CHAPTER 5

DISCRETE-TIME SYSTEM ANALYSIS USING THE z-TRANSFORM

The counterpart of the Laplace transform for discrete-time systems is the z-transform. The Laplace transform converts integro-differential equations into algebraic equations. In the same way, the z-transforms changes difference equations into algebraic equations, thereby simplifying the analysis of discrete-time systems. The z-transform method of analysis of discrete-time systems parallels the Laplace transform method of analysis of continuous-time systems, with some minor differences. In fact, we shall see that *the z-transform is the Laplace transform in disguise.*

The behavior of discrete-time systems is similar to that of continuous-time systems (with some differences). The frequency-domain analysis of discrete-time systems is based on the fact (proved in Section 3.8-3) that the response of a linear, time-invariant, discrete-time (LTID) system to an everlasting exponential z^n is the same exponential (within a multiplicative constant) given by $H[z]z^n$. We then express an input $x[n]$ as a sum of (everlasting) exponentials of the form z^n. The system response to $x[n]$ is then found as a sum of the system's responses to all these exponential components. The tool that allows us to represent an arbitrary input $x[n]$ as a sum of (everlasting) exponentials of the form z^n is the z-transform.

5.1 THE z-TRANSFORM

We define $X[z]$, the direct z-transform of $x[n]$, as

$$X[z] = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{5.1}$$

where z is a complex variable. The signal $x[n]$, which is the inverse z-transform of $X[z]$, can be obtained from $X[z]$ by using the following inverse z-transformation:

$$x[n] = \frac{1}{2\pi j} \oint X[z]z^{n-1}\, dz \tag{5.2}$$

The symbol $\oint$ indicates an integration in counterclockwise direction around a closed path in the complex plane (see Fig. 5.1). We derive this z-transform pair later, in Chapter 9, as an extension of the discrete-time Fourier transform pair.

As in the case of the Laplace transform, we need not worry about this integral at this point because inverse z-transforms of many signals of engineering interest can be found in a z-transform table. The direct and inverse z-transforms can be expressed symbolically as

$$X[z] = \mathcal{Z}\{x[n]\} \qquad \text{and} \qquad x[n] = \mathcal{Z}^{-1}\{X[z]\}$$

or simply as

$$x[n] \Longleftrightarrow X[z]$$

Note that

$$\mathcal{Z}^{-1}[\mathcal{Z}\{x[n]\}] = x[n] \qquad \text{and} \qquad \mathcal{Z}[\mathcal{Z}^{-1}\{X[z]\}] = X[z]$$

LINEARITY OF THE z-TRANSFORM

Like the Laplace transform, the z-transform is a linear operator. If

$$x_1[n] \Longleftrightarrow X_1[z] \qquad \text{and} \qquad x_2[n] \Longleftrightarrow X_2[z]$$

then

$$a_1 x_1[n] + a_2 x_2[n] \Longleftrightarrow a_1 X_1[z] + a_2 X_2[z] \tag{5.3}$$

The proof is trivial and follows from the definition of the z-transform. This result can be extended to finite sums.

THE UNILATERAL z-TRANSFORM

For the same reasons discussed in Chapter 4, we find it convenient to consider the unilateral z-transform. As seen for the Laplace case, the bilateral transform has some complications because of nonuniqueness of the inverse transform. In contrast, the unilateral transform has a unique inverse. This fact simplifies the analysis problem considerably, but at a price: the unilateral version can handle only causal signals and systems. Fortunately, most of the practical cases are causal. The more general *bilateral z-transform* is discussed later, in Section 5.9. In practice, the term *z-transform* generally means *the unilateral z-transform.*

In a basic sense, there is no difference between the unilateral and the bilateral z-transform. The unilateral transform is the bilateral transform that deals with a subclass of signals starting at $n = 0$ (causal signals). Hence, the definition of the unilateral transform is the same as that of the bilateral [Eq. (5.1)], except that the limits of the sum are from 0 to ∞

$$X[z] = \sum_{n=0}^{\infty} x[n] z^{-n} \tag{5.4}$$

The expression for the inverse z-transform in Eq. (5.2) remains valid for the unilateral case also.

THE REGION OF CONVERGENCE (ROC) OF X[z]

The sum in Eq. (5.1) [or (5.4)] defining the direct z-transform $X[z]$ may not converge (exist) for all values of z. The values of z (the region in the complex plane) for which the sum in Eq. (5.1) converges (or exists) is called the *region of existence,* or more commonly the *region of convergence* (ROC), for $X[z]$. This concept will become clear in the following example.

EXAMPLE 5.1

Find the z-transform and the corresponding ROC for the signal $\gamma^n u[n]$.

