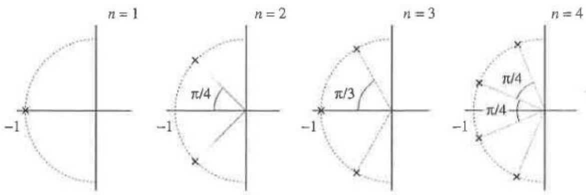
- Have a pole on the Real Axis

➔Even orders (n=2,4,6…):

- Have a pole on the off axis

Angle between poles: $\dfrac{\pi}{n}$

Butterworth Filters: Pole-Zero Diagram



- Since H(s) is stable and causal, its poles must lie in the LHP
- Poles of -H(s) are those in the RHP
- Poles lie on the unit circle (for a normalized filter)

n is the order of the filter

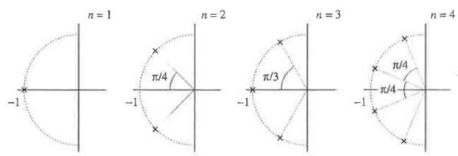
$$\rightarrow H(s) = \frac{1}{(s - s_1)(s - s_2)\ldots(s - s_n)}$$

Where:

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}$$

$$= \cos\frac{\pi}{2n}(2k+n-1) + j\sin\frac{\pi}{2n}(2k+n-1) \qquad k = 1, 2, 3, \ldots, n$$

Butterworth Filters: 4^{th} Order Filter Example



- Plugging in for n=4, k=1,…4:

$$\mathcal{H}(s) = \frac{1}{(s+0.3827-j0.9239)(s+0.3827+j0.9239)(s+0.9239-j0.3827)(s+0.9239+j0.3827)}$$

$$= \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}$$

$$= \frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}$$

- We can generalize → Butterworth Table

n	a_1	a_2	a_3	a_4	a_5
2	1.41421356				
3	2.00000000	2.00000000			
4	2.61312593	3.41421356	2.61312593		
5	3.23606798	5.23606798	5.23606798	3.23606798	
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331

This is for 3dB bandwidth at ω_c=1

Butterworth Filters: Scaling Back (from Normalized)

- Start with Normalized equation & Table
- Replace ω with $\frac{\omega}{\omega_c}$ in the filter equation

- For example:
  for f_c=100Hz → ω_c=200π rad/sec

From the Butterworth table: for n=2, $a_1=\sqrt{2}$
Thus:

$$H(s) = \frac{1}{(\frac{s}{200\pi})^2 + \sqrt{2}(\frac{s}{200\pi}) + 1}$$
$$= \frac{1}{s^2 + 200\pi\sqrt{2} + 40{,}000\pi^2}$$

Butterworth: Determination of Filter Order

- Define $\mathbf{G_x}$ as the gain of a lowpass Butterworth filter at $\omega = \omega_x$
- Then:

$$\hat{G}_x = 20\log_{10}|H(j\omega_x)| = -10\log\left[1 + \left(\frac{\omega_x}{\omega_c}\right)^{2n}\right]$$

And thus:

$$\hat{G}_p = -10\log\left[1 + \left(\frac{\omega_p}{\omega_c}\right)^{2n}\right]$$

$$\hat{G}_s = -10\log\left[1 + \left(\frac{\omega_s}{\omega_c}\right)^{2n}\right]$$

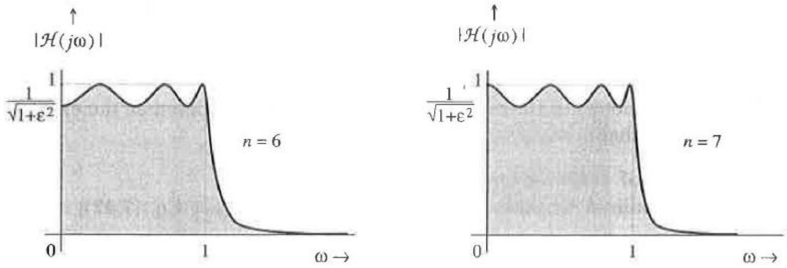
Or alternatively: $\omega_c = \frac{\omega_p}{\left[10^{-\hat{G}_p/10} - 1\right]^{1/2n}}$ & $\omega_c = \frac{\omega_s}{\left[10^{-\hat{G}_s/10} - 1\right]^{1/2n}}$

