Signals and Vectors
Systems as Maps

ELEC 3004: **Digital Linear Systems**: Signals & Controls

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Lecture 2

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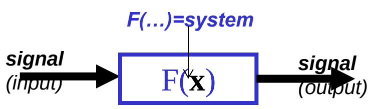
Lecture Schedule:

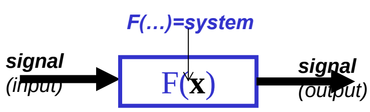
Week	Date	Lecture Title
1	4-Mar	Introduction & Systems Overview
	6-Mar	[Linear Dynamical Systems]
2	11-Mar	Signals as Vectors & Systems as Maps
	13-Mar	[Signals]
3	18-Mar	Sampling & Data Acquisition & Antialiasing Filters
	20-Mar	[Discrete Signals]
4	25-Mar	Filter Analysis & Filter Design
	27-Mar	[Filters]
5	1-Apr	Digital Filters
	3-Apr	[Digital Filters]
6	8-Apr	Discrete Systems & Z-Transforms
	10-Apr	[Z-Transforms]
7	15-Apr	Convolution & FT & DFT
	17-Apr	Frequency Response
8	29-Apr	Introduction to Control
	1-May	[Feedback]
9	6-May	Introduction to Digital Control
	8-May	[Digitial Control]
10	13-May	Stability of Digital Systems
	15-May	[Stability]
11	20-May	State-Space
	22-May	Controllability & Observability
12	27-May	PID Control & System Identification
	29-May	Digitial Control System Hardware
13	3-Jun	Applications in Industry & Information Theory & Communications
	5-Jun	Summary and Course Review

Signals as Vectors

- Back to the beginning!



Signals as Vectors



- There is a perfect analogy between signals and vectors …

Signals are vectors!

- A vector can be represented as a sum of its components in a variety of ways, depending upon the choice of coordinate system. A signal can also be represented as a sum of its components in a variety of ways.

Signals as Vectors

- Represent them as Column Vectors

$$x = \begin{bmatrix} x[1] \\ x[2] \\ x[3] \\ \vdots \\ x[N] \end{bmatrix}.$$

Signals as Vectors

- Can represent phenomena of interest in terms of signals
- Natural vector space structure (addition/substraction/norms)
- Can use norms to describe and quantify properties of signals

Signals as vectors

Signals can take real or complex values.

In both cases, a natural vector space structure:

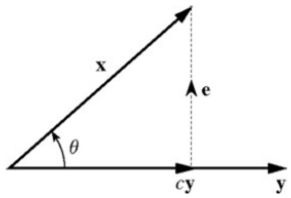
- Can add two signals: $x_1[n] + x_2[n]$
- Can multiply a signal by a scalar number: $C \cdot x[n]$
- Form linear combinations: $C_1 \cdot x_1[n] + C_2 \cdot x_2[n]$

Various Types

- Audio signal (sound pressure on microphone)
- B/W video signal (light intensity on
- photosensor)
- Voltage/current in a circuit (measure with
- multimeter)
- Car speed (from tachometer)
- Robot arm position (from rotary encoder)
- Daily prices of books / air tickets / stocks
- Hourly glucose level in blood (from glucose monitor)
- Heart rate (from heart rate sensor)

Vector Refresher

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos \theta \qquad (6.46)$$



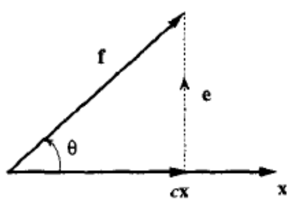
- Length: $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$
- Decomposition: $\mathbf{x} = c_1\mathbf{y} + \mathbf{e}_1 = c_2\mathbf{y} + \mathbf{e}_2$
- Dot Product of $\perp$ is 0: $\mathbf{x} \cdot \mathbf{y} = 0$

Vectors [2]

- Magnitude and Direction

$$f \cdot x = |f||x| \cos(\theta)$$

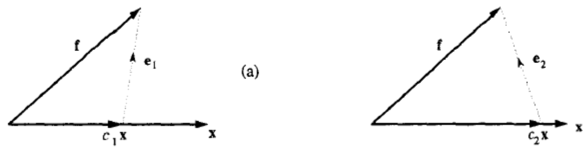
- Component (projection) of a vector along another vector



