Sampling & Data Acquisition & Antialiasing Filters

http://elec3004.com

ELEC 3004: **Digital Linear Systems**: Signals & Controls

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Lecture 3

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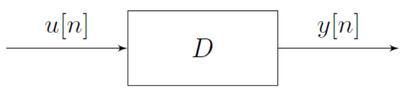
Lecture Schedule:

Week	Date	Lecture Title
1	4-Mar	Introduction & Systems Overview
	6-Mar	[Linear Dynamical Systems]
2	11-Mar	Signals as Vectors & Systems as Maps
	13-Mar	[Signals]
3	18-Mar	Sampling & Data Acquisition & Antialiasing Filters
	20-Mar	[Discrete Signals]
4	25-Mar	Filter Analysis & Filter Design
	27-Mar	[Filters]
5	1-Apr	Digital Filters
	3-Apr	[Digital Filters]
6	8-Apr	Discrete Systems & Z-Transforms
	10-Apr	[Z-Transforms]
7	15-Apr	Convolution & FT & DFT
	17-Apr	Frequency Response
8	29-Apr	Introduction to Control
	1-May	[Feedback]
9	6-May	Introduction to Digital Control
	8-May	[Digitial Control]
10	13-May	Stability of Digital Systems
	15-May	[Stability]
11	20-May	State-Space
	22-May	Controllability & Observability
12	27-May	PID Control & System Identification
	29-May	Digitial Control System Hardware
13	3-Jun	Applications in Industry & Information Theory & Communications
	5-Jun	Summary and Course Review

Systems as Maps

Then a System is a **MATRIX**



$$y = Du.$$

$$\begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[M] \end{bmatrix} = \begin{bmatrix} D_{11} & D12 & \cdots & D_{1N} \\ D_{21} & D_{22} & \cdots & D_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ D_{M1} & D_{M2} & \cdots & D_{MN} \end{bmatrix} \begin{bmatrix} u[1] \\ u[2] \\ \vdots \\ u[N] \end{bmatrix}.$$

$$y[i] = \sum_j D_{ij} u[j].$$

Recall From Last Time …
Classifications of Systems

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Causality:
Causal (physical or nonanticipative) systems

- Is one for which the output at any instant t_0 depends only on the value of the input x(t) for $t \leq t_0$. Ex:

$u(t) = x(t-2) \Rightarrow \text{causal}$ $u(t) = x(t-2) + x(t+2) \Rightarrow \text{noncausal}$

- A "*real-time*" system must be causals
  - How can it respond to future inputs?
- A prophetic system: knows future inputs and acts on it (now)
  - The output would begin before t_0
- In some cases Noncausal maybe modelled as causal with delay
- Noncausal systems provide **an upper bound** on the performance of causal systems

Causality: Looking at this from the output's perspective…

- **Causal** = The output *before* some time *t* does not depend on the input *after* time *t*.

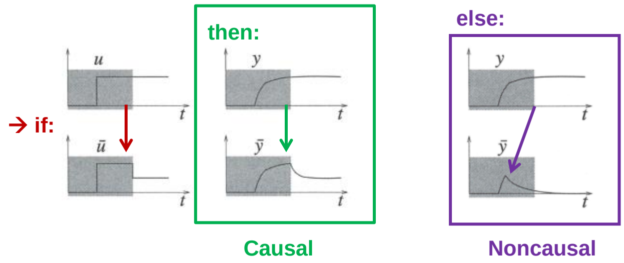
Given: $y(t) = F(u(t))$

For:

$\hat{u}(t) = u(t)\,, \forall 0 \leq t < T \ or\ [0, T)$

Then for a T>0:

$\rightarrow \hat{y}(t) = y(t)\,,\ \forall 0 \leq t < T$

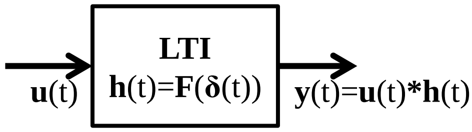


Dynamical Systems…

- A system with a memory
  - Where past history (or derivative states) are **<u>relevant</u>** in determining the response
- Ex:
  - RC circuit: Dynamical
    - Clearly a function of the "capacitor's past" (initial state) and
    - Time! (charge / discharge)
  - R circuit: is memoryless ∵ the output of the system (recall V=IR) at some time **t** only depends on the input at time **t**

- Lumped/Distributed
  - Lumped: Parameter is constant through the process & can be treated as a "point" in space
- Distributed: System dimensions ≠ small over signal
  - Ex: waveguides, antennas, microwave tubes, etc.

Linear Time Invariant



- Linear & Time-invariant (of course - tautology!)
- Impulse response: **h**(t)=**F**(**δ**(t))
- Why?
  - Since it is linear the output response (**y**) to any input (**x**) is:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$
$$y(t) = F\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] \overset{linear}{\rightarrow} \int_{-\infty}^{\infty} x(\tau)\,F\left[\delta(t-\tau)\right]d\tau$$
$$h(t-\tau) \overset{TI}{\equiv} F\left[\delta(t-\tau)\right]$$
$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

- The output of any continuous-time LTI system is the **convolution** of input **u**(t) with the impulse response **F**(**δ**(t)) of the system.

Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

$$a_0 y + a_1\frac{dy}{dt} + \cdots + a_n\frac{d^n y}{dt^n} = b_0 x + b_1\frac{dx}{dt} + \cdots + b_m\frac{d^m x}{dt^m}$$

Laplace:

$$a_0 Y(s) + a_1 sY(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 sX(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

- Total response = Zero-input response + Zero-state response

Initial conditions (Zero-input response)

External Input (Zero-state response)

Linear Systems and ODE's

- Linear system described by differential equation

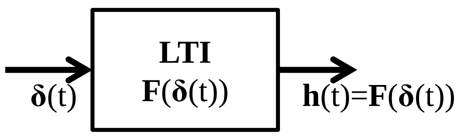
$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

- Which using Laplace Transforms can be written as

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s) Y(s) = B(s) X(s)$$

where *A(s)* and *B(s)* are polynomials in *s*

Unit Impulse Response



- **δ**(t): Impulsive excitation
- h(t): characteristic mode terms

Ex:

EXAMPLE 2.4

Determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)\, y(t) = Dx(t) \qquad (2.25)$$

This is a second-order system ($N = 2$) having the characteristic polynomial
$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2)$

The characteristic roots of this system are $\lambda = -1$ and $\lambda = -2$. Therefore

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t} \qquad (2.26a)$$

Differentiation of this equation yields

$$\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \qquad (2.26b)$$

The initial conditions are [see Eq. (2.24b) for $N = 2$]
$\dot{y}_n(0) = 1$ and $y_n(0) = 0$

Setting $t = 0$ in Eqs. (2.26a) and (2.26b), and substituting the initial conditions just given, we obtain
$0 = c_1 + c_2$

$1 = -c_1 - 2c_2$

Solution of these two simultaneous equations yields
$c_1 = 1$ and $c_2 = -1$

Therefore
$y_n(t) = e^{-t} - e^{-2t}$

Moreover, according to Eq. (2.25), $P(D) = D$, so that
$P(D) y_n(t) = D y_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}$

Also in this case, $b_0 = 0$ [the second-order term is absent in $P(D)$]. Therefore
$h(t) = [P(D) y_n(t)] u(t) = (-e^{-t} + 2e^{-2t}) u(t)$

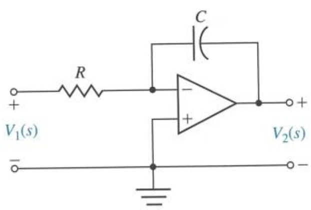
System Models

- Various things – all the same!

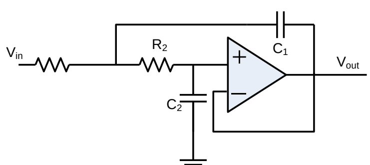
Table 2.1 Summary of Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Circuits



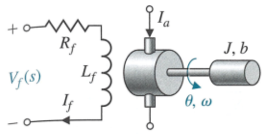
$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$



$$\frac{v_{out}}{v_{in}} = \frac{1}{C_1C_2R_1R_2s^2+C_2(R_1+R_2)s+1}$$

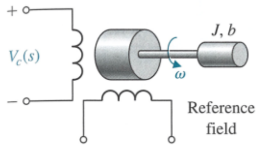
Motors

5. DC motor, field-controlled, rotational actuator



$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$$

7. AC motor, two-phase control field, rotational actuator



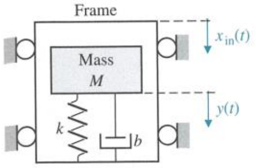
$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

Mechanical Systems

15. Accelerometer, acceleration sensor



$$x_o(t) = y(t) - x_{in}(t),$$

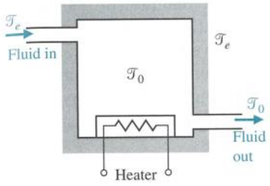
$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$

For low-frequency oscillations, where $\omega < \omega_n$,

$$\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

Thermal Systems

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$ = temperature difference due to thermal process