By definition

$$X[z] = \sum_{n=0}^{\infty} \gamma^n u[n] z^{-n}$$

Since $u[n] = 1$ for all $n \geq 0$,

$$X[z] = \sum_{n=0}^{\infty} \left(\frac{\gamma}{z}\right)^n$$

$$= 1 + \left(\frac{\gamma}{z}\right) + \left(\frac{\gamma}{z}\right)^2 + \left(\frac{\gamma}{z}\right)^3 + \cdots + \cdots \tag{5.5}$$

It is helpful to remember the following well-known geometric progression and its sum:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \qquad \text{if} \quad |x| < 1 \tag{5.6}$$

Use of Eq. (5.6) in Eq. (5.5) yields

$$X[z] = \frac{1}{1 - \dfrac{\gamma}{z}} \qquad \left|\frac{\gamma}{z}\right| < 1$$

$$= \frac{z}{z - \gamma} \qquad |z| > |\gamma| \tag{5.7}$$

Observe that $X[z]$ exists only for $|z| > |\gamma|$. For $|z| < |\gamma|$, the sum in Eq. (5.5) may not converge; it goes to infinity. Therefore, the ROC of $X[z]$ is the shaded region outside the circle of radius $|\gamma|$, centered at the origin, in the z-plane, as depicted in Fig. 5.1b.

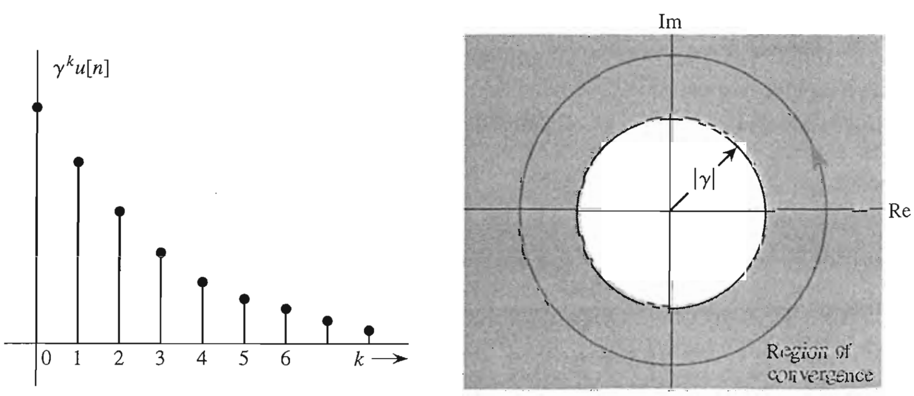

Figure 5.1 $\gamma^n u[n]$ and the region of convergence of its z-transform.

Later in Eq. (5.85), we show that the z-transform of another signal $-\gamma^n u[-(n+1)]$ is also $z/(z-\gamma)$. However, the ROC in this case is $|z| < |\gamma|$. Clearly, the inverse z-transform of $z/(z-\gamma)$ is not unique. However, if we restrict the inverse transform to be causal, then the inverse transform is unique, namely, $\gamma^n u[n]$.

The ROC is required for evaluating $x[n]$ from $X[z]$, according to Eq. (5.2). The integral in Eq. (5.2) is a contour integral, implying integration in a counterclockwise direction along a closed path centered at the origin and satisfying the condition $|z| > |\gamma|$. Thus, any circular path centered at the origin and with a radius greater than $|\gamma|$ (Fig. 5.1b) will suffice. We can show that the integral in Eq. (5.2) along any such path (with a radius greater than $|\gamma|$) yields the same result, namely $x[n]$.† Such integration in the complex plane requires a background in the theory of functions of complex variables. We can avoid this integration by compiling a table of z-transforms (Table 5.1), where z-transform pairs are tabulated for a variety of signals. To find the inverse z-transform of say, $z/(z-\gamma)$, instead of using the complex integration in Eq. (5.2), we consult the table and find the inverse z-transform of $z/(z-\gamma)$ as $\gamma^n u[n]$. Because of uniqueness property of the unilateral z-transform, there is only one inverse for each $X[z]$. Although the table given here is rather short, it comprises the functions of most practical interest.

The situation of the z-transform regarding the uniqueness of the inverse transform is parallel to that of the Laplace transform. For the bilateral case, the inverse z-transform is not unique unless the ROC is specified. For the unilateral case, the inverse transform is unique; the region of convergence need not be specified to determine the inverse z-transform. For this reason, we shall ignore the ROC in the unilateral z-transform Table 5.1.