$\mathbf{f} = c\mathbf{x} + \mathbf{e}$ ← Error Vector

Vectors [3]

- ∞ bases given $\vec{\mathbf{x}}$



- Which is the best one?

$$\mathbf{f} \simeq c\mathbf{x}$$

$$c|\mathbf{x}| = |\mathbf{f}| \cos\theta$$

$$c|\mathbf{x}|^2 = |\mathbf{f}||\mathbf{x}| \cos\theta = \mathbf{f}\cdot\mathbf{x}$$

$$c = \frac{\mathbf{f}\cdot\mathbf{x}}{\mathbf{x}\cdot\mathbf{x}} = \frac{1}{|\mathbf{x}|^2}\mathbf{f}\cdot\mathbf{x}$$

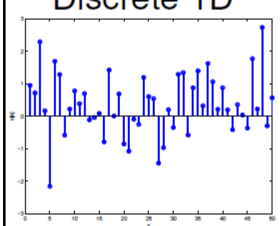
$$\mathbf{f}\cdot\mathbf{x} = 0$$

- Can I allow more basis vectors than I have dimensions?

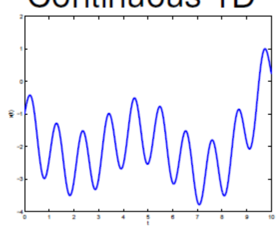
Vectors / Signals Can Be Multidimensional

- A signal is a quantity that varies as a function of an index set
- They can be multidimensional:
  - 1-dim, discrete index (time): x[n]
  - 1-dim, continuous index (time): x(t)
  - 2-dim, discrete (e.g., a B/W or RGB image): x[j; k]
  - 3-dim, video signal (e.g, video): x[j; k; n]

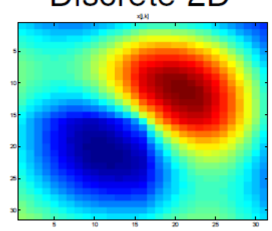
Discrete 1D



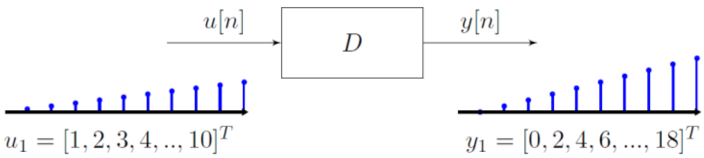
Continuous 1D



Discrete 2D



It’s Just a Linear Map

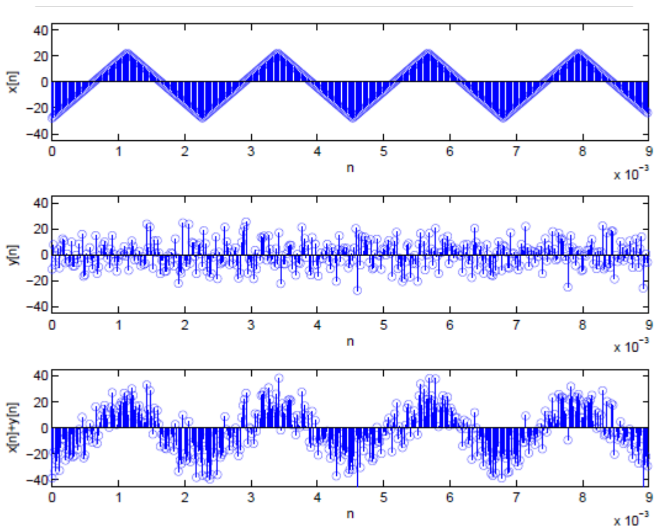


- y[n]=2u[n-1] is a linear map
- BUT y[n]=2(u[n]-1) is **NOT Why?**

- **Because of homogeneity!**

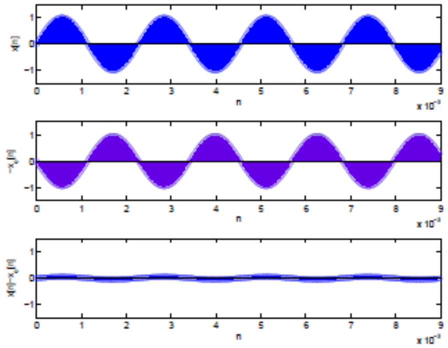
  T(au)=aT(u)

Linear combinations of signals



Application example: active noise cancellation

A “noise” signal, that we want to get rid of.