C_t = thermal capacitance

Q = fluid flow rate = constant

S = specific heat of water

R_t = thermal resistance of insulation

$q(s)$ = transform of rate of heat flow of heating element

First Order Systems

First order systems

$$ay' + by = 0 \quad (\text{with } a \neq 0)$$

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

$$Ty' + y = 0 \quad \text{or} \quad y' + ry = 0$$

where

- $T = a/b$ is a *time* (units: seconds)
- $r = b/a = 1/T$ is a *rate* (units: 1/sec)

First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT+1} = \frac{y(0)}{s+1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, y *decays* exponentially

- T gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half $(0.693 = \log 2)$
- $4.6T$ gives time to decay by 0.01 $(4.6 = \log 100)$

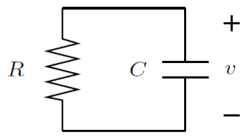
if $T < 0$, y *grows* exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100

First Order Systems

Examples

simple RC circuit:



circuit equation: $RCv' + v = 0$

solution: $v(t) = v(0)e^{-t/(RC)}$

population dynamics:

- $y(t)$ is population of some bacteria at time t
- growth (or decay if negative) rate is $y' = by - dy$ where b is birth rate, d is death rate
- $y(t) = y(0)e^{(b-d)t}$ (grows if $b > d$; decays if $b < d$)

Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a\underbrace{(s^2Y(s) - sy(0) - y'(0))}_{\mathcal{L}(y'')} + b\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

Second Order Systems

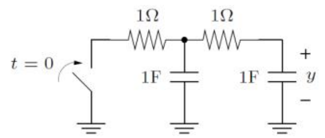
so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

Ex: RC Circuit

Example: second-order RC circuit



at $t = 0$, the voltage across each capacitor is 1V

- for $t \geq 0$, y satisfies LCCODE (from page 2-18)

$$y'' + 3y' + y = 0$$

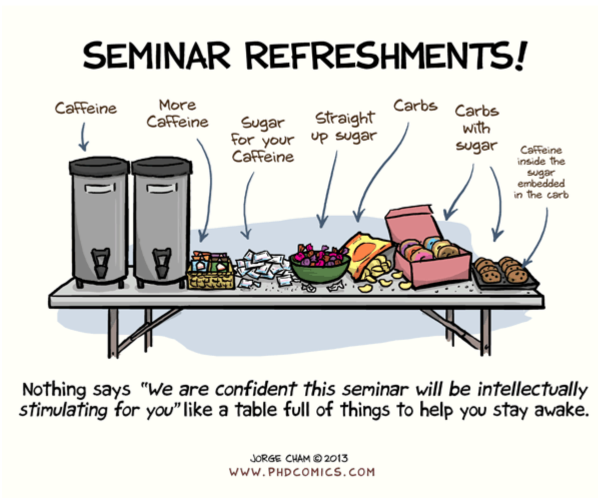
- initial conditions:

$$y(0) = 1, \quad y'(0) = 0$$

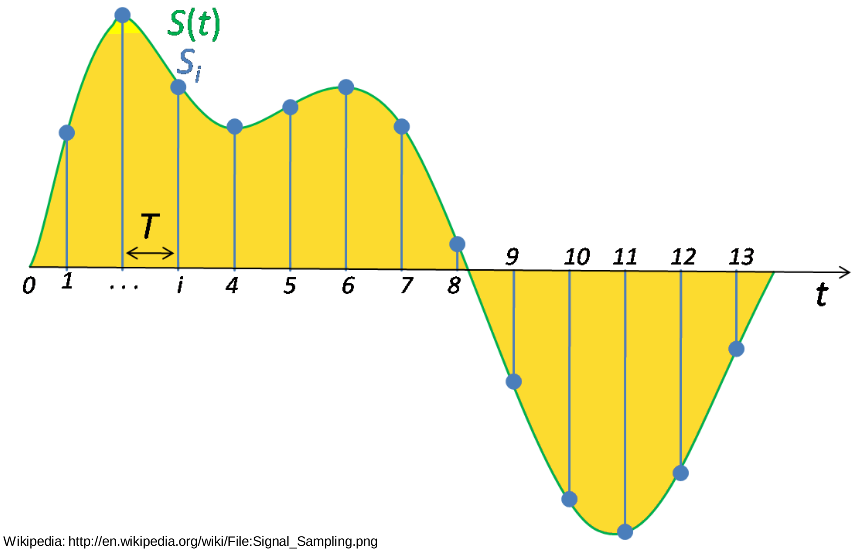
(at $t = 0$, voltage across righthand capacitor is one; current through righthand resistor is zero)

Sampling!

Not this type of sampling …



This type of sampling…

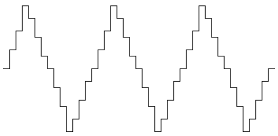


Source: Wikipedia: http://en.wikipedia.org/wiki/File:Signal_Sampling.png

Analog vs Digital

- Analog Signal: An analog or analogue signal is any variable signal continuous in both time and amplitude

- Digital Signal: A digital signal is a signal that is both discrete and quantized



E.g. Music stored in a CD: 44,100 Samples per second and 16 bits to represent amplitude

Digital Signal

- Representation of a signal against a discrete set
- The set is fixed in by computing hardware

$$s \in \mathbb{Z}$$

- Can be scaled or normalized … but is limited

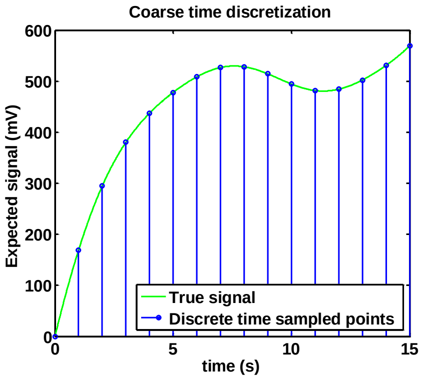
$$s \in \mathbb{Z}(0, \ldots, 2^{16})$$

- Time is also discretized

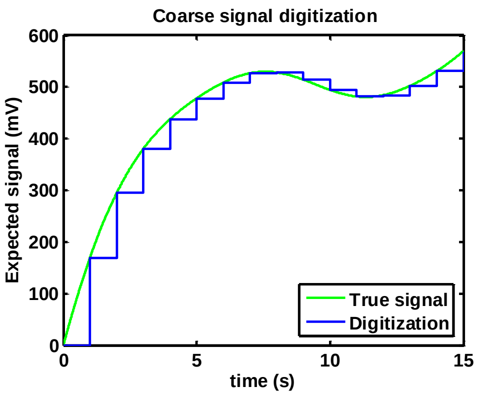
$$s' \in \frac{\mathbb{Z}(0,\ldots,2^{16})}{2^{16}}$$

Representation of Signal

- Time Discretization



- Digitization



Signal: A carrier of (desired) information [1]

- Need **NOT** be electrical:
    - Thermometer
    - Clock hands
    - Automobile speedometer

- Need **NOT** always being given
    - "Abnormal" sounds/operations
    - Ex: "pitch" or "engine hum" during machining as an indicator for feeds and speeds

Signal: A carrier of (desired) information [2]

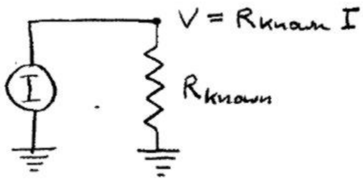
- Electrical signals
    - Voltage
    - Current

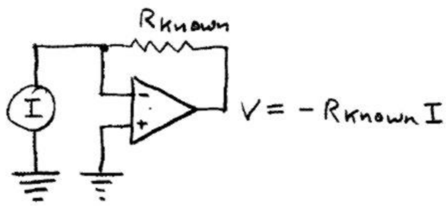
- **Digital signals**
    - **Convert analog electrical signals to an appropriate digital electrical message**
    - **Processing by a microcontroller or microprocessor**

