System Analysis & Convolution

ELEC 3004: **Digital Linear Systems**: Signals & Controls

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(with material from Kumaresan, Continuous-Time Fourier Transform, URI)

Lecture 4

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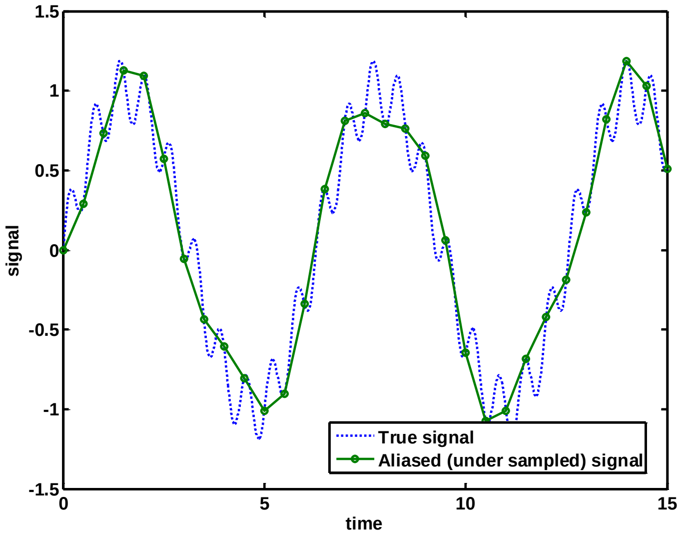
Lecture Schedule:

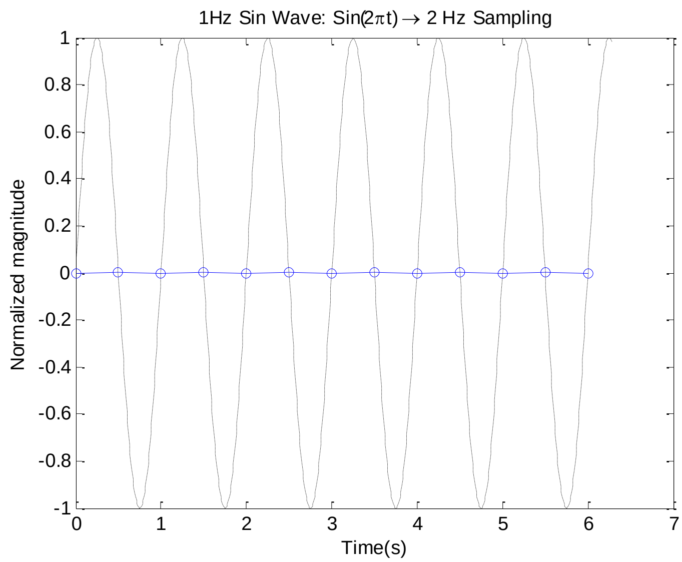
Week	Date	Lecture Title
1	4-Mar	Introduction & Systems Overview
	6-Mar	[Linear Dynamical Systems]
2	11-Mar	Signals as Vectors & Systems as Maps
	13-Mar	[Signals]
3	18-Mar	Sampling & Data Acquisition & Antialiasing Filters
	20-Mar	[Sampling]
4	25-Mar	System Analysis & Convolution
	27-Mar	[Convolution & FT]
5	1-Apr	Frequency Response & Filter Analysis
	3-Apr	[Filters]
6	8-Apr	Discrete Systems & Z-Transforms
	10-Apr	[Z-Transforms]
7	15-Apr	Introduction to Control
	17-Apr	[Feedback]
8	29-Apr	Digital Filters
	1-May	[Digital Filters]
9	6-May	Introduction to Digital Control
	8-May	[Digitial Control]
10	13-May	Stability of Digital Systems
	15-May	[Stability]
11	20-May	State-Space
	22-May	Controllability & Observability
12	27-May	PID Control & System Identification
	29-May	Digitial Control System Hardware
13	3-Jun	Applications in Industry & Information Theory & Communications
	5-Jun	Summary and Course Review