- At subject location, signal is

$$x[n]$$

- Microphone picks up signal

$$x_c[n]$$

- Subtract the two signals:

$$y(t) = x(t) - x_c(t)$$

Notice careful synchronization is needed!

Norms of signals

Can introduce a notion of signals being ”nearby.”

This is characterized by a metric (or distance function).

$$d(\mathrm{x}, \mathrm{y})$$

If compatible with the vector space structure, we have a norm.

$$\|\mathrm{x} - \mathrm{y}\|$$

Examples of Norms

Can use many different norms, depending on what we want to do.

The following are particularly important:

- ℓ_2 (Euclidean) norm:

$$\|x\|_2 = \left(\sum_{k=1}^{n} |x[k]|^2\right)^{\frac{1}{2}} \qquad \texttt{norm(x,2)}$$

- ℓ_1 norm:

$$\|x\|_1 = \sum_{k=1}^{n} |x[k]| \qquad \texttt{norm(x,1)}$$

- ℓ_∞ norm:

$$\|x\|_\infty = \max_k |x[k]| \qquad \texttt{norm(x,inf)}$$

What are the differences?

Properties of norms

For any norm $\|\cdot\|$, and any signal x, we have:

1. Linearity: if C is a scalar,

$$\|C \cdot \mathrm{x}\| = |C| \cdot \|\mathrm{x}\|$$

2. Subadditivity (triangle inequality):

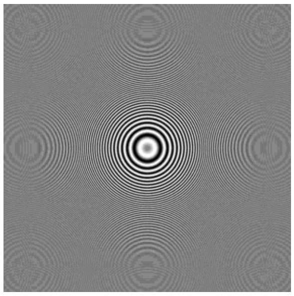
$$\|\mathrm{x} + \mathrm{y}\| \leq \|\mathrm{x}\| + \|\mathrm{y}\|$$

Can use norms:

- To detect whether a signal is (approximately) zero.
- To compare two signals, and determine if they are “close.”

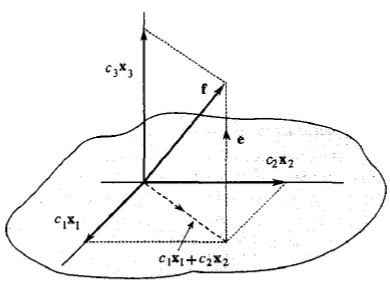
$$\|\mathrm{x} - \mathrm{y}\| \approx 0$$

Where are we going with this?



Signal representation by Orthogonal Signal Set

- **Orthogonal Vector Space**



➔ A signal may be thought of as having components.

Component of a Signal

$$f(t) \simeq cx(t) \qquad t_1 \le t \le t_2$$

$$c = \frac{\int_{t_1}^{t_2} f(t)x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt} = \frac{1}{E_x}\int_{t_1}^{t_2} f(t)x(t)\,dt$$

$$\int_{t_1}^{t_2} f(t)x(t)\,dt = 0$$

- Let's take an example:

$$f(t) \simeq c \sin t \qquad 0 \le t \le 2\pi$$

$$x(t) = \sin t \quad \text{and} \quad E_x = \int_0^{2\pi} \sin^2(t)\,dt = \pi$$

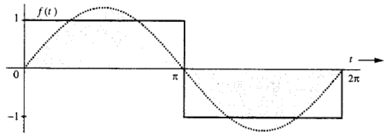


Fig. 3.3 Approximation of square signal in terms of a single sinusoid.