Ex: Current-to-voltage conversion

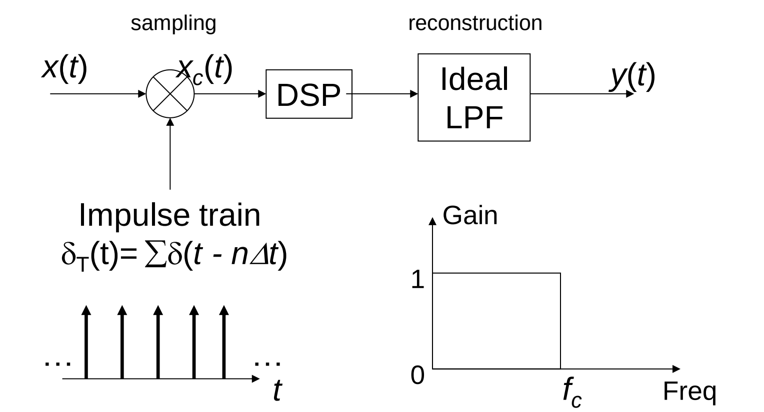
- simple:
  Precision Resistor
- better:
  Use an "op amp"

$$i = \frac{V_{\text{measured}}}{R_{\text{known}}}$$





Mathematics of Sampling and Reconstruction



Sampling frequency $f_s = 1/\Delta t$

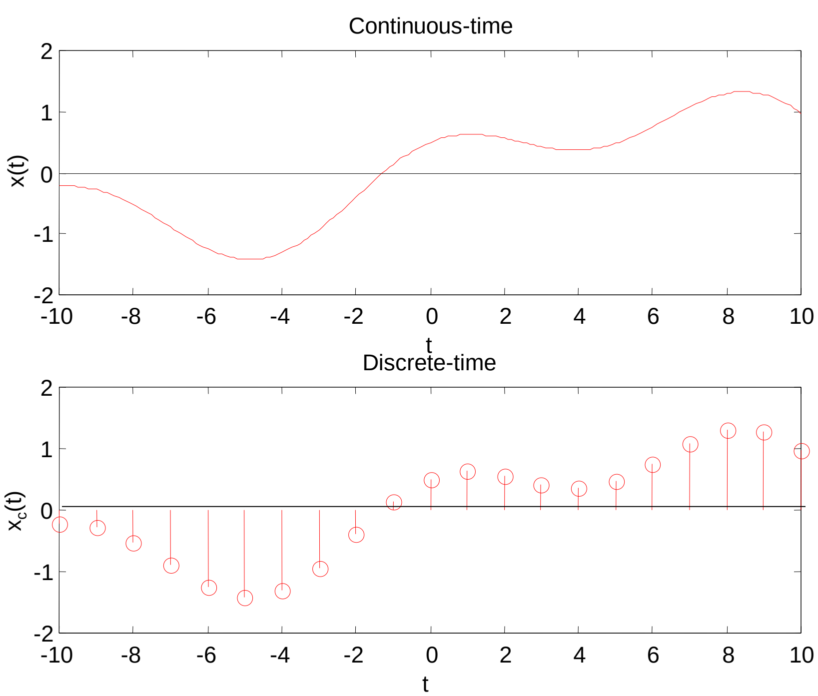
Cut-off frequency = f_c

Mathematical Model of Sampling

- x(t) multiplied by impulse train δT(t)

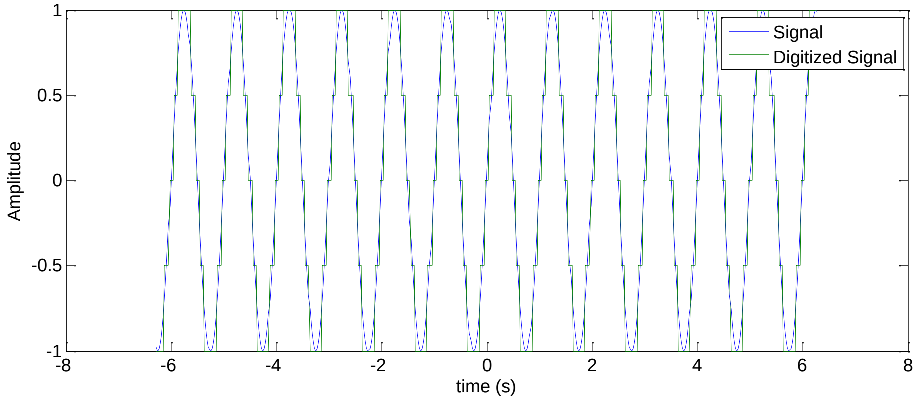
$$
\begin{aligned}
x_c(t) &= x(t)\delta_T(t) \\
&= x(t)\left[\delta(t)+\delta(t-\Delta t)+\delta(t-2\Delta t)+\cdots\right] \\
&= \sum_n x(n\Delta t)\delta(t-n\Delta t)
\end{aligned}
$$

- $x_c(t)$ is a train of impulses of height $x(t)|_{t=n\Delta t}$



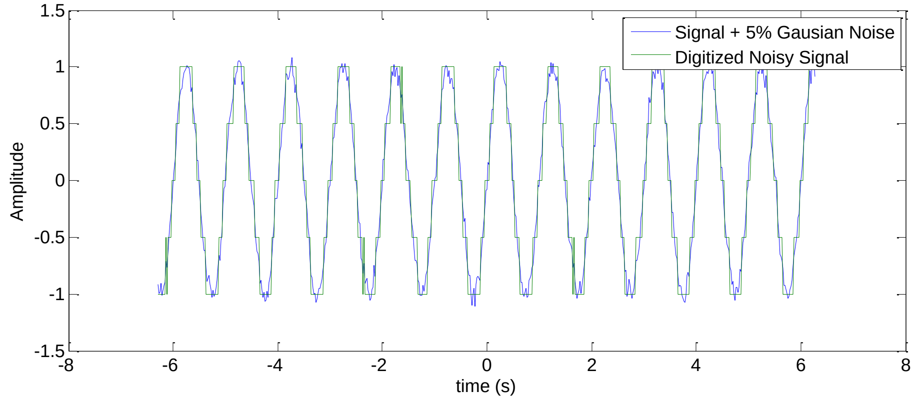
Discrete Time Signal

- Image a signal…



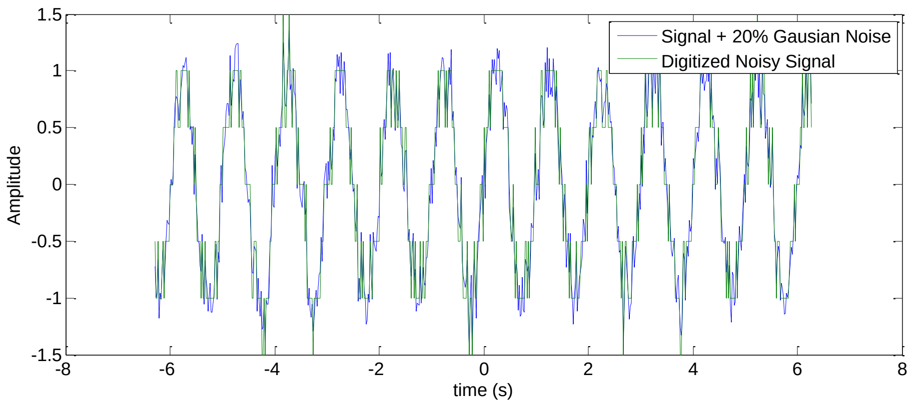
Discrete Time Signals

- Digitization helps beat the Noise!



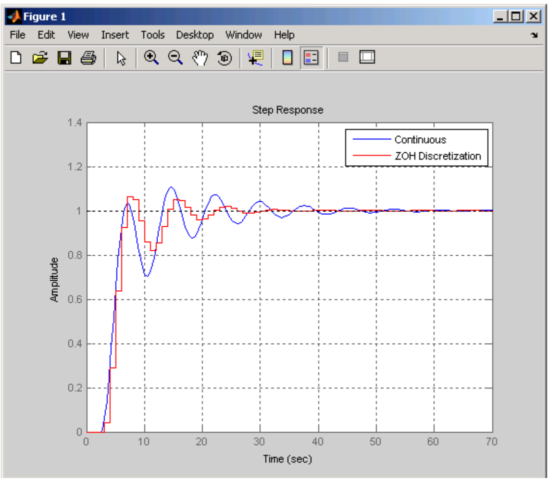
Discrete Time Signals

- But only so much…



Discrete Time Signals

- Can make control tricky!



Signal Manipulations

- Shifting

$$y(n) = x(n - n_0)$$

- Reversal

$$y(n) = x(-n)$$

- Time Scaling

(Down Sampling)

$$y(M) = x(Mn)$$

(Up Sampling)

$$y(n) = x\left(\frac{n}{N}\right)$$

Frequency Domain Analysis of Sampling

- Consider the case where the DSP performs no filtering operations
  - i.e., only passes xc(t) to the reconstruction filter
- To understand we need to look at the frequency domain
- Sampling: we know
  - multiplication in time ≡ convolution in frequency
  - F{x(t)} = X(w)
  - F{δT(t)} = ∑δ(w - 2πn/Δt),
  - i.e., an impulse train in the frequency domain

Frequency Domain Analysis of Sampling

- In the frequency domain we have

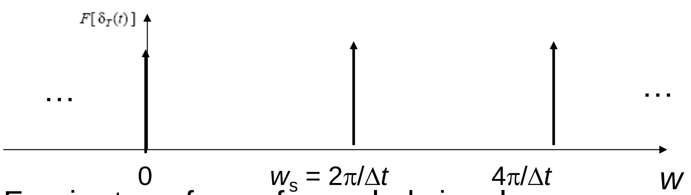
$$X_c(w) = \frac{1}{2\pi}\left(X(w) * \frac{2\pi}{\Delta t}\sum_n \delta\left(w - \frac{2\pi n}{\Delta t}\right)\right)$$
$$= \frac{1}{\Delta t}\sum_n X\left(w - \frac{2\pi n}{\Delta t}\right)$$

Remember convolution with an impulse? Same idea for an impulse train

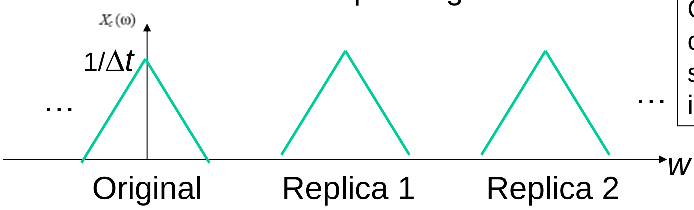
- Let's look at an example
  - where X(w) is triangular function
  - with maximum frequency w_m rad/s
  - being sampled by an impulse train, of frequency w_s rad/s