Sampling Recap

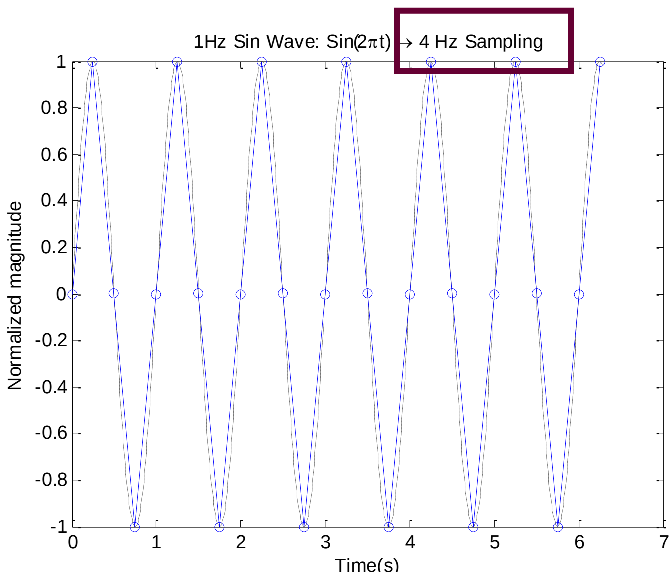
Sampling < Nyquist → Aliasing

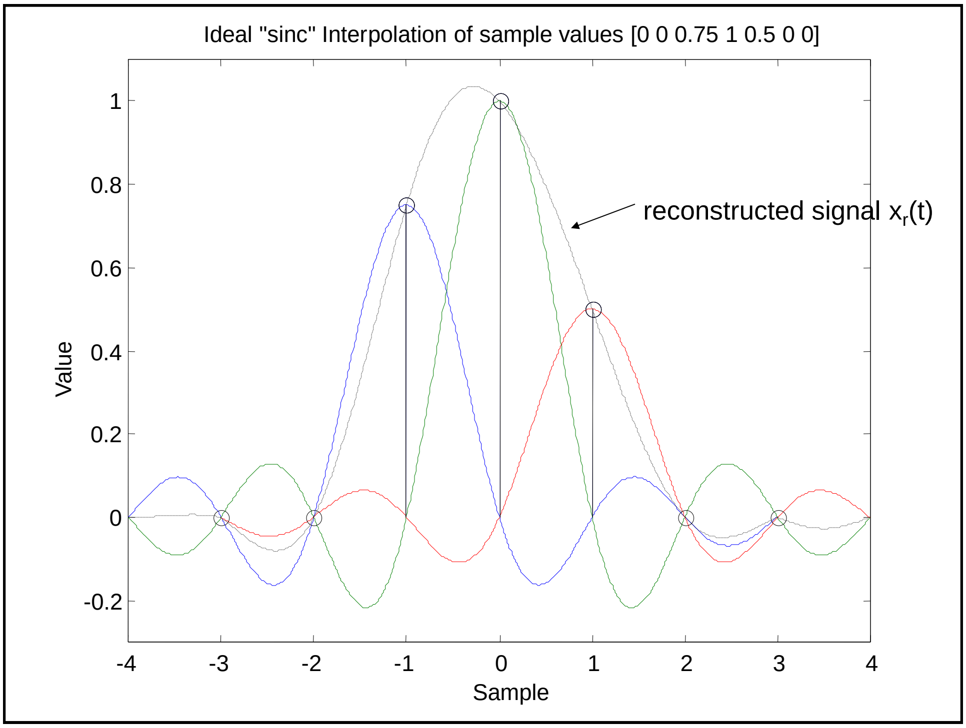


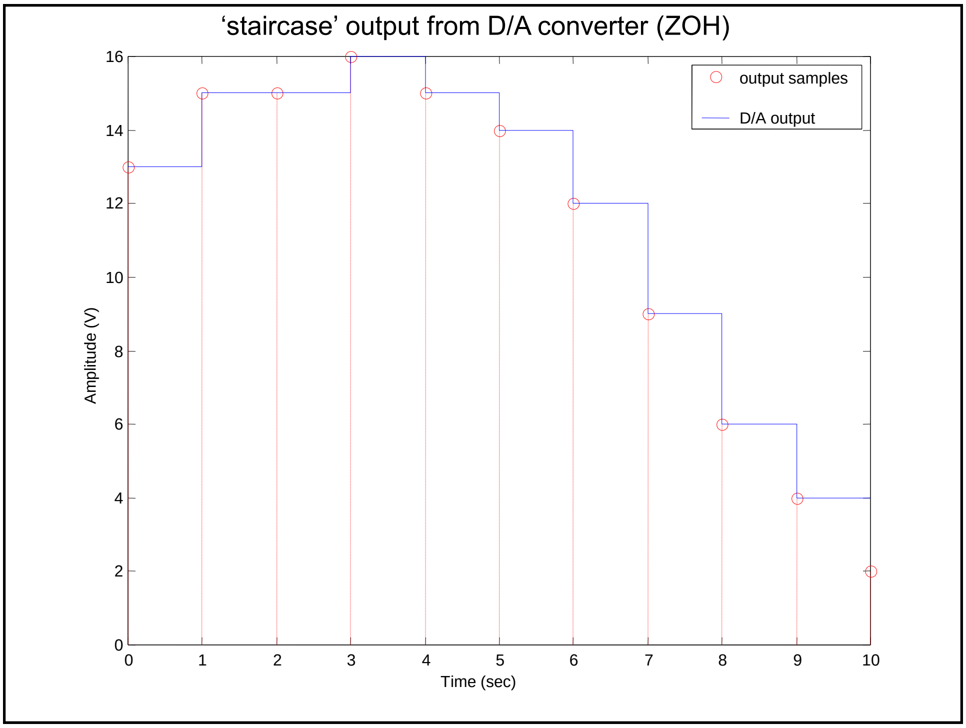
Nyquist is not enough …



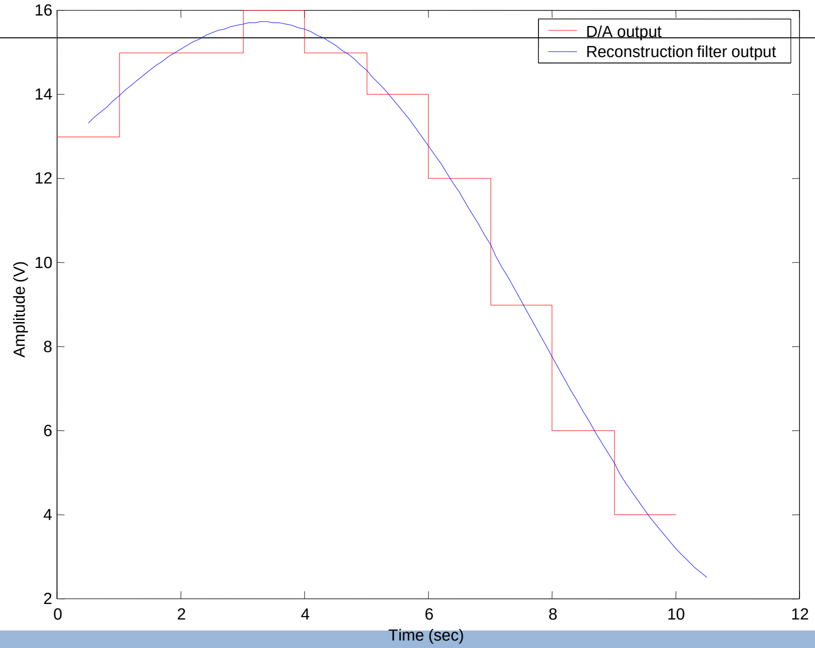
A little more than Nyquist is not enough …