Thus

$$f(t) \simeq \frac{4}{\pi}\sin t \qquad (3.14)$$

Basis Spaces of a Signal

$$\int_{t_1}^{t_2} x_m(t)x_n(t)\,dt = \begin{cases} 0 & m \ne n \\ E_n & m = n \end{cases}$$

$$\begin{aligned} f(t) &\simeq c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t) \\ &= \sum_{n=1}^{N} c_nx_n(t) \end{aligned}$$

$$e(t) = f(t) - \sum_{n=1}^{N} c_nx_n(t)$$

$$\begin{aligned} c_n &= \frac{\int_{t_1}^{t_2} f(t)x_n(t)\,dt}{\int_{t_1}^{t_2} {x_n}^2(t)\,dt} \\ &= \frac{1}{E_n}\int_{t_1}^{t_2} f(t)x_n(t)\,dt \qquad n = 1, 2, \ldots, N \end{aligned}$$

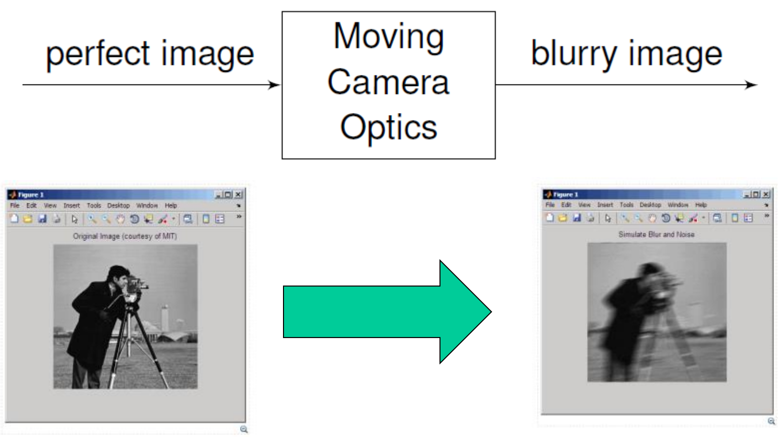
$$\begin{aligned} f(t) &= c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + \cdots \\ &= \sum_{n=1}^{\infty} c_nx_n(t) \qquad t_1 \le t \le t_2 \end{aligned}$$

Basis Spaces of a Signal

$$f(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_n x_n(t) + \cdots$$
$$= \sum_{n=1}^{\infty} c_n x_n(t) \qquad t_1 \leq t \leq t_2$$

- Observe that the error energy *Ee* generally decreases as *N*, the number of terms, is increased because the term *Ck* 2 *Ek* is nonnegative. Hence, it is possible that the error energy -> 0 as *N* -> 00. When this happens, the orthogonal signal set is said to be complete.
- In this case, it’s no more an approximation but an equality

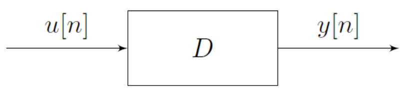
Ex: Deblurring



- Matlab: **deconvwnr**

Systems as Maps

Then a System is a **MATRIX**



$$y = Du.$$

$$\begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[M] \end{bmatrix} = \begin{bmatrix} D_{11} & D12 & \cdots & D_{1N} \\ D_{21} & D_{22} & \cdots & D_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ D_{M1} & D_{M2} & \cdots & D_{MN} \end{bmatrix} \begin{bmatrix} u[1] \\ u[2] \\ \vdots \\ u[N] \end{bmatrix}.$$

$$y[i] = \sum_j D_{ij} u[j].$$

Recall From Last Time …
Classifications of Systems

1. Linear and nonlinear systems
2. Constant-parameter and time-varying-parameter systems
3. Instantaneous (memoryless) and dynamic (with memory) systems
4. Causal and noncausal systems
5. Continuous-time and discrete-time systems
6. Analog and digital systems
7. Invertible and noninvertible systems
8. Stable and unstable systems

Causality:
Causal (physical or nonanticipative) systems

- Is one for which the output at any instant t_0 depends only on the value of the input x(t) for $t \le t_0$. Ex:

$\boxed{u(t) = x(t-2) \Rightarrow \text{causal}}$ $\boxed{u(t) = x(t-2) + x(t+2) \Rightarrow \text{noncausal}}$

- A "*real-time*" system must be causals
  - How can it respond to future inputs?
- A prophetic system: knows future inputs and acts on it (now)
  - The output would begin before t_0
- In some cases Noncausal maybe modelled as causal with delay
- Noncausal systems provide **an upper bound** on the performance of causal systems

Causality:
Looking at this from the output's perspective…

- **Causal** = The output *before* some time *t* does not depend on the input *after* time *t.*