Fourier transform of original signal X(ω) (signal spectrum)

Fourier transform of impulse train $\delta_T(\omega/2\pi)$ (sampling signal)

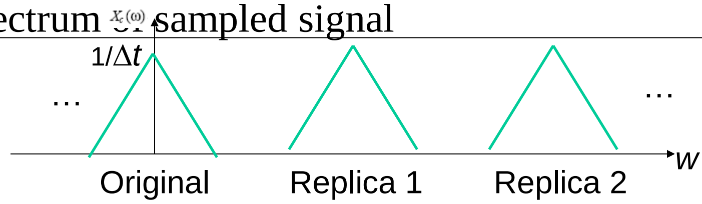


Fourier transform of sampled signal

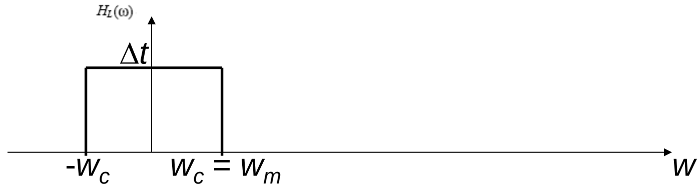


Original spectrum convolved with spectrum of impulse train

Spectrum of sampled signal



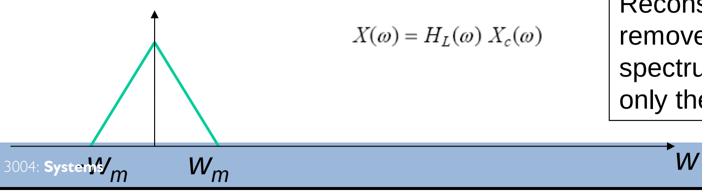
Reconstruction filter (ideal lowpass filter)



Spectrum of reconstructed signal

$$X(\omega) = H_L(\omega)\, X_c(\omega)$$

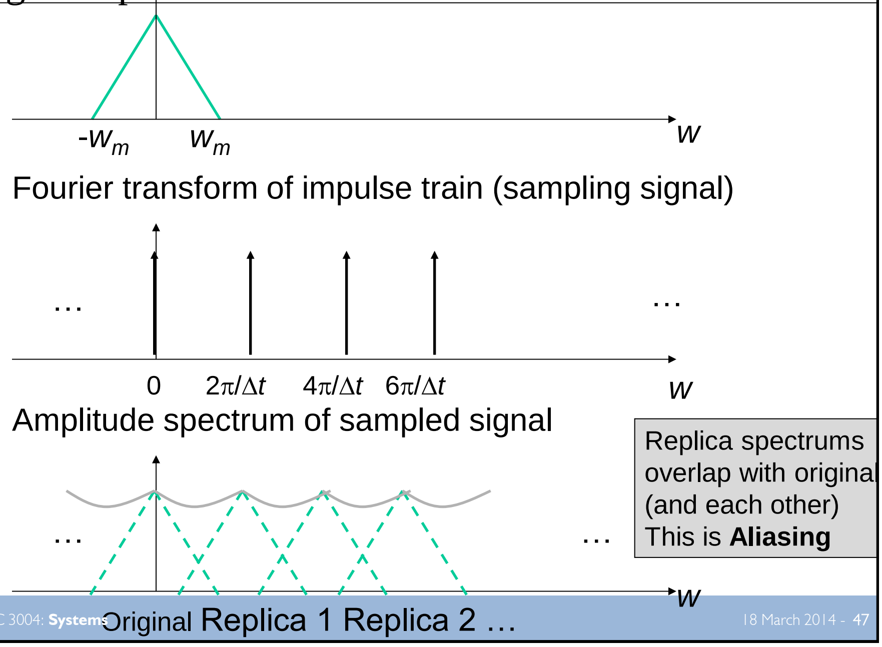
Reconstruction filter removes the replica spectrums & leaves only the original



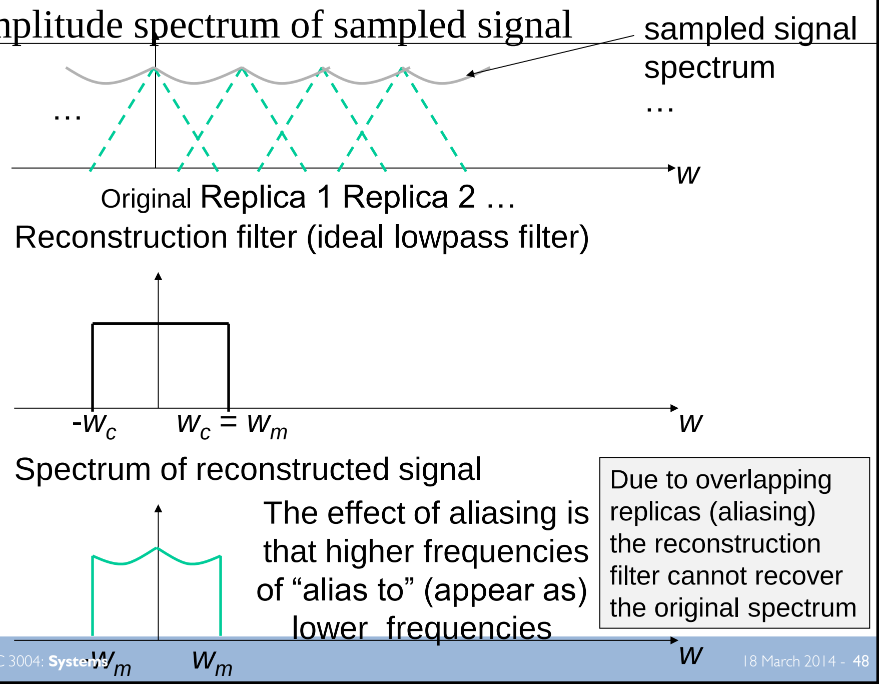
Sampling Frequency

- In this example it was possible to recover the original signal from the discrete-time samples
- But is this always the case?
- Consider an example where the sampling frequency w_s is reduced
  - i.e., Δt is increased

Original Spectrum



Amplitude spectrum of sampled signal



Sampling Theorem

- The Nyquist criterion states:

To prevent aliasing, a ***bandlimited*** *signal of bandwidth* w_B *rad/s must be sampled at a rate greater than* $2w_B$ *rad/s*

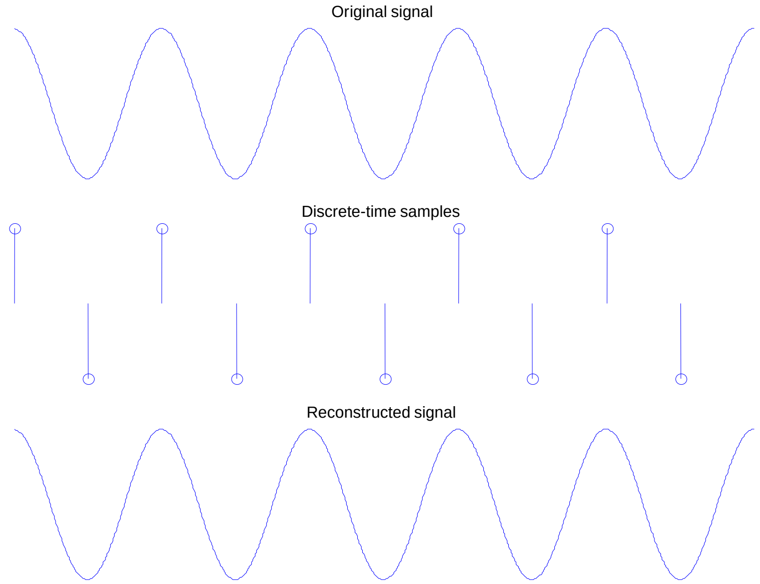
$$-w_s > 2w_B$$

Note: this is a > sign not a ≥

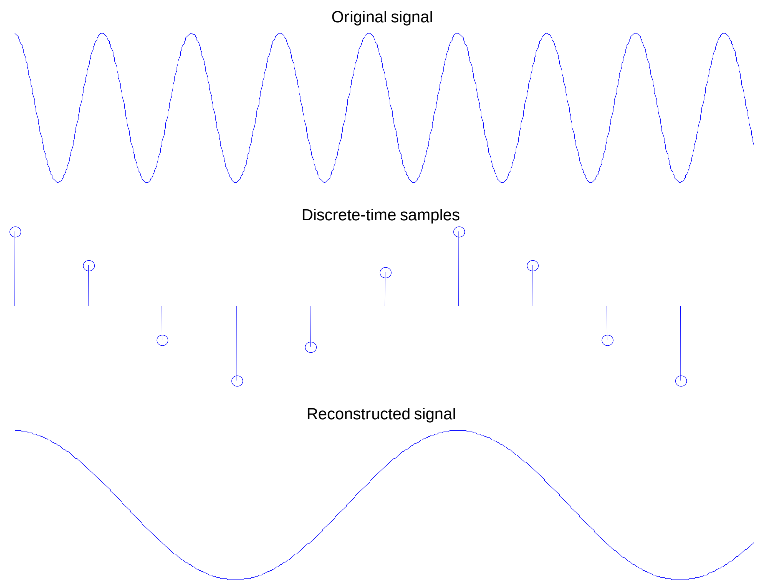
Also note: Most real world signals require band-limiting with a lowpass (anti-aliasing) filter