EXISTENCE OF THE z-TRANSFORM

By definition

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} \frac{x[n]}{z^n}$$

The existence of the z-transform is guaranteed if

$$|X[z]| \leq \sum_{n=0}^{\infty} \frac{|x[n]|}{|z|^n} < \infty$$

for some $|z|$. Any signal $x[n]$ that grows no faster than an exponential signal r_0^n, for some r_0, satisfies this condition. Thus, if

$$|x[n]| \leq r_0^n \qquad \text{for some } r_0 \tag{5.8}$$

then

$$|X[z]| \leq \sum_{n=0}^{\infty} \left(\frac{r_0}{|z|}\right)^n = \frac{1}{1 - \dfrac{r_0}{|z|}} \qquad |z| > r_0$$

†Indeed, the path need not even be circular. It can have any odd shape, as long as it encloses the pole(s) of $X[z]$ and the path of integration is counterclockwise.

TABLE 5.1 (Unilateral) z-Transform Pairs

No.	$x[n]$	$X[z]$
1	$\delta[n-k]$	z^{-k}
2	$u[n]$	$\dfrac{z}{z-1}$
3	$nu[n]$	$\dfrac{z}{(z-1)^2}$
4	$n^2u[n]$	$\dfrac{z(z+1)}{(z-1)^3}$
5	$n^3u[n]$	$\dfrac{z(z^2+4z+1)}{(z-1)^4}$
6	$\gamma^n u[n]$	$\dfrac{z}{z-\gamma}$
7	$\gamma^{n-1}u[n-1]$	$\dfrac{1}{z-\gamma}$
8	$n\gamma^n u[n]$	$\dfrac{\gamma z}{(z-\gamma)^2}$
9	$n^2\gamma^n u[n]$	$\dfrac{\gamma z(z+\gamma)}{(z-\gamma)^3}$
10	$\dfrac{n(n-1)(n-2)\cdots(n-m+1)}{\gamma^m m!}\gamma^n u[n]$	$\dfrac{z}{(z-\gamma)^{m+1}}$
11a	$\lvert\gamma\rvert^n \cos\beta n\, u[n]$	$\dfrac{z(z-\lvert\gamma\rvert\cos\beta)}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
11b	$\lvert\gamma\rvert^n \sin\beta n\, u[n]$	$\dfrac{z\lvert\gamma\rvert\sin\beta}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12a	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$	$\dfrac{rz[z\cos\theta-\lvert\gamma\rvert\cos(\beta-\theta)]}{z^2-(2\lvert\gamma\rvert\cos\beta)z+\lvert\gamma\rvert^2}$
12b	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$ $\quad \gamma=\lvert\gamma\rvert e^{j\beta}$	$\dfrac{(0.5re^{j\theta})z}{z-\gamma}+\dfrac{(0.5re^{-j\theta})z}{z-\gamma^*}$
12c	$r\lvert\gamma\rvert^n\cos(\beta n+\theta)u[n]$	$\dfrac{z(Az+B)}{z^2+2az+\lvert\gamma\rvert^2}$

$$r=\sqrt{\frac{A^2\lvert\gamma\rvert^2+B^2-2AaB}{\lvert\gamma\rvert^2-a^2}}$$

$$\beta=\cos^{-1}\frac{-a}{\lvert\gamma\rvert}$$

$$\theta=\tan^{-1}\frac{Aa-B}{A\sqrt{\lvert\gamma\rvert^2-a^2}}$$

Therefore, $X[z]$ exists for $|z| > r_0$. Almost all practical signals satisfy condition (5.8) and are therefore z-transformable. Some signal models (e.g., γ^{n^2}) grow faster than the exponential signal r_0^n (for any r_0) and do not satisfy Eq. (5.8) and therefore are not z-transformable. Fortunately, such signals are of little practical or theoretical interest. Even such signals over a finite interval are z-transformable.

EXAMPLE 5.2

Find the z-transforms of

(a) $\delta[n]$

(b) $u[n]$

(c) $\cos \beta n\, u[n]$

(d) The signal shown in Fig. 5.2

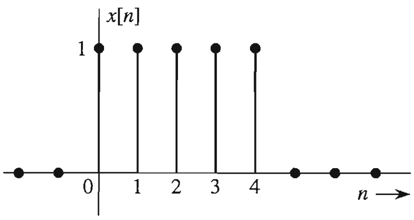

Figure 5.2

Recall that by definition

$$X[z] = \sum_{n=0}^{\infty} x[n]z^{-n}$$
$$= x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \frac{x[3]}{z^3} + \cdots \tag{5.9}$$

(a) For $x[n] = \delta[n]$, $x[0] = 1$ and $x[2] = x[3] = x[4] = \cdots = 0$. Therefore

$$\delta[n] \Longleftrightarrow 1 \qquad \text{for all } z \tag{5.10}$$

(b) For $x[n] = u[n]$, $x[0] = x[1] = x[3] = \cdots = 1$. Therefore

$$X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$$

From Eq. (5.6) it follows that

$$X[z] = \frac{1}{1 - \frac{1}{z}} \qquad \left|\frac{1}{z}\right| < 1$$
$$= \frac{z}{z-1} \qquad |z| > 1$$