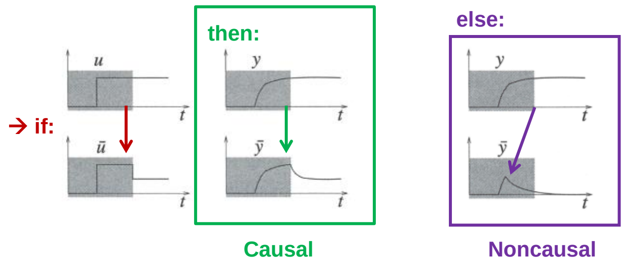
Given: $y(t) = F(u(t))$

For:

$\hat{u}(t) = u(t), \forall 0 \le t < T \; or \; [0, T)$

Then for a T>0:

$\rightarrow \hat{y}(t) = y(t), \; \forall 0 \le t < T$

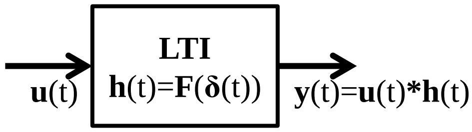


Dynamical Systems…

- A system with a memory
  - Where past history (or derivative states) are **relevant** in determining the response
- Ex:
  - RC circuit: Dynamical
    - Clearly a function of the "capacitor's past" (initial state) and
    - Time! (charge / discharge)
  - R circuit: is memoryless ∵ the output of the system (recall V=IR) at some time **t** only depends on the input at time **t**

- Lumped/Distributed
  - Lumped: Parameter is constant through the process & can be treated as a "point" in space
- Distributed: System dimensions ≠ small over signal
  - Ex: waveguides, antennas, microwave tubes, etc.

Linear Time Invariant



- Linear & Time-invariant (of course - tautology!)
- Impulse response: **h**(t)=**F**(**δ**(t))
- Why?
  - Since it is linear the output response (**y**) to any input (**x**) is:

$$
\begin{aligned}
&x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \\
&y(t) = F\left[\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right] \overset{linear}{\rightarrow} \int_{-\infty}^{\infty} x(\tau)\,F[\delta(t-\tau)]\,d\tau \\
&h(t-\tau) \overset{TI}{\equiv} F[\delta(t-\tau)] \\
&\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)
\end{aligned}
$$

- The output of any continuous-time LTI system is the **convolution** of input **u**(t) with the impulse response **F**(**δ**(t)) of the system.

Linear Dynamic [Differential] System

≡ LTI systems for which the input & output are linear ODEs

$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

Laplace:

$$
\begin{aligned}
&a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s) \\
&A(s)Y(s) = B(s)X(s)
\end{aligned}
$$

- Total response = $\underbrace{\text{Zero-input response}}_{\text{Initial conditions}}$ + $\underbrace{\text{Zero-state response}}_{\text{External Input}}$

Linear Systems and ODE's

- Linear system described by differential equation

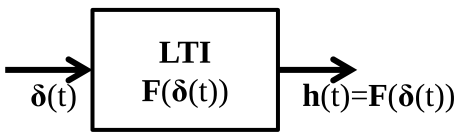
$$a_0 y + a_1 \frac{dy}{dt} + \cdots + a_n \frac{d^n y}{dt^n} = b_0 x + b_1 \frac{dx}{dt} + \cdots + b_m \frac{d^m x}{dt^m}$$

- Which using Laplace Transforms can be written as

$$a_0 Y(s) + a_1 s Y(s) + \cdots + a_n s^n Y(s) = b_0 X(s) + b_1 s X(s) + \cdots + b_m s^m X(s)$$
$$A(s)Y(s) = B(s)X(s)$$

where *A(s)* and *B(s)* are polynomials in *s*

Unit Impulse Response



- $\boldsymbol{\delta}$(t): Impulsive excitation
- h(t): characteristic mode terms

Ex:

EXAMPLE 2.4

Determine the unit impulse response $h(t)$ for a system specified by the equation

$$(D^2 + 3D + 2)\, y(t) = Dx(t) \qquad (2.25)$$

This is a second-order system ($N = 2$) having the characteristic polynomial
$(\lambda^2 + 3\lambda + 2) = (\lambda + 1)(\lambda + 2)$

The characteristic roots of this system are $\lambda = -1$ and $\lambda = -2$. Therefore

$$y_n(t) = c_1 e^{-t} + c_2 e^{-2t} \qquad (2.26a)$$

Differentiation of this equation yields

$$\dot{y}_n(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \qquad (2.26b)$$