Time Domain Analysis of Sampling

- Frequency domain analysis of sampling is very useful to understand
  - sampling (X(w)*$\sum \delta(w - 2\pi n/\Delta t)$)
  - reconstruction (lowpass filter removes replicas)
  - aliasing (if $w_s \leq 2w_B$)
- Time domain analysis can also illustrate the concepts
  - sampling a sinewave of increasing frequency
  - sampling images of a rotating wheel

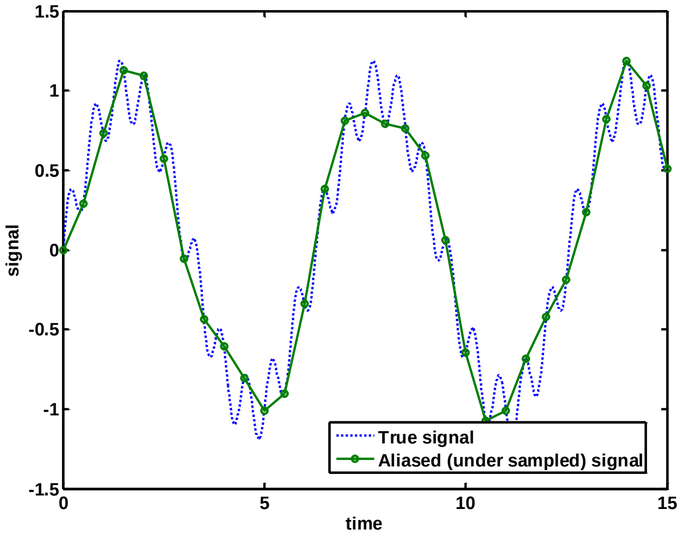


A signal of the original frequency is reconstructed

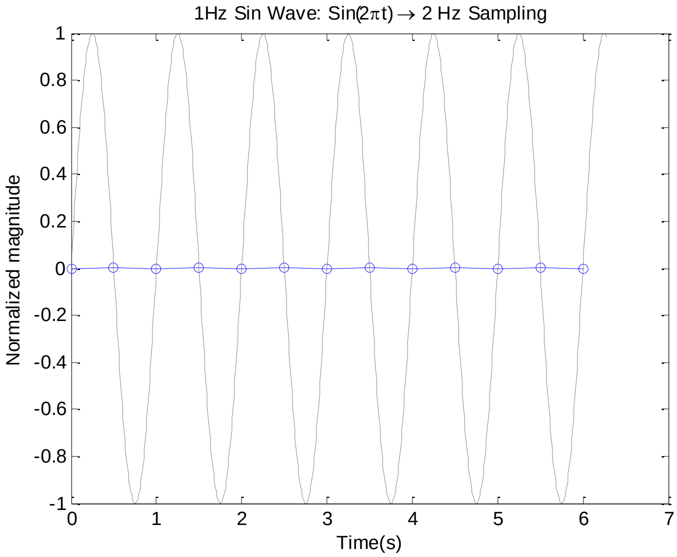


A signal with a reduced frequency is recovered, i.e., the signal is aliased to a lower frequency (we recover a replica)

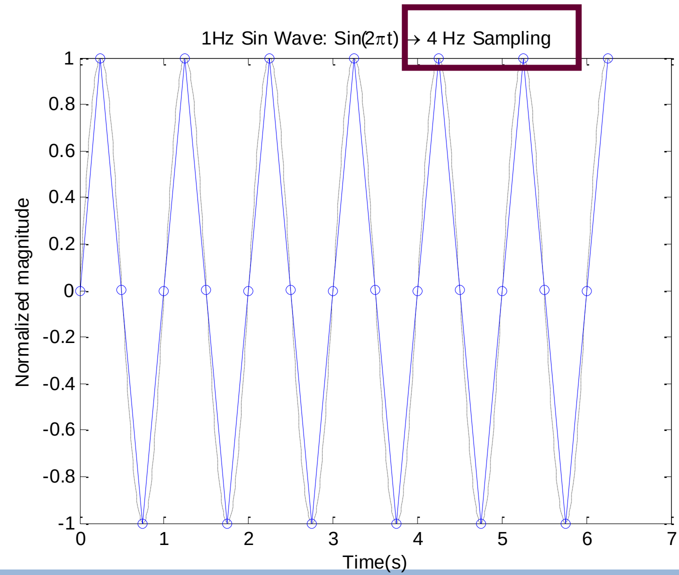
Sampling < Nyquist → Aliasing



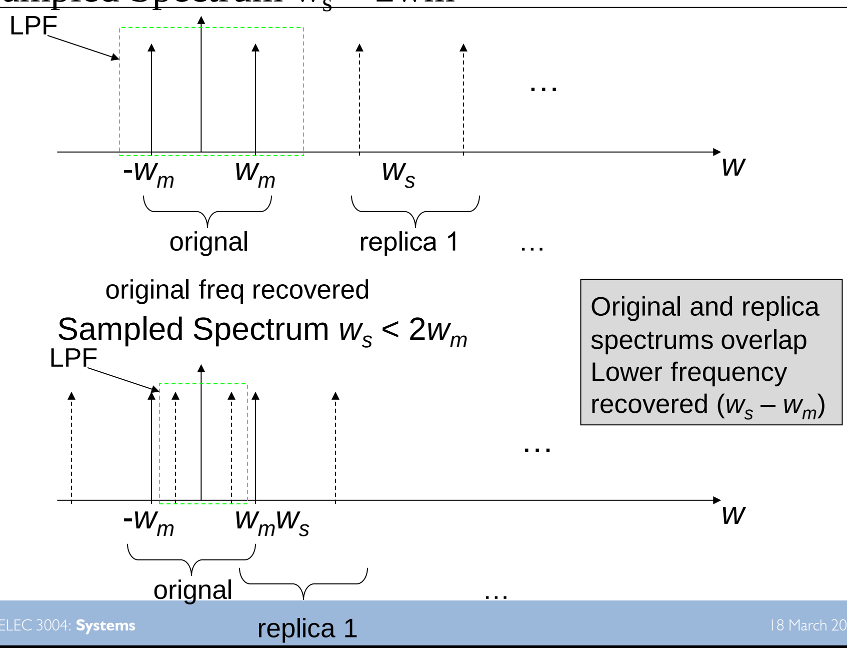
Nyquist is not enough …



A little more than Nyquist is not enough …



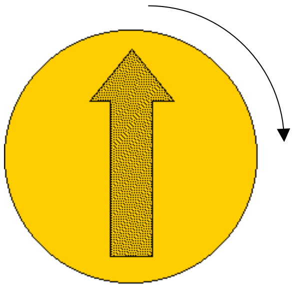
Sampled Spectrum $w_s > 2wm$

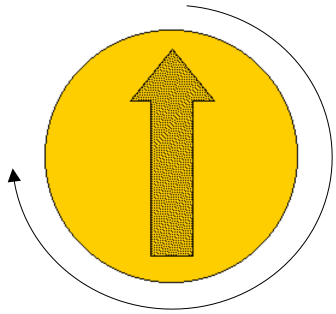


Temporal Aliasing

90° clockwise rotation/frame clockwise rotation perceived

270° clockwise rotation/frame (90°) anticlockwise rotation perceived i.e., aliasing





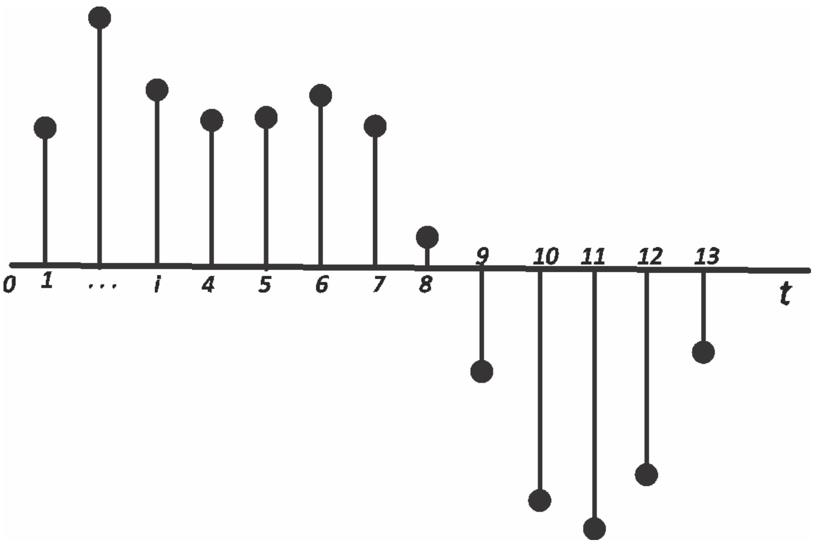
Require LPF to 'blur' motion

Time Domain Analysis of Reconstruction

- Frequency domain: multiply by ideal LPF
  - ideal LPF: 'rect' function (gain Δt, cut off w_c)
  - removes replica spectrums, leaves original
- Time domain: this is equivalent to
  - convolution with 'sinc' function
  - as $F^{-1}\{\Delta t \ \text{rect}(w/w_c)\} = \Delta t \, w_c \, \text{sinc}(w_c t/\pi)$
  - i.e., weighted sinc on every sample
- Normally, $w_c = w_s/2$