Therefore

$$u[n] \Longleftrightarrow \frac{z}{z-1} \qquad |z| > 1 \tag{5.11}$$

(c) Recall that $\cos \beta n = (e^{j\beta n} + e^{-j\beta n})/2$. Moreover, according to Eq. (5.7),

$$e^{\pm j\beta n}u[n] \Longleftrightarrow \frac{z}{z - e^{\pm j\beta}} \qquad |z| > |e^{\pm j\beta}| = 1$$

Therefore

$$X[z] = \frac{1}{2}\left[\frac{z}{z-e^{j\beta}} + \frac{z}{z-e^{-j\beta}}\right]$$
$$= \frac{z(z-\cos\beta)}{z^2 - 2z\cos\beta + 1} \qquad |z| > 1$$

(d) Here $x[0]=x[1]=x[2]=x[3]=x[4]=1$ and $x[5]=x[6]=\cdots=0$. Therefore, according to Eq. (5.9)

$$X[z] = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$$
$$= \frac{z^4+z^3+z^2+z+1}{z^4} \qquad \text{for all } z \neq 0$$

We can also express this result in a more compact form by summing the geometric progression on the right-hand side of the foregoing equation. From the result in Section B.7-4 with $r = 1/z$, $m = 0$, and $n = 4$, we obtain

$$X[z] = \frac{\left(\frac{1}{z}\right)^5 - \left(\frac{1}{z}\right)^0}{\frac{1}{z} - 1} = \frac{z}{z-1}(1 - z^{-5})$$

EXERCISE E5.1

(a) Find the z-transform of a signal shown in Fig. 5.3.

(b) Use pair 12a (Table 5.1) to find the z-transform of $x[n] = 20.65(\sqrt{2})^n \cos((\pi/4)n - 1.415)u[n]$.

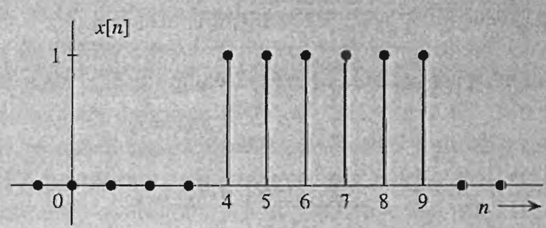

Figure 5.3

ANSWERS

(a) $X[z] = \dfrac{z^5 + z^4 + +z^3 + z^2 + z + 1}{z^9}$ or $\dfrac{z}{z-1}(z^{-4} - z^{-10})$

(b) $\dfrac{z(3.2z + 17.2)}{z^2 - 2z + 2}$

5.1-1 Finding the Inverse Transform

As in the Laplace transform, we shall avoid the integration in the complex plane required to find the inverse z-transform [Eq. (5.2)] by using the (unilateral) transform table (Table 5.1). Many of the transforms $X[z]$ of practical interest are rational functions (ratio of polynomials in z), which can be expressed as a sum of partial fractions, whose inverse transforms can be readily found in a table of transform. The partial fraction method works because for every transformable $x[n]$ defined for $n \geq 0$, there is a corresponding unique $X[z]$ defined for $|z| > r_0$ (where r_0 is some constant), and vice versa.

EXAMPLE 5.3

Find the inverse z-transform of

(a) $\dfrac{8z - 19}{(z-2)(z-3)}$

(b) $\dfrac{z(2z^2 - 11z + 12)}{(z-1)(z-2)^3}$

(c) $\dfrac{2z(3z + 17)}{(z-1)(z^2 - 6z + 25)}$

(a) Expanding $X[z]$ into partial fractions yields

$$X[z] = \frac{8z - 19}{(z-2)(z-3)} = \frac{3}{z-2} + \frac{5}{z-3}$$

From Table 5.1, pair 7, we obtain

$$x[n] = [3(2)^{n-1} + 5(3)^{n-1}]u[n-1] \tag{5.12a}$$

If we expand rational $X[z]$ into partial fractions directly, we shall always obtain an answer that is multiplied by $u[n-1]$ because of the nature of pair 7 in Table 5.1. This form is rather awkward as well as inconvenient. We prefer the form that contains $u[n]$ rather than $u[n-1]$. A glance at Table 5.1 shows that the z-transform of every signal that is multiplied by $u[n]$ has a factor z in the numerator. This observation suggests that we expand $X[z]$ into *modified partial fractions*, where each term has a factor z in the numerator. This goal can be accomplished by expanding $X[z]/z$ into partial fractions and then multiplying both sides by z. We shall demonstrate this procedure by reworking part **(a)**. For this case

$$\frac{X[z]}{z} = \frac{8z-19}{z(z-2)(z-3)} = \frac{(-19/6)}{z} + \frac{(3/2)}{z-2} + \frac{(5/3)}{z-3}$$