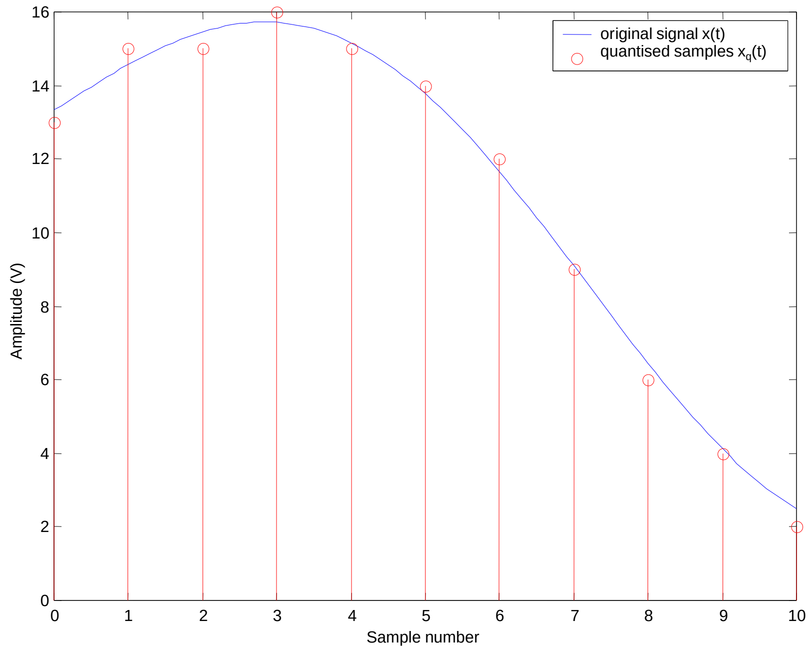

Ideal "sinc" Interpolation of sample values [0 0 0.75 1 0.5 0 0]
reconstructed signal $x_r(t)$
Value
Sample
1
0.8
0.6
0.4
0.2
0
-0.2
-4
-3
-2
-1
0
1
2
3
4


'staircase' output from D/A converter (ZOH)
output samples
D/A output
Amplitude (V)
Time (sec)
16
14
12
10
8
6
4
2
0
0
1
2
3
4
5
6
7
8
9
10

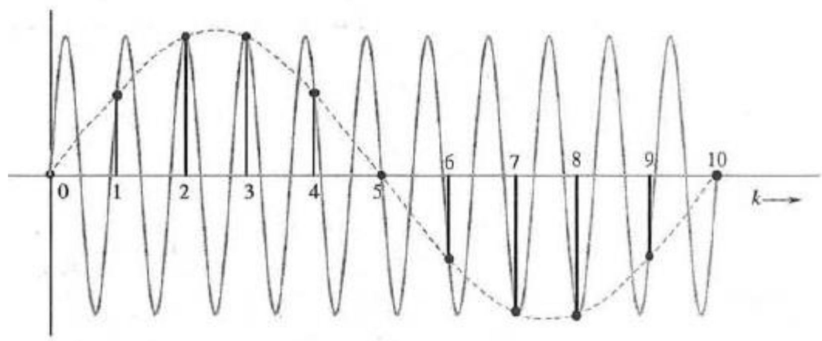
Smooth output from reconstruction filter



Example: error due to signal quantisation



Aliasing: Another view of this



Alliasing

- Aliasing - through sampling, two entirely different analog sinusoids take on the same "discrete time" identity

For f[k]=cosΩk, Ω=ωT:

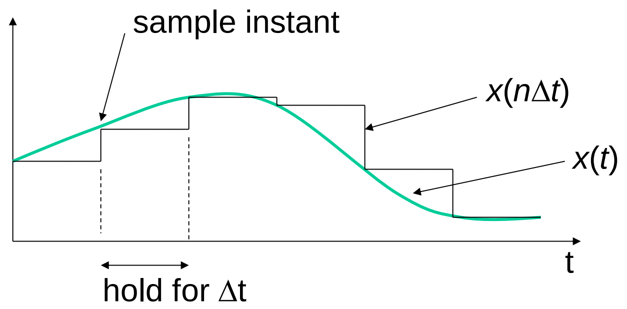
The period has to be less than Fh (highest frequency): $T \leq \frac{1}{2\mathcal{F}_h}$

Thus: $0 \leq \mathcal{F} \leq \frac{\mathcal{F}_s}{2}$

ω_f: aliased frequency: $\omega T = \omega_f T + 2\pi m$

Practical Sampling

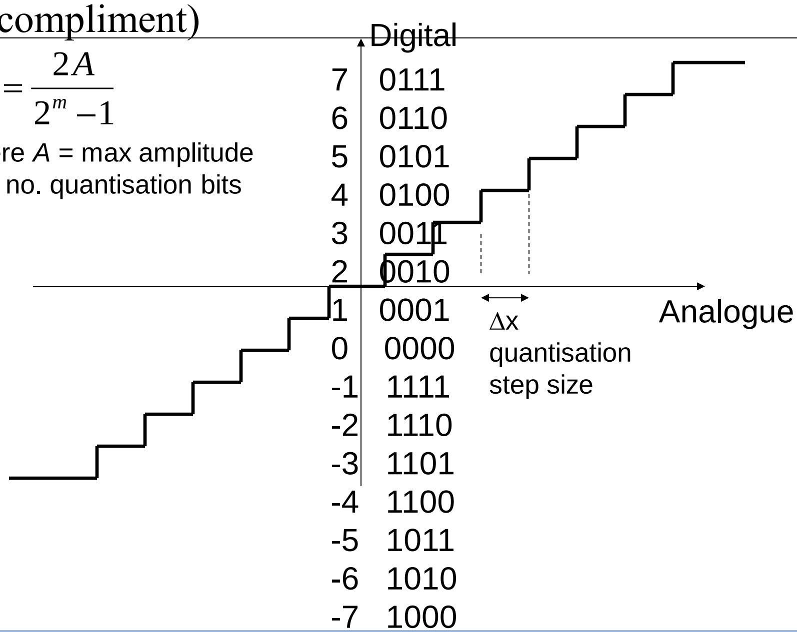
- Sample and Hold (S/H)
  1. takes a sample every Δt seconds
  2. holds that value constant until next sample
- Produces 'staircase' waveform, $x(n\Delta t)$



Input-output for 4-bit quantiser (two's compliment)

$$\Delta x = \frac{2A}{2^m - 1}$$

where *A* = max amplitude
m = no. quantisation bits



Quantisation

- Analogue to digital converter (A/D)
  - Calculates nearest binary number to $x(n\Delta t)$
    - $x_q[n] = q(x(n\Delta t))$, where q() is non-linear rounding fctn
  - output modeled as $x_q[n] = x(n\Delta t) + e[n]$
- Approximation process
  - therefore, loss of information (unrecoverable)
  - known as 'quantisation noise' ($e[n]$)
  - error reduced as number of bits in A/D increased
    - i.e., Δx, quantisation step size reduces

$$|e[n]| \leq \frac{\Delta x}{2}$$

Practical Anti-aliasing Filter

- Non-ideal filter
  - wc = ws /2
- Filter usually 4th – 6th order (e.g., Butterworth)
  - so frequencies > wc may still be present
  - not higher order as phase response gets worse
- Luckily, most real signals
  - are lowpass in nature
    - signal power reduces with increasing frequency
  - e.g., speech naturally bandlimited (say < 8KHz)
  - Natural signals have a (approx) 1/f spectrum
  - so, in practice aliasing is not (usually) a problem