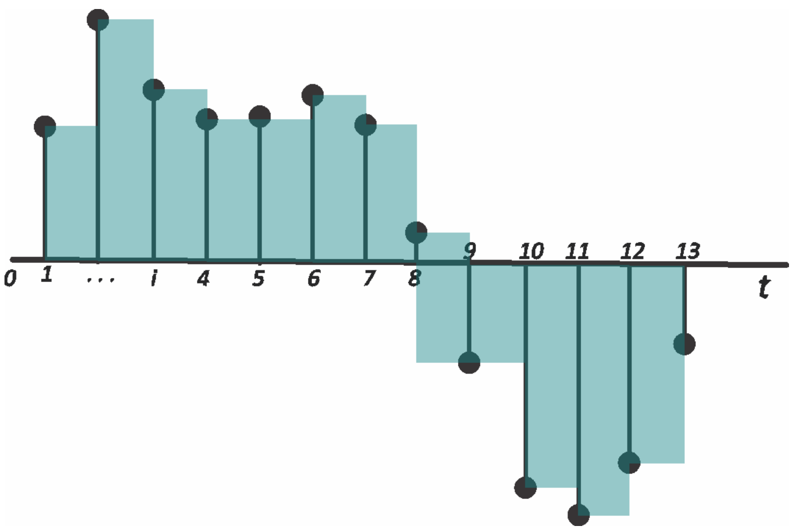
$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n\Delta t)\Delta t w_c \,\text{sinc}\left(\frac{w_c(t-n\Delta t)}{\pi}\right)$$

Reconstruction



Reconstruction

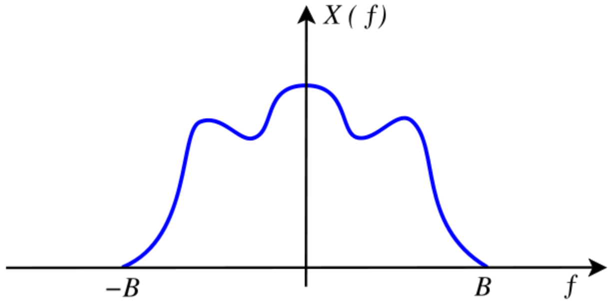
- Zero-Order Hold [ZOH]



Reconstruction

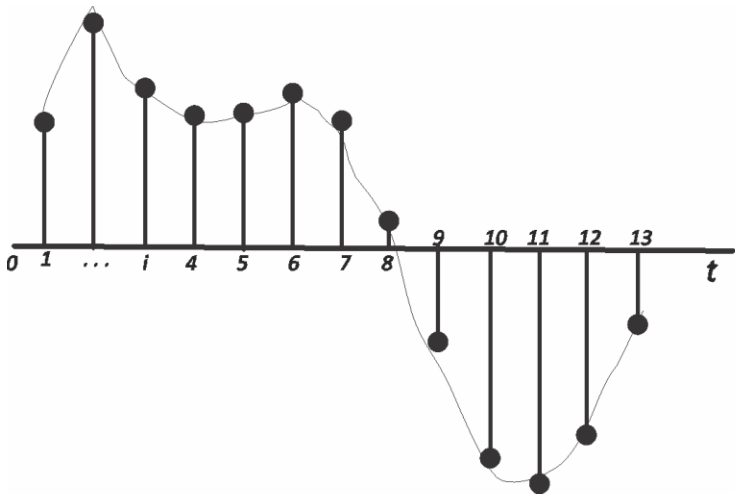
- Whittaker–Shannon interpolation formula

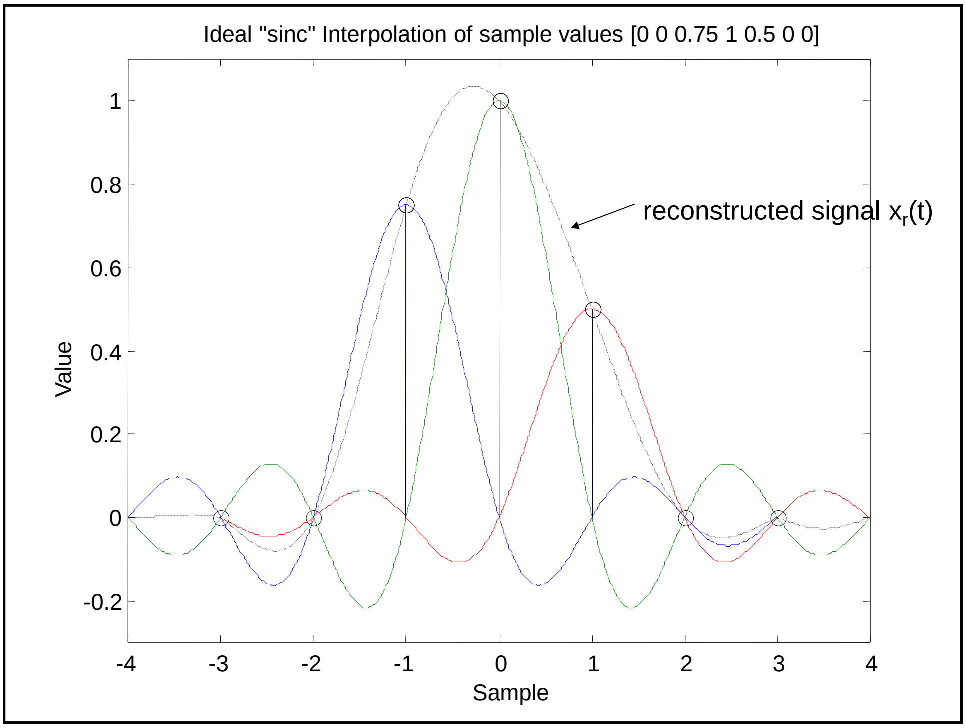
$$x(t) = \sum_{n=-\infty}^{\infty} x[n] \cdot \mathrm{sinc}\left(\frac{t-nT}{T}\right)$$



Reconstruction

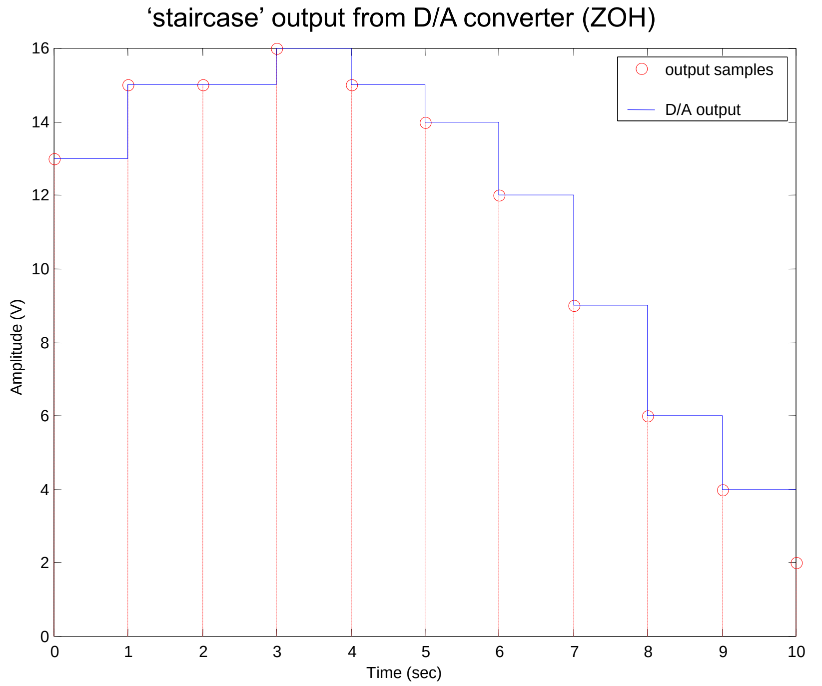
- Whittaker–Shannon interpolation formula