Multiplying both sides by z yields

$$X[z] = -\frac{19}{6} + \frac{3}{2}\left(\frac{z}{z-2}\right) + \frac{5}{3}\left(\frac{z}{z-3}\right)$$

From pairs 1 and 6 in Table 5.1, it follows that

$$x[n] = -\tfrac{19}{6}\delta[n] + \left[\tfrac{3}{2}(2)^n + \tfrac{5}{3}(3)^n\right]u[n] \tag{5.12b}$$

The reader can verify that this answer is equivalent to that in Eq. (5.12a) by computing $x[n]$ in both cases for $n = 0, 1, 2, 3, \ldots,$ and comparing the results. The form in Eq. (5.12b) is more convenient than that in Eq. (5.12a). For this reason, we shall always expand $X[z]/z$ rather than $X[z]$ into partial fractions and then multiply both sides by z to obtain modified partial fractions of $X[z]$, which have a factor z in the numerator.

(b)

$$X[z] = \frac{z(2z^2-11z+12)}{(z-1)(z-2)^3}$$

and

$$\frac{X[z]}{z} = \frac{2z^2-11z+12}{(z-1)(z-2)^3} = \frac{k}{z-1} + \frac{a_0}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)}$$

where

$$k = \left.\frac{2z^2-11z+12}{(z-1)(z-2)^3}\right|_{z=1} = -3$$

$$a_0 = \left.\frac{2z^2-11z+12}{(z-1)(z-2)^3}\right|_{z=2} = -2$$

Therefore

$$\frac{X[z]}{z} = \frac{2z^2-11z+12}{(z-1)(z-2)^3} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} + \frac{a_1}{(z-2)^2} + \frac{a_2}{(z-2)} \tag{5.13}$$

We can determine a_1 and a_2 by clearing fractions. Or we may use a shortcut. For example, to determine a_2, we multiply both sides of Eq. (5.13) by z and let $z \to \infty$. This yields

$$0 = -3 - 0 + 0 + a_2 \implies a_2 = 3$$

This result leaves only one unknown, a_1, which is readily determined by letting z take any convenient value, say $z = 0$, on both sides of Eq. (5.13). This step yields

$$\frac{12}{8} = 3 + \frac{1}{4} + \frac{a_1}{4} - \frac{3}{2}$$

which yields $a_1 = -1$. Therefore

$$\frac{X[z]}{z} = \frac{-3}{z-1} - \frac{2}{(z-2)^3} - \frac{1}{(z-2)^2} + \frac{3}{z-2}$$

and

$$X[z] = -3\frac{z}{z-1} - 2\frac{z}{(z-2)^3} - \frac{z}{(z-2)^2} + 3\frac{z}{z-2}$$

Now the use of Table 5.1, pairs 6 and 10, yields

$$\begin{aligned} x[n] &= \left[-3 - 2\frac{n(n-1)}{8}(2)^n - \frac{n}{2}(2)^n + 3(2)^n\right]u[n] \\ &= -\left[3 + \frac{1}{4}(n^2 + n - 12)2^n\right]u[n] \end{aligned}$$

(c) Complex Poles.

$$\begin{aligned} X[z] &= \frac{2z(3z+17)}{(z-1)(z^2-6z+25)} \\ &= \frac{2z(3z+17)}{(z-1)(z-3-j4)(z-3+j4)} \end{aligned}$$

Poles of $X[z]$ are 1, $3 + j4$, and $3 - j4$. Whenever there are complex conjugate poles, the problem can be worked out in two ways. In the first method we expand $X[z]$ into (modified) first-order partial fractions. In the second method, rather than obtain one factor corresponding to each complex conjugate pole, we obtain quadratic factors corresponding to each pair of complex conjugate poles. This procedure is explained next.