Practical Reconstruction

Two stage process:

1. Digital to analogue converter (D/A)
   - zero order hold filter
   - produces 'staircase' analogue output
2. Reconstruction filter
   - non-ideal filter: $w_c = w_s/2$
   - further reduces replica spectrums
   - usually 4^{th} – 6^{th} order e.g., Butterworth
     - for acceptable phase response

System Analysis

[Chapter 2, Lathi]

Linear Differential Systems

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y(t) = b_m\frac{d^m f}{dt^m} + b_{m-1}\frac{d^{m-1}f}{dt^{m-1}} + \cdots + b_1\frac{df}{dt} + b_0 f(t) \qquad (2.1a)$$

where all the coefficients a_i and b_i are constants. Using operational notation D to represent d/dt, we can express this equation as

$$(D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0)\, y(t) = (b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0)\, f(t) \qquad (2.1b)$$

or

$$Q(D)y(t) = P(D)f(t) \qquad (2.1c)$$

where the polynomials $Q(D)$ and $P(D)$ are

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0 \qquad (2.2a)$$

$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0 \qquad (2.2b)$$

Linear Differential System Order

$$Q(D)y(t) = P(D)f(t)$$

$$Q(D) = D^n + a_{n-1}D^{n-1} + \cdots + a_1 D + a_0$$

$$P(D) = b_m D^m + b_{m-1}D^{m-1} + \cdots + b_1 D + b_0$$

→ *y(t)=P(D)/Q(D) f(t)*
P(D): M
Q(D): N
(yes, N is de**N**ominator)

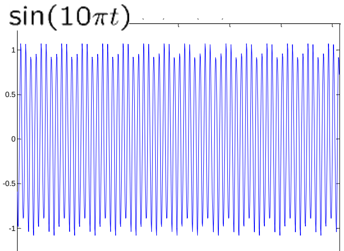
- In practice: m ≤ n

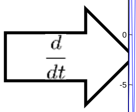
∵ **if** m > *n:*
then the system is an
(m - n)th -order differentiator of high-frequency signals!

- Derivatives magnify noise!

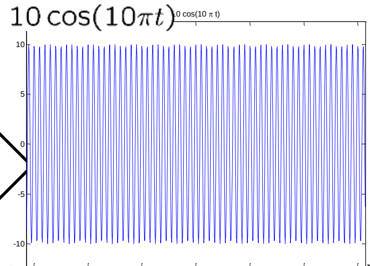
Derivatives magnify noise!

- $\sin(10\pi t)$

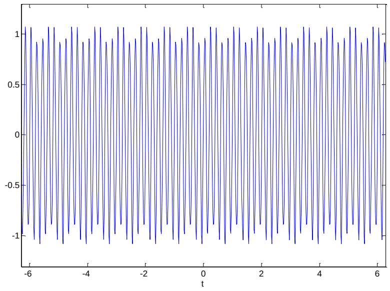




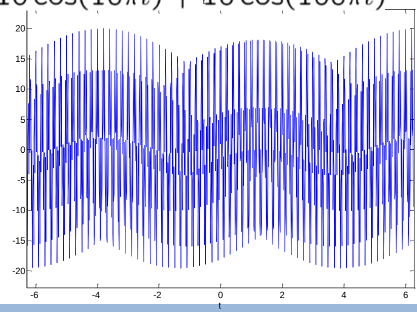
- $10\cos(10\pi t)$



- $\sin(10\pi t) + 0.1\sin(100\pi t)$



- $10\cos(10\pi t) + 10\cos(100\pi t)$



Zero-Input | Zero-State

Total response = zero-input response + zero-state response

Zero Input

- = The system response when the input *f(t)* = 0 so that it is the result of internal system conditions (such as energy storages, initial conditions) alone.
- It is **independent of the external input**.

Zero-State

- = the system response to the external input *f (t)* when the system is in zero state, meaning the absence of all internal energy storages; that is, all initial conditions are zero.

System Stability

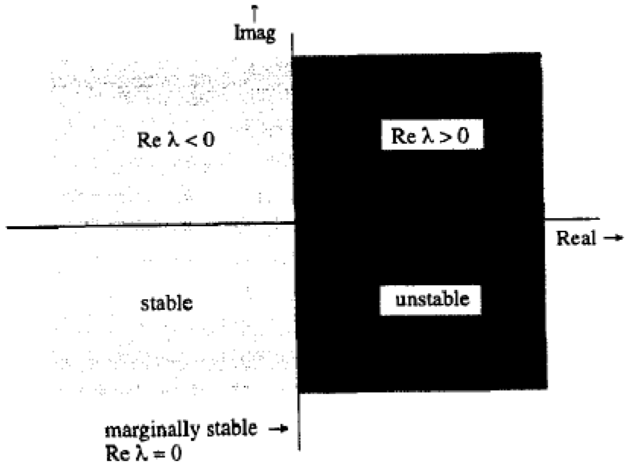
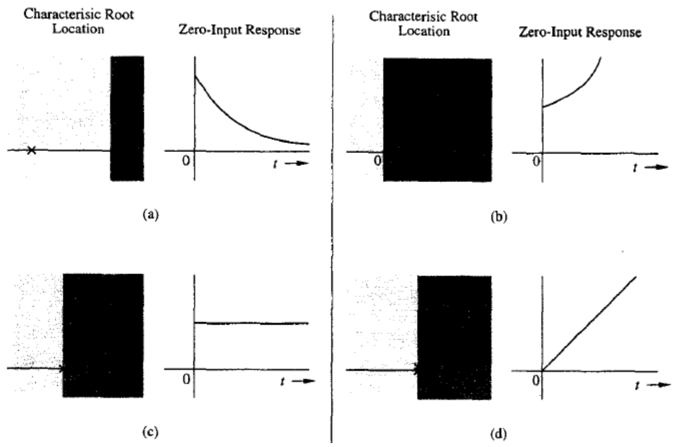


Fig. 2.15 Characteristic roots location and system stability.