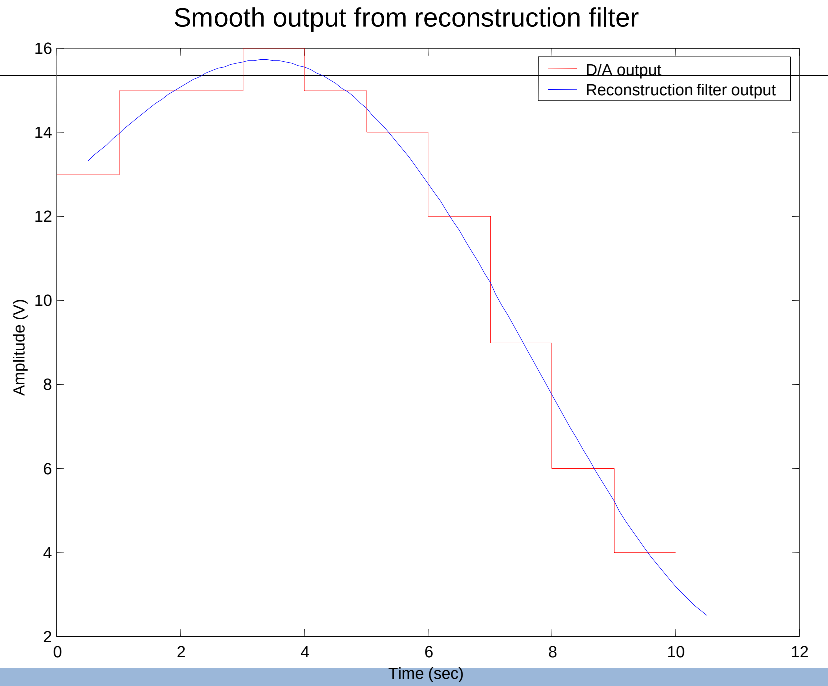


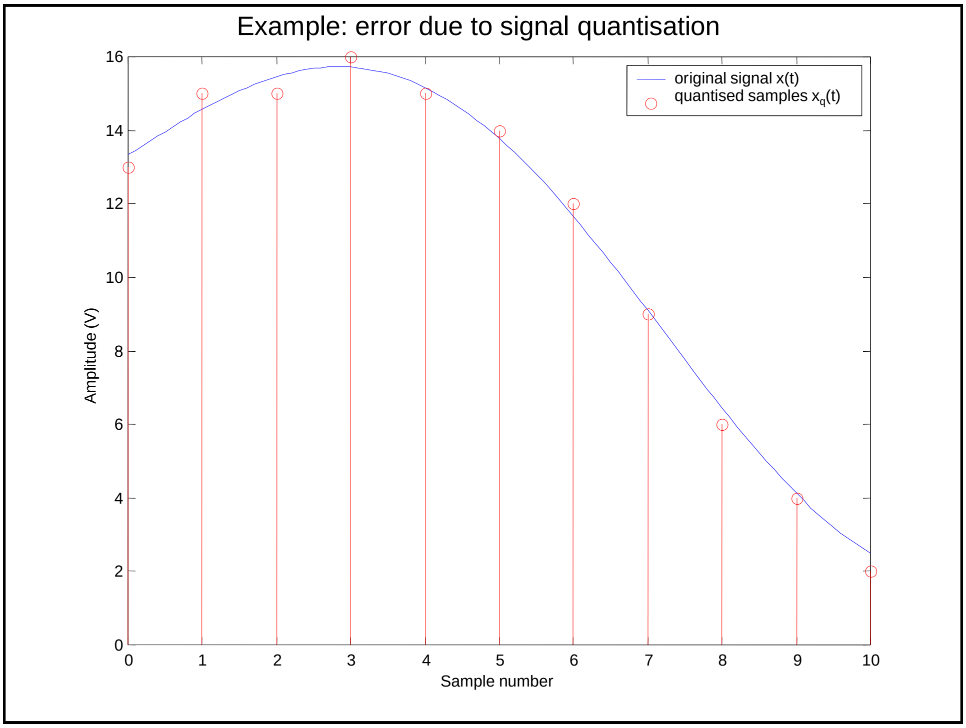


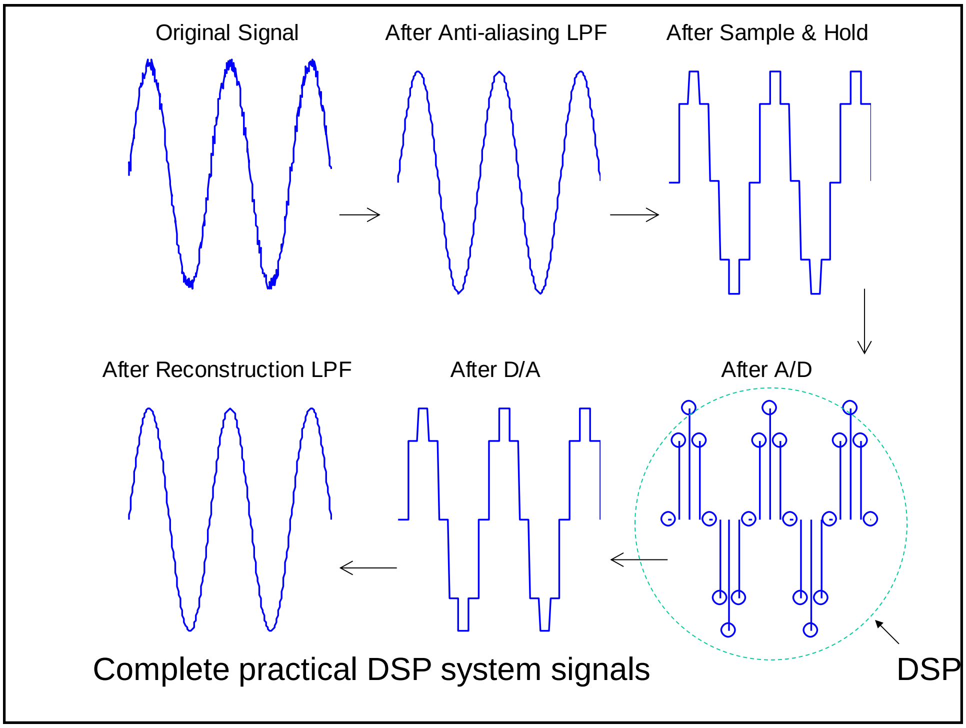
Sampling and Reconstruction Theory and Practice

- Signal is bandlimited to bandwidth WB
  - Problem: real signals are not bandlimited
    - Therefore, require (non-ideal) anti-aliasing filter
- Signal multiplied by ideal impulse train
  - problems: sample pulses have finite width
  - and not ⊗ in practice, but sample & hold circuit
- Samples discrete-time, continuous valued
  - Problem: require discrete values for DSP
    - Therefore, require A/D converter (quantisation)
- Ideal lowpass reconstruction (‘sinc’ interpolation)
  - problems: ideal lowpass filter not available
    - Therefore, use D/A converter and practical lowpass filter


'staircase' output from D/A converter (ZOH)
output samples
D/A output
Amplitude (V)
Time (sec)
0
1
2
3
4
5
6
7
8
9
10
16
14
12
10
8
6
4
2
0

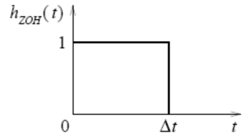

Smooth output from reconstruction filter
D/A output
Reconstruction filter output
Amplitude (V)
Time (sec)
0
2
4
6
8
10
12
16
14
12
10
8
6
4
2


Example: error due to signal quantisation
original signal x(t)
quantised samples $x_q(t)$
Amplitude (V)
Sample number
0
1
2
3
4
5
6
7
8
9
10
0
2
4
6
8
10
12
14
16

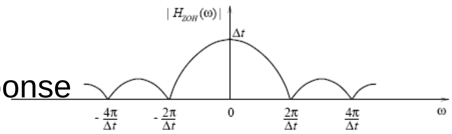

Original Signal
After Anti-aliasing LPF
After Sample & Hold
After Reconstruction LPF
After D/A
After A/D
Complete practical DSP system signals
DSP

Zero Order Hold (ZOH)

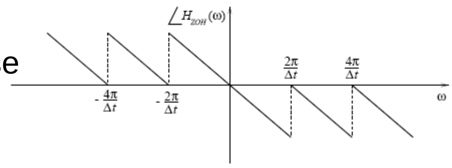
ZOH impulse response



ZOH amplitude response



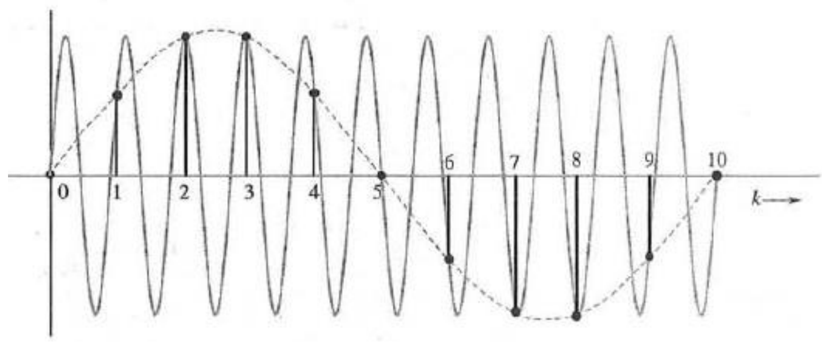
ZOH phase response



Finite Width Sampling

- Impulse train sampling not realisable
  - sample pulses have finite width (say nanosecs)
- This produces two effects,
- Impulse train has sinc envelope in frequency domain
  - impulse train is square wave with small duty cycle
  - Reduces amplitude of replica spectrums
    - smaller replicas to remove with reconstruction filter ☺
- Averaging of signal during sample time
  - effective low pass filter of original signal
    - can reduce aliasing, but can reduce fidelity ☹
    - negligible with most S/H ☺