METHOD OF FIRST-ORDER FACTORS

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2(3z+17)}{(z-1)(z-3-j4)(z-3+j4)}$$

We find the partial fraction of $X[z]/z$ using the Heaviside "cover-up" method:

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{1.6e^{-j2.246}}{z-3-j4} + \frac{1.6e^{j2.246}}{z-3+j4}$$

and

$$X[z] = 2\frac{z}{z-1} + (1.6e^{-j2.246})\frac{z}{z-3-j4} + (1.6e^{j2.246})\frac{z}{z-3+j4}$$

The inverse transform of the first term on the right-hand side is $2u[n]$. The inverse transform of the remaining two terms (complex conjugate poles) can be obtained from pair 12b (Table 5.1) by identifying $r/2 = 1.6$, $\theta = -2.246$ rad, $\gamma = 3 + j4 = 5e^{j0.927}$, so that $|\gamma| = 5$, $\beta = 0.927$. Therefore

$$x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]u[n]$$

METHOD OF QUADRATIC FACTORS

$$\frac{X[z]}{z} = \frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{Az+B}{z^2-6z+25}$$

Multiplying both sides by z and letting $z \to \infty$, we find

$$0 = 2 + A \Longrightarrow A = -2$$

and

$$\frac{2(3z+17)}{(z-1)(z^2-6z+25)} = \frac{2}{z-1} + \frac{-2z+B}{z^2-6z+25}$$

To find B, we let z take any convenient value, say $z = 0$. This step yields

$$\frac{-34}{25} = -2 + \frac{B}{25} \Longrightarrow B = 16$$

Therefore

$$\frac{X[z]}{z} = \frac{2}{z-1} + \frac{-2z+16}{z^2-6z+25}$$

and

$$X[z] = \frac{2z}{z-1} + \frac{z(-2z+16)}{z^2-6z+25}$$

We now use pair 12c, where we identify $A = -2$, $B = 16$, $|\gamma| = 5$, and $a = -3$. Therefore

$$r = \sqrt{\frac{100+256-192}{25-9}} = 3.2, \qquad \beta = \cos^{-1}\left(\frac{3}{5}\right) = 0.927 \text{ rad}$$

and

$$\theta = \tan^{-1}\left(\tfrac{-10}{-8}\right) = -2.246 \text{ rad}$$

so that

$$x[n] = [2 + 3.2(5)^n \cos(0.927n - 2.246)]u[n]$$

EXERCISE E5.2

Find the inverse z-transform of the following functions:

(a) $\dfrac{z(2z-1)}{(z-1)(z+0.5)}$

(b) $\dfrac{1}{(z-1)(z+0.5)}$

(c) $\dfrac{9}{(z+2)(z-0.5)^2}$

(d) $\dfrac{5z(z-1)}{z^2-1.6z+0.8}$

[Hint: $\sqrt{0.8} = 2/\sqrt{5}$.]

ANSWERS

(a) $\left[\frac{2}{3} + \frac{4}{3}(-0.5)^n\right]u[n]$

(b) $-2\delta[n] + \left[\frac{2}{3} + \frac{4}{3}(-0.5)^n\right]u[n]$

(c) $18\delta[n] - [0.72(-2)^n + 17.28(0.5)^n - 14.4n(0.5)^n]u[n]$

(d) $\frac{5\sqrt{5}}{2}\left(\frac{2}{\sqrt{5}}\right)^n \cos(0.464n + 0.464)u[n]$

Inverse Transform by Expansion of $X[z]$ in Power Series of z^{-1}

By definition

$$\begin{aligned} X[z] &= \sum_{n=0}^{\infty} x[n]z^{-n} \\ &= x[0] + \frac{x[1]}{z} + \frac{x[2]}{z^2} + \frac{x[3]}{z^3} + \cdots \\ &= x[0]z^0 + x[1]z^{-1} + x[2]z^{-2} + x[3]z^{-3} + \cdots \end{aligned}$$

This result is a power series in z^{-1}. Therefore, if we can expand $X[z]$ into the power series in z^{-1}, the coefficients of this power series can be identified as $x[0], x[1], x[2], x[3], \ldots$. A rational $X[z]$ can be expanded into a power series of z^{-1} by dividing its numerator by the denominator. Consider, for example,

$$\begin{aligned} X[z] &= \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} \\ &= \frac{7z^3-2z^2}{z^3-1.7z^2+0.8z-0.1} \end{aligned}$$

To obtain a series expansion in powers of z^{-1}, we divide the numerator by the denominator as follows:

$$\begin{array}{r|l} & 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots \\ \hline z^3 - 1.7z^2 + 0.8z - 0.1 & 7z^3 - 2z^2 \\ & 7z^3 - 11.9z^2 + 5.60z - 0.7 \\ \hline & 9.9z^2 - 5.60z + 0.7 \\ & 9.9z^2 - 16.83z + 7.92 - 0.99z^{-1} \\ \hline & 11.23z - 7.22 + 0.99z^{-1} \\ & 11.23z - 19.09 + 8.98z^{-1} \\ \hline & 11.87 - 7.99z^{-1} \end{array}$$

Thus

$$X[z] = \frac{z^2(7z-2)}{(z-0.2)(z-0.5)(z-1)} = 7 + 9.9z^{-1} + 11.23z^{-2} + 11.87z^{-3} + \cdots$$

Therefore

$$x[0] = 7, \; x[1] = 9.9, \; x[2] = 11.23, \; x[3] = 11.87, \ldots$$

Although this procedure yields $x[n]$ directly, it does not provide a closed-form solution. For this reason, it is not very useful unless we want to know only the first few terms of the sequence $x[n]$.