The initial conditions are [see Eq. (2.24b) for $N = 2$]
$\dot{y}_n(0) = 1$ and $y_n(0) = 0$

Setting $t = 0$ in Eqs. (2.26a) and (2.26b), and substituting the initial conditions just given, we obtain
$0 = c_1 + c_2$

$1 = -c_1 - 2c_2$

Solution of these two simultaneous equations yields
$c_1 = 1$ and $c_2 = -1$

Therefore
$y_n(t) = e^{-t} - e^{-2t}$

Moreover, according to Eq. (2.25), $P(D) = D$, so that
$P(D)y_n(t) = Dy_n(t) = \dot{y}_n(t) = -e^{-t} + 2e^{-2t}$

Also in this case, $b_0 = 0$ [the second-order term is absent in $P(D)$]. Therefore
$h(t) = [P(D)y_n(t)]u(t) = (-e^{-t} + 2e^{-2t})u(t)$

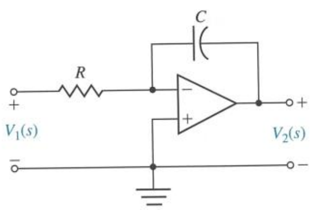
System Models

- Various things – all the same!

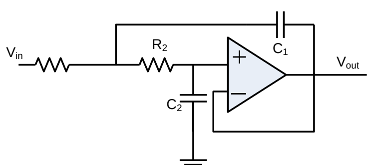
Table 2.1 Summary of Through- and Across-Variables for Physical Systems

System	Variable Through Element	Integrated Through-Variable	Variable Across Element	Integrated Across-Variable
Electrical	Current, i	Charge, q	Voltage difference, v_{21}	Flux linkage, λ_{21}
Mechanical translational	Force, F	Translational momentum, P	Velocity difference, v_{21}	Displacement difference, y_{21}
Mechanical rotational	Torque, T	Angular momentum, h	Angular velocity difference, ω_{21}	Angular displacement difference, θ_{21}
Fluid	Fluid volumetric rate of flow, Q	Volume, V	Pressure difference, P_{21}	Pressure momentum, γ_{21}
Thermal	Heat flow rate, q	Heat energy, H	Temperature difference, $\mathcal{T}_{21}$	

Circuits



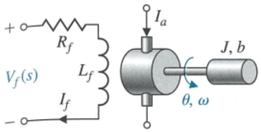
$$\frac{V_2(s)}{V_1(s)} = -\frac{1}{RCs}$$



$$\frac{v_{out}}{v_{in}} = \frac{1}{C_1C_2R_1R_2s^2+C_2(R_1+R_2)s+1}$$

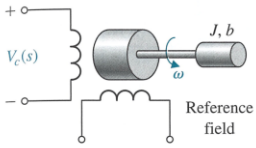
Motors

5. DC motor, field-controlled, rotational actuator



$$\frac{\theta(s)}{V_f(s)} = \frac{K_m}{s(Js + b)(L_f s + R_f)}$$

7. AC motor, two-phase control field, rotational actuator



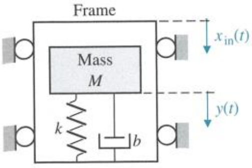
$$\frac{\theta(s)}{V_c(s)} = \frac{K_m}{s(\tau s + 1)}$$

$$\tau = J/(b - m)$$

m = slope of linearized torque-speed curve (normally negative)

Mechanical Systems

15. Accelerometer, acceleration sensor



$$x_o(t) = y(t) - x_{in}(t),$$

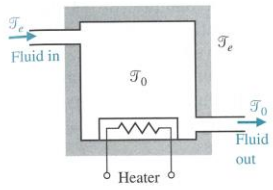
$$\frac{X_o(s)}{X_{in}(s)} = \frac{-s^2}{s^2 + (b/M)s + k/M}$$

For low-frequency oscillations, where $\omega < \omega_n$,

$$\frac{X_o(j\omega)}{X_{in}(j\omega)} \simeq \frac{\omega^2}{k/M}$$

Thermal Systems

16. Thermal heating system



$$\frac{\mathcal{T}(s)}{q(s)} = \frac{1}{C_t s + (QS + 1/R_t)}, \text{ where}$$

$\mathcal{T} = \mathcal{T}_o - \mathcal{T}_e$ = temperature difference due to thermal process