Lathi, p. 149

System Stability [II]



Lathi, p. 150

System Stability [III]

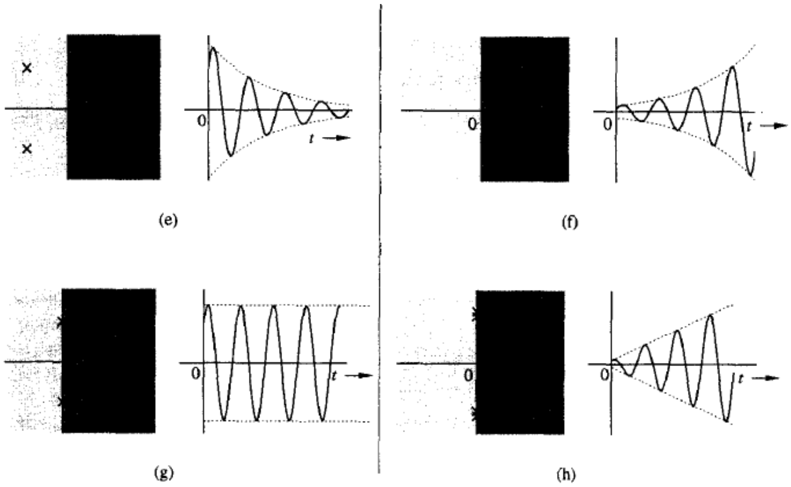


Fig. 2.16 Location of characteristic roots and the corresponding characteristic modes.

Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a(\underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathcal{L}(y'')}) + b(\underbrace{sY(s) - y(0)}_{\mathcal{L}(y')}) + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

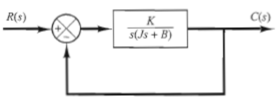
Second Order Systems

so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

Second Order Response



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

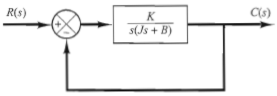
Three Types:

- I: Underdamped: $(0 < \zeta < 1)$:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}.$$

$$\begin{aligned}\mathcal{L}^{-1}[C(s)] &= c(t)\\ &= 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right)\\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right),\end{aligned}$$

Second Order Response



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

Three Types:

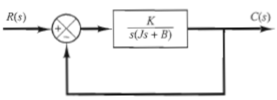
- II: Critically Damped: $(\zeta = 1)$

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

$$\lim_{\zeta \to 1} \frac{\sin \omega_d t}{\sqrt{1 - \zeta^2}} = \lim_{\zeta \to 1} \frac{\sin \omega_n \sqrt{1 - \zeta^2}\, t}{\sqrt{1 - \zeta^2}} = \omega_n t$$

Second Order Response



$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

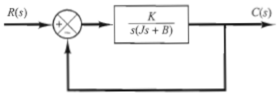
Three Types:

- III: Over Damped: $(\zeta > 1)$

For a unit-step input, $R(s) = 1/s$ and $C(s)$ can be written

$$C(s) = \frac{\omega_n^2}{\left(s + \zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\left(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)s}$$

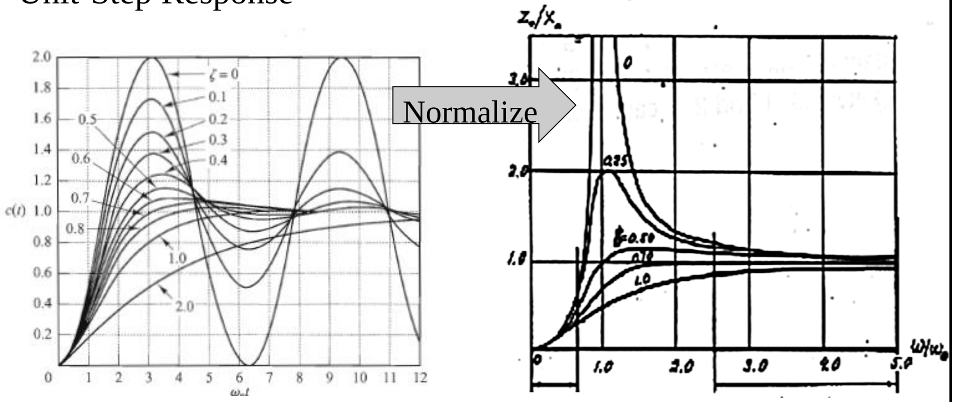
Second Order Response



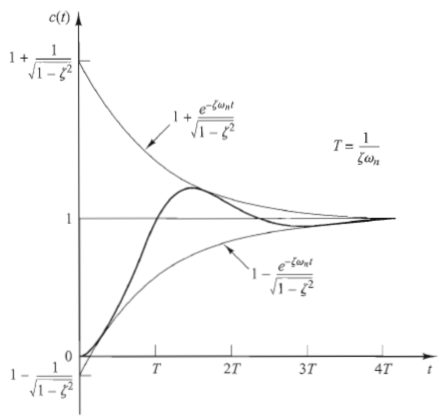
$$\frac{C(s)}{R(s)} = \frac{\frac{K}{J}}{\left[s + \frac{B}{2J} + \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]\left[s + \frac{B}{2J} - \sqrt{\left(\frac{B}{2J}\right)^2 - \frac{K}{J}}\right]}$$

$$\frac{K}{J} = \omega_n^2, \qquad \zeta = \frac{B}{B_c} = \frac{B}{2\sqrt{JK}}$$

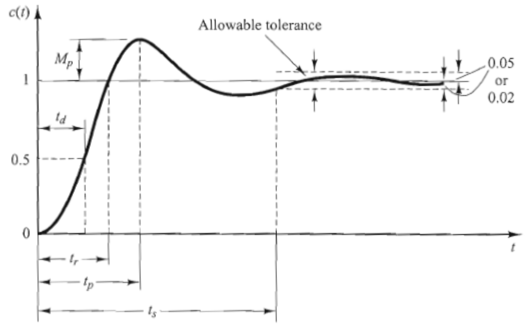
Unit-Step Response



Second Order Response
Envelope Curves



Second Order Response
Unit Step Response Terms



- Delay time, t_d: The time required for the response to reach half the final value
- Rise time, t_r: The time required for the response to rise from 10% to 90%
- Peak time, t_p:The time required for the response to reach the first peak of the overshoot
- Maximum (percent) overshoot, Mp:

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

- Settling time, t_s: The time to be within 2-5% of the final value

Second Order Response
Seeing this on the S-plane

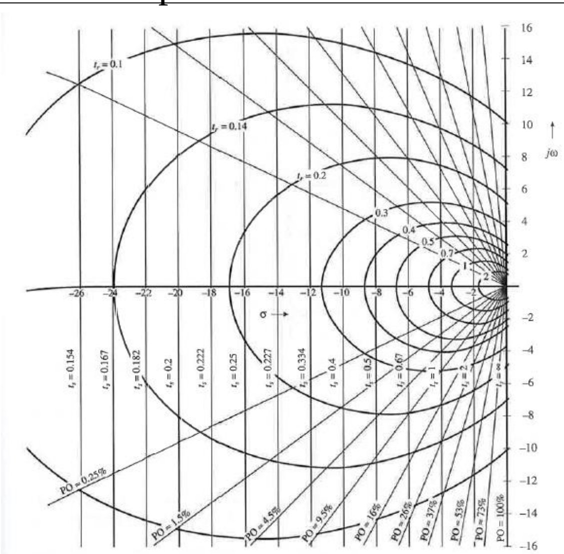
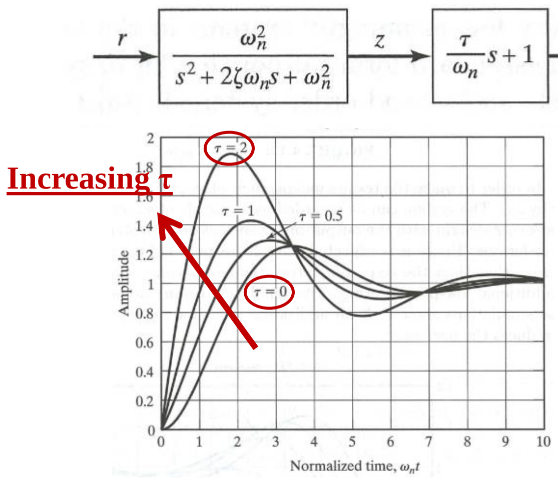


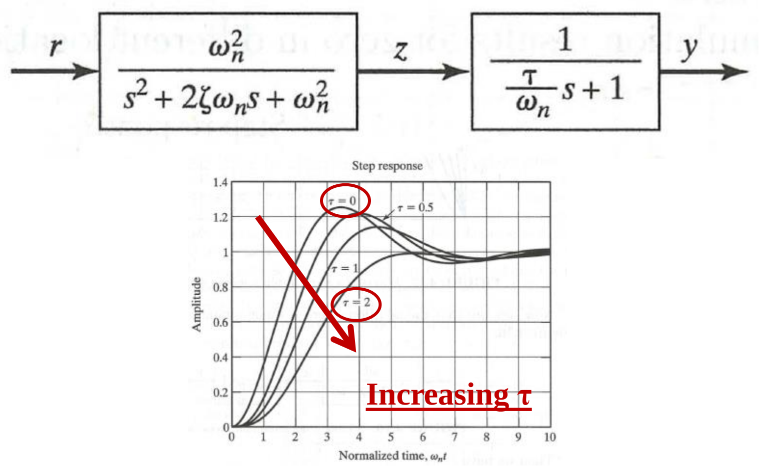
Fig. 6.40 Contours of second-order system pole location for constant PO, constant t_s, and constant t_r in s plane.

Second Order Response
The Case of **Adding a Zero**



- The addition of a zero (a **s** term) gives a system with a shorter rise time, a shorter peak time, and a larger overshoot

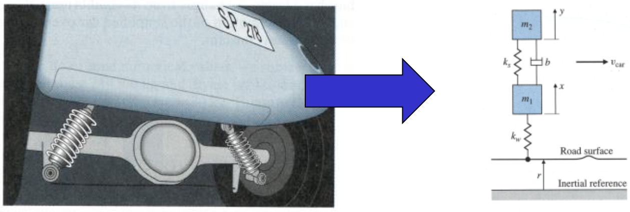
Second Order Response
The Case of **Adding a Zero**

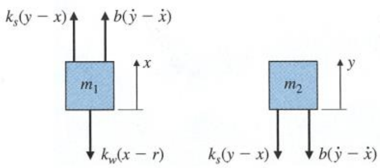


- The addition of a pole (a **1/s** term) **slows down** the system response and reduces the overshoot.

Adapted from Qui,
IFC 2010 – pp. 154-5

Example: Quarter-Car Model





Example: Quarter-Car Model (2)

$$\ddot{x} + \frac{b}{m_1}(\dot{x} - \dot{y}) + \frac{k_s}{m_1}(x - y) + \frac{k_w}{m_1}x = \frac{k_w}{m_1}r,$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y} - \dot{x}) + \frac{k_s}{m_2}(y - x) = 0.$$

$$s^2X(s) + s\frac{b}{m_1}(X(s) - Y(s)) + \frac{k_s}{m_1}(X(s) - Y(s)) + \frac{k_w}{m_1}X(s) = \frac{k_w}{m_1}R(s),$$

$$s^2Y(s) + s\frac{b}{m_2}(Y(s) - X(s)) + \frac{k_s}{m_2}(Y(s) - X(s)) = 0,$$

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}\left(s + \frac{k_s}{b}\right)}{s^4 + \left(\frac{b}{m_1} + \frac{b}{m_2}\right)s^3 + \left(\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1}\right)s^2 + \left(\frac{k_w b}{m_1 m_2}\right)s + \frac{k_w k_s}{m_1 m_2}}.$$

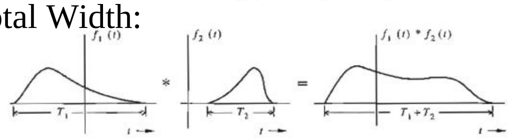
Convolution

Convolution & Properties

$$f_1(t) * f_2(t) \equiv \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau$$

Properties:

- Commutative: $f_1(t) * f_2(t) = f_2(t) * f_1(t)$
- Distributive: $f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t)$
- Associative: $f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t)$
- Shift:
  if $f_1(t)*f_2(t)=c(t)$, then $f_1(t-\mathbf{T})*f_2(t)= f_1(t)*f_2(t-\mathbf{T})=c(t-\mathbf{T})$
- Identity (Convolution with an Impulse):
  $f(t) * \delta(t) = f(t)$
- Total Width:



Based on Lathi, *SPLS*, Sec 2.4-1

Convolution & Properties [II]

- Convolution systems are **linear**:

$$h * (\alpha u_1 + \beta u_2) = \alpha(h * u_1) + \beta(h * u_2)$$

- Convolution systems are **causal**: the output y(t) at time t depends only on past inputs

- Convolution systems are **time-invariant** (if we shift the signal, the output similarly shifts)

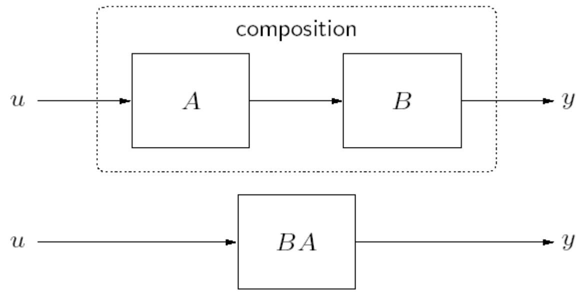
➔

$$\tilde{u}(t) = \begin{cases} 0 & t < T \\ u(t-T) & t \geq 0 \end{cases}$$

$$\tilde{y}(t) = \begin{cases} 0 & t < T \\ y(t-T) & t \geq 0 \end{cases}$$

Convolution & Properties [III]

- Composition of convolution systems corresponds to:
  - multiplication of transfer functions
  - convolution of impulse responses



- Thus:
  - We can manipulate block diagrams with transfer functions as if they were simple gains
  - convolution systems commute with each other

Convolution & Systems

- Convolution system with input u (u(t) = 0, t < 0) and output y:

$$y(t) = \int_0^t h(\tau)u(t-\tau)\,d\tau = \int_0^t h(t-\tau)u(\tau)\,d\tau$$

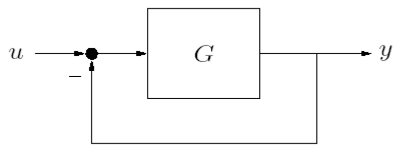
- abbreviated:

$$y = h * u$$

- in the frequency domain:

$$Y(s) = H(s)U(s)$$

Convolution & Feedback



- In the time domain:

$$y(t) = \int_0^t g(t-\tau)(u(\tau) - y(\tau))\,d\tau$$

- In the frequency domain:
  - Y=G(U-Y)

  ➔Y(s) = H(s)U(s)

$$H(s) = \frac{G(s)}{1+G(s)}$$

Graphical Understanding of Convolution

➔ For c(τ)= $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$:

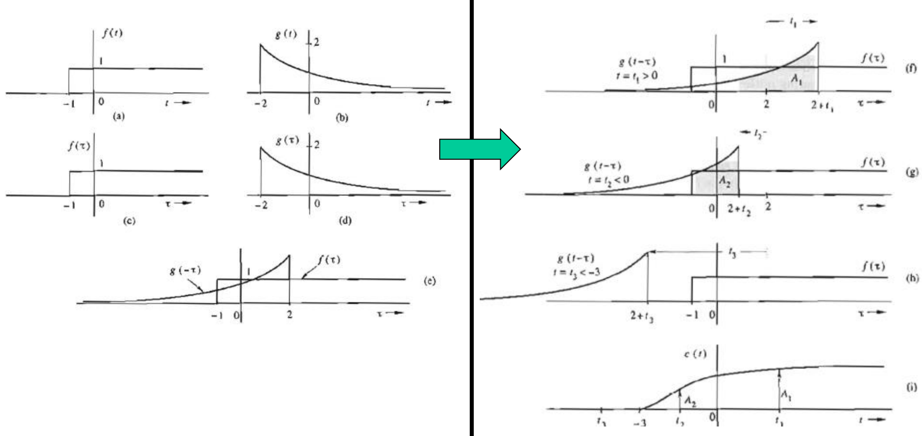
1. Keep the function ***f*** **(τ)** fixed
2. **Flip** (invert) the function ***g*(*τ*)** about the vertical axis (τ=0)
   = this is ***g*(-*τ*)**
3. **Shift** this frame (***g*(-*τ*)**) along τ (horizontal axis) by $\mathbf{t_0}$.
   = this is $\boldsymbol{g(t_0 - \tau)}$

➔ For c($\mathbf{t_0}$):

4. $\mathbf{c(t_0)}$ = the area under the product of $\boldsymbol{f(\tau)}$ **and** $\boldsymbol{g(t_0 - \tau)}$

5. Repeat this procedure, shifting the frame by different values (positive and negative) to obtain **c(t) for all values of t**.

Graphical Understanding of Convolution (Ex)



Another View

e.g. convolution

x(n) = 1 2 3 4 5
h(n) = 3 2 1

x(k)	0 0 1 | 2 3 4 5	0 0 1 2 | 3 4 5	0 0 1 2 3 | 4 5	
h(n,k)	1 2 3 | 0 0 0 0	0 1 2 3 | 0 0 0	0 0 1 2 3 | 0 0	h(n-k)
y(n,k)	3	2 6	1 4 9	
y(n)	3	8	14	

Sum over all k

Notice the gain

Matrix Formulation of Convolution

$$\mathbf{y} = \mathbf{H}\mathbf{x}$$

$$\begin{bmatrix} 3 \\ 8 \\ 14 \\ 20 \\ 26 \\ 14 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{bmatrix} . \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

Toeplitz Matrix

Convolution Definition

The convolution of two functions $f_1(t)$ and $f_2(t)$ is defined as:

$$f(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$= f_1(t) * f_2(t)$$

Source: URI ELE436

Properties of Convolution

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau = \int_{\tau=-\infty}^{\tau=\infty} f_1(\tau) f_2(t-\tau) d\tau$$

$$= \int_{t-\tau=-\infty}^{t-\tau=\infty} f_1(t-\tau) f_2[t-(t-\tau)] d(t-\tau)$$