EXERCISE E5.3

Using long division to find the power series in z^{-1}, show that the inverse z-transform of $z/(z-0.5)$ is $(0.5)^n u[n]$ or $(2)^{-n}u[n]$.

RELATIONSHIP BETWEEN $h[n]$ AND $H[z]$

For an LTID system, if $h[n]$ is its unit impulse response, then from Eq. (3.71b), where we defined $H[z]$, the system transfer function, we write

$$H[z] = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \tag{5.14a}$$

For causal systems, the limits on the sum are from $n = 0$ to ∞. This equation shows that the transfer function $H[z]$ is the z-transform of the impulse response $h[n]$ of an LTID system; that is,

$$h[n] \Longleftrightarrow H[z] \tag{5.14b}$$

This important result relates the time-domain specification $h[n]$ of a system to $H[z]$, the frequency-domain specification of a system. The result is parallel to that for LTIC systems.

EXERCISE E5.4

Redo Exercise E3.14 by taking the inverse z-transform of $H[z]$, as given by Eq. (3.73).

5.2 SOME PROPERTIES OF THE z-TRANSFORM

The z-transform properties are useful in the derivation of z-transforms of many functions and also in the solution of linear difference equations with constant coefficients. Here we consider a few important properties of the z-transform.

In our discussion, the variable n appearing in signals, such as $x[n]$ and $y[n]$, may or may not stand for time. However, in most applications of our interest, n is proportional to time. For this reason, we shall loosely refer to the variable n as time.

In the following discussion of the shift property, we deal with shifted signals $x[n]u[n]$, $x[n-k]u[n-k]$, $x[n-k]u[n]$, and $x[n+k]u[n]$. Unless we physically understand the meaning of such shifts, our understanding of the shift property remains mechanical rather than intuitive or heuristic. For this reason using a hypothetical signal $x[n]$, we have illustrated various shifted signals for $k = 1$ in Fig. 5.4.

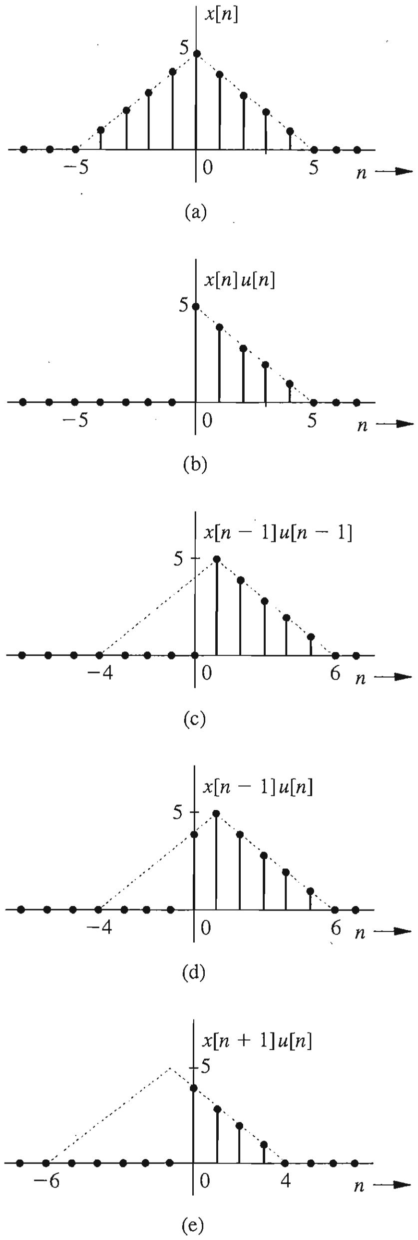

Figure 5.4 A signal $x[n]$ and its shifted versions.

RIGHT SHIFT (DELAY)

If

$$x[n]u[n] \Longleftrightarrow X[z]$$

then

$$x[n-1]u[n-1] \Longleftrightarrow \frac{1}{z}X[z] \tag{5.15a}$$

In general,

$$x[n-m]u[n-m] \Longleftrightarrow \frac{1}{z^m}X[z] \tag{5.15b}$$

Moreover,

$$x[n-1]u[n] \Longleftrightarrow \frac{1}{z}X[z] + x[-1] \tag{5.16a}$$

Repeated application of this property yields

$$\begin{aligned} x[n-2]u[n] \Longleftrightarrow & \frac{1}{z}\left[\frac{1}{z}X[z] + x[-1]\right] + x[-2] \\ &= \frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2] \end{aligned} \tag{5.16b}$$

In general, for integer value of m

$$x[n-m]u[n] \Longleftrightarrow z^{-m}X[z] + z^{-m}\sum_{n=1}^{m} x[-n]z^n \tag{5.16c}$$

A look at Eqs. (5.15a) and (5.16a) shows that they are identical except for the extra term $x[-1]$ in Eq. (5.16a). We see from Fig. 5.4c and 5.4d that $x[n-1]u[n]$ is the same as $x[n-1]u[n-1]$ plus $x[-1]\delta[n]$. Hence, the difference between their transforms is $x[-1]$.