Solving for n gives:

$$n = \frac{\log\left[\left(10^{-\hat{G}_s/10} - 1\right) / \left(10^{-\hat{G}_p/10} - 1\right)\right]}{2\log(\omega_s/\omega_p)}$$

PS. See Lathi 4.10 (p. 453) for an example in MATLAB

Chebyshev Filters



- **equal-ripple:**
  Because all the ripples in the passband are of equal height
- If we reduce the ripple, the passband behaviour improves, but it does so at the cost of stopband behaviour

Chebyshev Filters

- Chebyshev Filters: Provide tighter transition bands (sharper cutoff) than the same-order Butterworth filter, but this is achieved at the expense of inferior passband behavior (rippling)

➔ For the lowpass (LP) case: at higher frequencies (in the stopband), the Chebyshev filter gain is smaller than the comparable Butterworth filter gain by about ***6(n* - 1) dB**

- The amplitude response of a normalized Chebyshev lowpass filter is:

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1 + \epsilon^2 {C_n}^2(\omega)}}$$

Where Cn(ω), the nth-order Chebyshev polynomial, is given by:

$$C_n(\omega) = \cos\left(n \cos^{-1} \omega\right)$$
$$C_n(\omega) = \cosh\left(n \cosh^{-1} \omega\right)$$

and where C_n is given by:

n	$C_n(\omega)$
0	1
1	ω
2	$2\omega^2 - 1$
3	$4\omega^3 - 3\omega$
4	$8\omega^4 - 8\omega^2 + 1$
5	$16\omega^5 - 20\omega^3 + 5\omega$
6	$32\omega^6 - 48\omega^4 + 18\omega^2 - 1$

Normalized Chebyshev Properties

- It's normalized: The passband is 0<ω<1
- **Amplitude response**: has **ripples** in the passband and is **smooth** (monotonic) in the stopband
- **Number of ripples**: there is a total of ***n*** maxima and minima over the passband 0<ω<1
- $C_n^2(0) = \begin{cases} 0, & n: odd \\ 1, & n: even \end{cases}$ ⇨ $|H(0)| = \begin{cases} 1, & n: odd \\ \frac{1}{\sqrt{1+\epsilon^2}}, & n: even \end{cases}$
- ϵ: ripple height → $r = \sqrt{1+\epsilon^2}$
- The Amplitude at ω=1: $\frac{1}{r} = \frac{1}{\sqrt{1+\epsilon^2}}$
- For Chebyshev filters, the ripple ***r*** dB takes the place of G_p

Determination of Filter Order

- The gain is given by: $\hat{G} = -10\log\left[1+\epsilon^2 {C_n}^2(\omega)\right]$

Thus, the gain at ω_s is: $\epsilon^2 {C_n}^2(\omega_s) = 10^{-\hat{G}_s/10} - 1$

- Solving:

$$n = \frac{1}{\cosh^{-1}(\omega_s)} \cosh^{-1}\left[\frac{10^{-\hat{G}_s/10} - 1}{10^{\hat{r}/10} - 1}\right]^{1/2}$$

- General Case:

$$n = \frac{1}{\cosh^{-1}(\omega_s/\omega_p)} \cosh^{-1}\left[\frac{10^{-\hat{G}_s/10} - 1}{10^{\hat{r}/10} - 1}\right]^{1/2}$$

Chebyshev Pole Zero Diagram

- Whereas Butterworth poles lie on a semi-circle,
  The poles of an nth-order normalized Chebyshev filter lie on a **semiellipse** of the major and minor semiaxes:

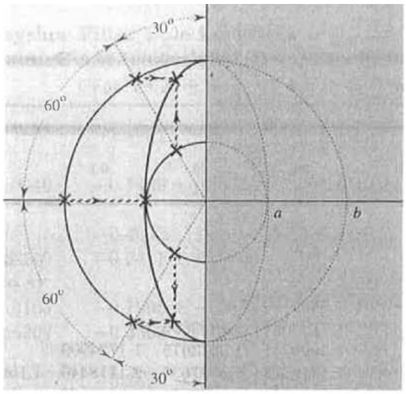
$$a = \sinh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right) \quad \& \quad b = \cosh\left(\frac{1}{n}\sinh^{-1}\left(\frac{1}{\epsilon}\right)\right)$$

And the poles are at the locations:

$$H(s) = \frac{1}{(s-s_1)(s-s_2)\dots(s-s_n)}$$

$$s_k = -\sin\left[\frac{(2k-1)\pi}{2n}\right]\sinh x + j\cos\left[\frac{(2k-1)\pi}{2n}\right]\cosh x, \; k = 1, \dots, n$$

Ex: Chebyshev Pole Zero Diagram for **n=3**



Procedure:

1. Draw two semicircles of radii **a** and **b** (from the previous slide).
2. Draw radial lines along the corresponding Butterworth angles (π/n) and locate the nth-order Butterworth poles (shown by crosses) on the two circles.
3. The location of the kth Chebyshev pole is the intersection of the horizontal projection and the vertical projection from the corresponding kth Butterworth poles on the outer and the inner circle, respectively.

Chebyshev Values / Table

$$\mathcal{H}(s) = \frac{K_n}{C'_n(s)} = \frac{K_n}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$K_n = \begin{cases} a_0 & n \text{ odd} \\ \frac{a_0}{\sqrt{1+\epsilon^2}} = \frac{a_0}{10^{\hat{r}/20}} & n \text{ even} \end{cases}$$

n	a_0	a_1	a_2	a_3
1	1.9652267			
2	1.1025103	1.0977343		
3	0.4913067	1.2384092	0.9883412	
4	0.2756276	0.7426194	1.4539248	0.9528114

1 db ripple
($\hat{r} = 1$)

Other Filter Types:
Chebyshev Type II = Inverse Chebyshev Filters

- Chebyshev filters passband has ripples and the stopband is smooth.
- **Instead:** this has **passband** have **smooth** response and **ripples** in the stopband.

➔ Exhibits maximally flat passband response and equi-ripple stopband

➔ `Cheby2` **in MATLAB**

$$|\mathcal{H}(\omega)|^2 = 1 - |\mathcal{H}_C(1/\omega)|^2 = \frac{\epsilon^2 C_n^2(1/\omega)}{1+\epsilon^2 C_n^2(1/\omega)}$$

Where: H_c is the Chebyshev filter system from before

- Passband behavior, especially for small ω, is **better** than Chebyshev
- **Smallest transition band** of the 3 filters (Butter, Cheby, Cheby2)
- Less time-delay (or phase loss) than that of the **Chebyshev**
- Both needs the **same order *n*** to meet a set of specifications.
- **$$$** (or number of elements):
  Cheby < Inverse Chebyshev < Butterworth (of the same **performance** [not order])

Other Filter Types:
Elliptic Filters (or Cauer) Filters

- Allow **ripple** in **both** the passband and the stopband,
  ➔ we can achieve **tighter** transition band

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1+\epsilon^2 {R_n}^2(\omega)}}$$

Where: $\mathbf{R_n}$ is the n^{th}-order Chebyshev rational function determined from a given ripple spec.
ϵ controls the ripple
$Gp = \frac{1}{\sqrt{1+\epsilon^2}}$

- Most efficient (η)
  - the **largest ratio** of the passband gain to stopband gain
  - **or** for a given ratio of passband to stopband gain, it requires the **smallest transition band**

➔ **in MATLAB:** **`ellipord`** followed by **`ellip`**

In Summary

Filter Type	Passband Ripple	Stopband Ripple	Transition Band	MATLAB Design Command
Butterworth	No	No	Loose	`butter`
Chebyshev	Yes	No	Tight	`cheby`
Chebyshev Type II (Inverse Chebyshev)	No	Yes	Tight	`cheby2`
Eliptic	Yes	Yes	Tightest	`ellip`