Aliasing: Another view of this



Alliasing

- Aliasing - through sampling, two entirely different analog sinusoids take on the same “discrete time” identity

For f[k]=cosΩk, Ω=ωT:

The period has to be less than Fh (highest frequency): $T \le \frac{1}{2\mathcal{F}_h}$

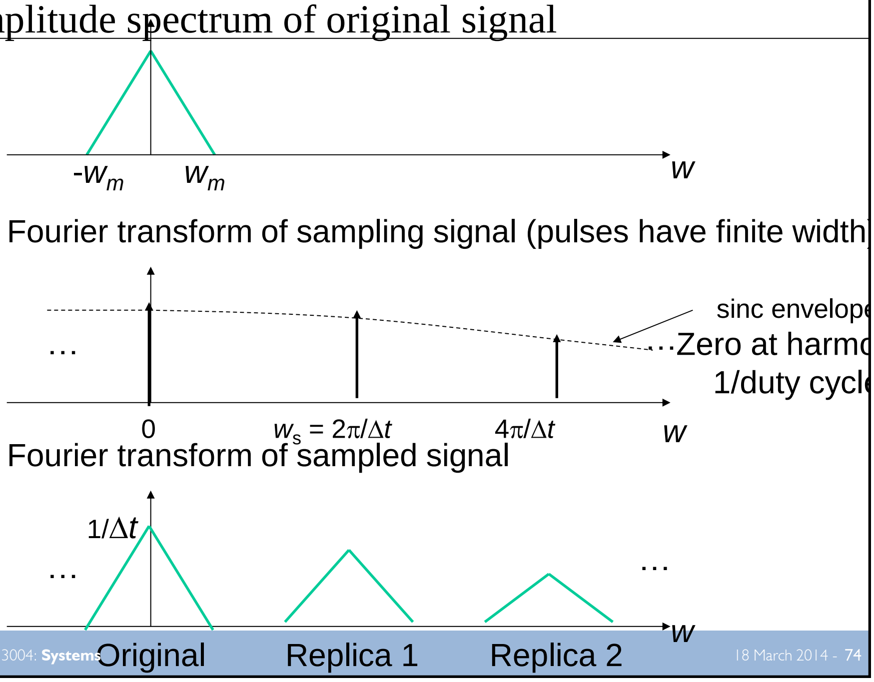
Thus: $0 \le \mathcal{F} \le \frac{\mathcal{F}_s}{2}$

ω_f: aliased frequency: $\omega T = \omega_f T + 2\pi m$

Practical Anti-aliasing Filter

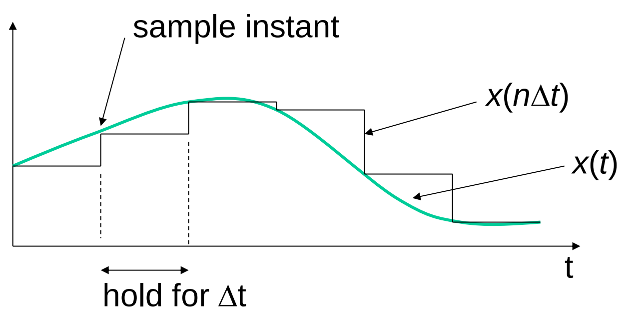
- Non-ideal filter
  - wc = ws /2
- Filter usually 4th – 6th order (e.g., Butterworth)
  - so frequencies > wc may still be present
  - not higher order as phase response gets worse
- Luckily, most real signals
  - are lowpass in nature
    - signal power reduces with increasing frequency
  - e.g., speech naturally bandlimited (say < 8KHz)
  - Natural signals have a (approx) 1/f spectrum
  - so, in practice aliasing is not (usually) a problem

Amplitude spectrum of original signal



Practical Sampling

- Sample and Hold (S/H)
  1. takes a sample every Δt seconds
  2. holds that value constant until next sample
- Produces 'staircase' waveform, $x(n\Delta t)$



Quantisation

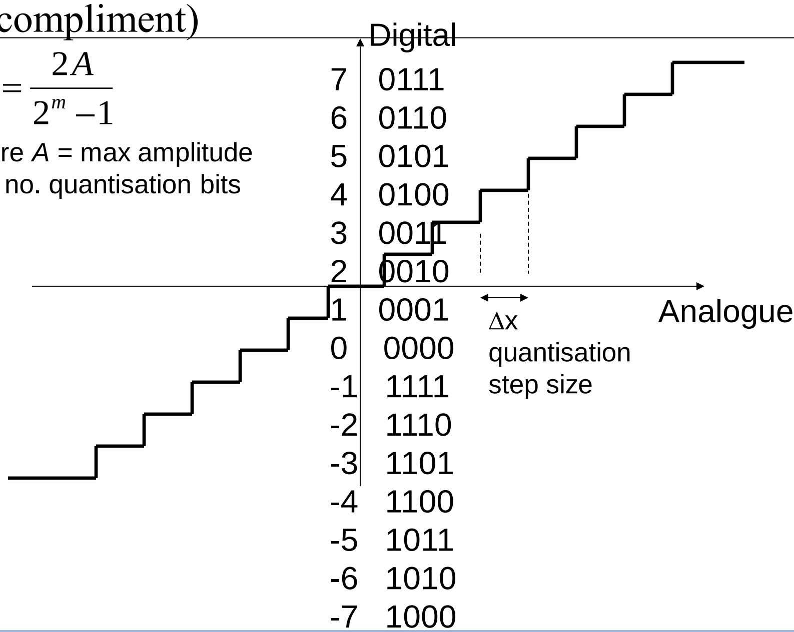
- Analogue to digital converter (A/D)
  - Calculates nearest binary number to $x(n\Delta t)$
    - $x_q[n] = q(x(n\Delta t))$, where q() is non-linear rounding fctn
  - output modeled as $x_q[n] = x(n\Delta t) + e[n]$
- Approximation process
  - therefore, loss of information (unrecoverable)
  - known as 'quantisation noise' ($e[n]$)
  - error reduced as number of bits in A/D increased
    - i.e., Δx, quantisation step size reduces

$$|e[n]| \leq \frac{\Delta x}{2}$$

Input-output for 4-bit quantiser (two's compliment)

$$\Delta x = \frac{2A}{2^m - 1}$$

where *A* = max amplitude
m = no. quantisation bits



Signal to Quantisation Noise

- To estimate SQNR we assume
  - $e[n]$ is uncorrelated to signal and is a
  - uniform random process
- assumptions not always correct!
  - not the only assumptions we could make…
- Also known a 'Dynamic range' (R_D)
  - expressed in decibels (dB)
  - ratio of power of largest signal to smallest (noise)

$$R_D = 10\log_{10}\left(\frac{P_{signal}}{P_{noise}}\right)$$

Dynamic Range

Need to estimate:

1. Noise power
   – uniform random process: $P_{noise} = \Delta x^2/12$
2. Signal power
   – (at least) two possible assumptions
   1. sinusoidal: $P_{signal} = A^2/2$
   2. zero mean Gaussian process: $P_{signal} = \sigma^2$
      - Note: as $\sigma \approx A/3$: $P_{signal} \approx A^2/9$
      - where σ^2 = variance, A = signal amplitude

1 extra bit halves Δx
i.e., 20log10(1/2) = 6dB

Regardless of assumptions: R_D increases by 6dB for every bit that is added to the quantiser

Practical Reconstruction

Two stage process:

1. Digital to analogue converter (D/A)
   – zero order hold filter
   – produces 'staircase' analogue output
2. Reconstruction filter
   – non-ideal filter: $w_c = w_s/2$
   – further reduces replica spectrums
   – usually $4^{th} – 6^{th}$ order e.g., Butterworth
      - for acceptable phase response

D/A Converter

- Analogue output y(t) is
  - convolution of output samples $y(n\Delta t)$ with $h_{ZOH}(t)$

$$y(t) = \sum_n y(n\Delta t) h_{ZOH}(t - n\Delta t)$$

$$h_{ZOH}(t) = \begin{cases} 1, & 0 \le t < \Delta t \\ 0, & \text{otherwise} \end{cases}$$

$$H_{ZOH}(w) = \Delta t \exp\left(\frac{-jw\Delta t}{2}\right) \frac{\sin(w\Delta t/2)}{w\Delta t/2}$$

D/A is lowpass filter with sinc type frequency response
It does not completely remove the replica spectrums
Therefore, additional reconstruction filter required

Summary

- Theoretical model of Sampling
  - bandlimited signal (wB)
  - multiplication by ideal impulse train (ws > 2wB)
    - convolution of frequency spectrums (creates replicas)
  - Ideal lowpass filter to remove replica spectrums
    - wc = ws /2
    - Sinc interpolation
- Practical systems
  - Anti-aliasing filter (wc < ws /2)
  - A/D (S/H and quantisation)
  - D/A (ZOH)
  - Reconstruction filter (wc = ws /2)

Don't confuse theory and practice!