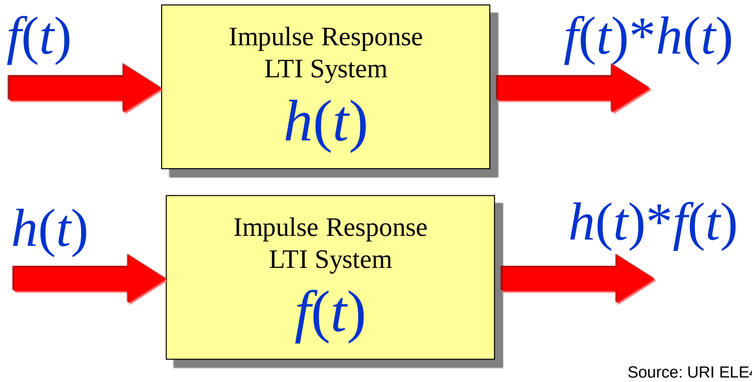
$$= -\int_{\tau=\infty}^{\tau=-\infty} f_1(t-\tau) f_2(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} f_1(t-\tau) f_2(\tau) d\tau \;\; = f_2(t) * f_1(t)$$

Source: URI ELE436

Properties of Convolution

$$f_1(t) * f_2(t) = f_2(t) * f_1(t)$$



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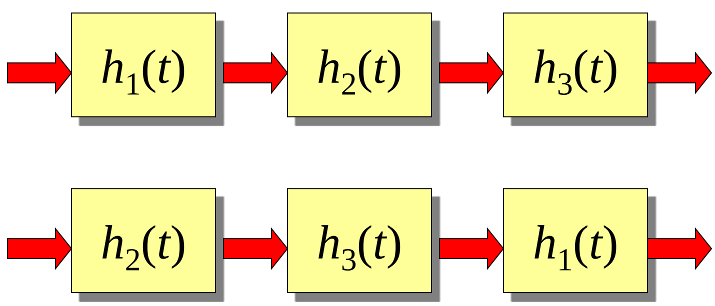
Properties of Convolution

$$[f_1(t) * f_2(t)] * f_3(t) = f_1(t) * [f_2(t) * f_3(t)]$$



- The two systems are identical!

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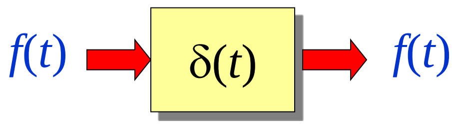
Properties of Convolution

$$f(t) * \delta(t) = f(t)$$



$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$$
$$= \int_{-\infty}^{\infty} f(t-\tau)\delta(\tau)d\tau$$
$$= f(t)$$

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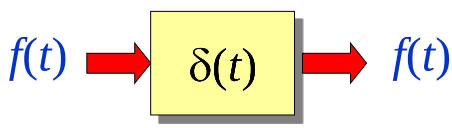
Properties of Convolution

$$f(t) * \delta(t) = f(t)$$



$$f(t) * \delta(t-T) = f(t-T)$$

$$f(t) * \delta(t-T) = \int_{-\infty}^{\infty} f(\tau)\delta(t-T-\tau)d\tau$$
$$= \int_{-\infty}^{\infty} f(t-T-\tau)\delta(\tau)d\tau$$
$$= f(t-T)$$

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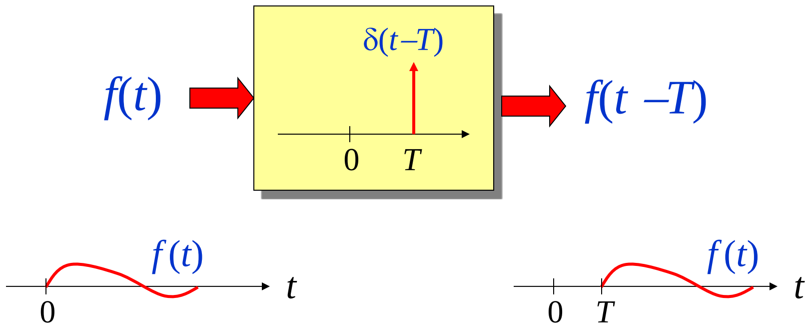
Properties of Convolution

$$f(t) * \delta(t-T) = f(t-T)$$



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Properties of Convolution

$$f_1(t) * f_2(t) \xleftrightarrow{\mathcal{F}} F_1(j\omega)F_2(j\omega)$$

$$F[f_1(t) * f_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} f_1(\tau) \left[\int_{-\infty}^{\infty} f_2(t-\tau) e^{-j\omega t} dt \right] d\tau$$

$$= \int_{-\infty}^{\infty} f_1(\tau) F_2(j\omega) e^{-j\omega\tau} d\tau$$

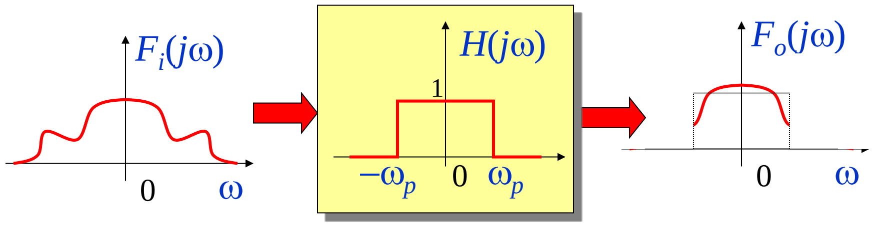
$$= F_2(j\omega) \int_{-\infty}^{\infty} f_1(\tau) e^{-j\omega\tau} d\tau \quad = F_1(j\omega)F_2(j\omega)$$

Time Domain	Frequency Domain
convolution	multiplication

LE436

Properties of Convolution

$$f_1(t) * f_2(t) \xleftrightarrow{\mathcal{F}} F_1(j\omega)F_2(j\omega)$$



An Ideal Low-Pass Filter

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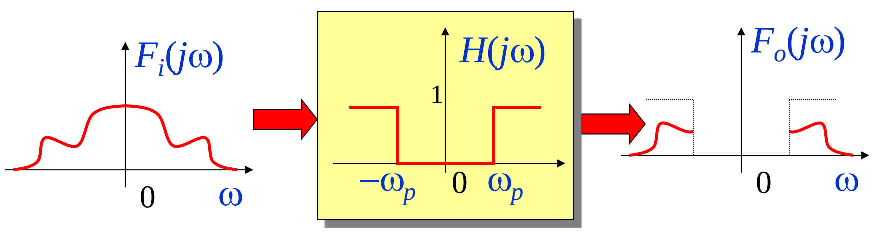
Properties of Convolution

$$f_1(t) * f_2(t) \xleftrightarrow{\mathcal{F}} F_1(j\omega)F_2(j\omega)$$



An Ideal High-Pass Filter

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