C_t = thermal capacitance

Q = fluid flow rate = constant

S = specific heat of water

R_t = thermal resistance of insulation

$q(s)$ = transform of rate of heat flow of heating element

First Order Systems

First order systems

$$ay' + by = 0 \quad (\text{with } a \neq 0)$$

righthand side is zero:

- called *autonomous system*
- solution is called *natural* or *unforced response*

can be expressed as

$$Ty' + y = 0 \quad \text{or} \quad y' + ry = 0$$

where

- $T = a/b$ is a *time* (units: seconds)
- $r = b/a = 1/T$ is a *rate* (units: 1/sec)

First Order Systems

Solution by Laplace transform

take Laplace transform of $Ty' + y = 0$ to get

$$T\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + Y(s) = 0$$

solve for $Y(s)$ (algebra!)

$$Y(s) = \frac{Ty(0)}{sT+1} = \frac{y(0)}{s+1/T}$$

and so $y(t) = y(0)e^{-t/T}$

First Order Systems

solution of $Ty' + y = 0$: $y(t) = y(0)e^{-t/T}$

if $T > 0$, y *decays* exponentially

- T gives time to decay by $e^{-1} \approx 0.37$
- $0.693T$ gives time to decay by half $(0.693 = \log 2)$
- $4.6T$ gives time to decay by 0.01 $(4.6 = \log 100)$

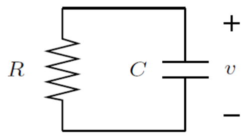
if $T < 0$, y *grows* exponentially

- $|T|$ gives time to grow by $e \approx 2.72$;
- $0.693|T|$ gives time to double
- $4.6|T|$ gives time to grow by 100

First Order Systems

Examples

simple RC circuit:



circuit equation: $RCv' + v = 0$

solution: $v(t) = v(0)e^{-t/(RC)}$

population dynamics:

- $y(t)$ is population of some bacteria at time t
- growth (or decay if negative) rate is $y' = by - dy$ where b is birth rate, d is death rate
- $y(t) = y(0)e^{(b-d)t}$ (grows if $b > d$; decays if $b < d$)

Second Order Systems

Second order systems

$$ay'' + by' + cy = 0$$

assume $a > 0$ (otherwise multiply equation by -1)

solution by Laplace transform:

$$a\underbrace{(s^2Y(s) - sy(0) - y'(0))}_{\mathcal{L}(y'')} + b\underbrace{(sY(s) - y(0))}_{\mathcal{L}(y')} + cY(s) = 0$$

solve for Y (just algebra!)

$$Y(s) = \frac{asy(0) + ay'(0) + by(0)}{as^2 + bs + c} = \frac{\alpha s + \beta}{as^2 + bs + c}$$

where $\alpha = ay(0)$ and $\beta = ay'(0) + by(0)$

Second Order Systems

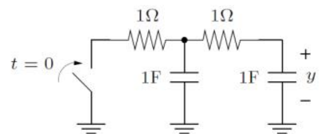
so solution of $ay'' + by' + cy = 0$ is

$$y(t) = \mathcal{L}^{-1}\left(\frac{\alpha s + \beta}{as^2 + bs + c}\right)$$

- $\chi(s) = as^2 + bs + c$ is called *characteristic polynomial* of the system
- form of $y = \mathcal{L}^{-1}(Y)$ depends on roots of characteristic polynomial χ
- coefficients of numerator $\alpha s + \beta$ come from initial conditions

Ex: RC Circuit

Example: second-order RC circuit



at $t = 0$, the voltage across each capacitor is $1\mathrm{V}$

- for $t \geq 0$, y satisfies LCCODE (from page 2-18)

$$y'' + 3y' + y = 0$$

- initial conditions:

$$y(0) = 1, \quad y'(0) = 0$$

(at $t = 0$, voltage across righthand capacitor is one; current through righthand resistor is zero)

Next Time…

- Sampling
  - Measurements at regular intervals of a continuous signal
  - Not to be confused with
    “ How to try regional dishes without indigestion”

- Review:
  - Chapter 8 of Lathi

- Send (and you shall receive) a positive signal ☺