Proof. For integer value of m

$$\mathcal{Z}\{x[n-m]u[n-m]\} = \sum_{n=0}^{\infty} x[n-m]u[n-m]z^{-n}$$

Recall that $x[n-m]u[n-m] = 0$ for $n < m$, so that the limits on the summation on the right-hand side can be taken from $n = m$ to ∞. Therefore

$$\begin{aligned} \mathcal{Z}\{x[n-m]u[n-m]\} &= \sum_{n=m}^{\infty} x[n-m]z^{-n} \\ &= \sum_{r=0}^{\infty} x[r]z^{-(r+m)} \\ &= \frac{1}{z^m}\sum_{r=0}^{\infty} x[r]z^{-r} = \frac{1}{z^m}X[z] \end{aligned}$$

To prove Eq. (5.16c), we have

$$\begin{aligned}\mathcal{Z}\{x[n-m]u[n]\} &= \sum_{n=0}^{\infty} x[n-m]z^{-n} = \sum_{r=-m}^{\infty} x[r]z^{-(r+m)} \\ &= z^{-m}\left[\sum_{r=-m}^{-1} x[r]z^{-r} + \sum_{r=0}^{\infty} x[r]z^{-r}\right] \\ &= z^{-m}\sum_{n=1}^{m} x[-n]z^{n} + z^{-m}X[z]\end{aligned}$$

LEFT SHIFT (ADVANCE)

If

$$x[n]u[n] \Longleftrightarrow X[z]$$

then

$$x[n+1]u[n] \Longleftrightarrow zX[z] - zx[0] \tag{5.17a}$$

Repeated application of this property yields

$$\begin{aligned}x[n+2]u[n] &\Longleftrightarrow z\{z(X[z]-zx[0]) - x[1]\} \\ &= z^2X[z] - z^2x[0] - zx[1]\end{aligned} \tag{5.17b}$$

and for integer value of m

$$x[n+m]u[n] \Longleftrightarrow z^m X[z] - z^m \sum_{n=0}^{m-1} x[n]z^{-n} \tag{5.17c}$$

Proof. By definition

$$\begin{aligned}\mathcal{Z}\{x[n+m]u[n]\} &= \sum_{n=0}^{\infty} x[n+m]z^{-n} \\ &= \sum_{r=m}^{\infty} x[r]z^{-(r-m)} \\ &= z^m \sum_{r=m}^{\infty} x[r]z^{-r} \\ &= z^m\left[\sum_{r=0}^{\infty} x[r]z^{-r} - \sum_{r=0}^{m-1} x[r]z^{-r}\right] \\ &= z^m X[z] - z^m \sum_{r=0}^{m-1} x[r]z^{-r}\end{aligned}$$

EXAMPLE 5.4

Find the z-transform of the signal $x[n]$ depicted in Fig. 5.5.

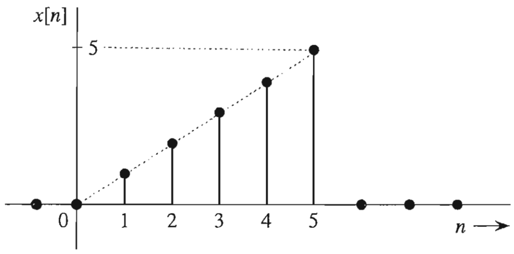

Figure 5.5

The signal $x[n]$ can be expressed as a product of n and a gate pulse $u[n]-u[n-6]$. Therefore

$$\begin{aligned} x[n] &= n\{u[n]-u[n-6]\} \\ &= nu[n]-nu[n-6] \end{aligned}$$

We cannot find the z-transform of $nu[n-6]$ directly by using the right-shift property [Eq. (5.15b)]. So we rearrange it in terms of $(n-6)u[n-6]$ as follows:

$$\begin{aligned} x[n] &= nu[n]-(n-6+6)u[n-6] \\ &= nu[n]-(n-6)u[n-6]-6u[n-6] \end{aligned}$$

We can now find the z-transform of the bracketed term by using the right-shift property [Eq. (5.15b)]. Because $u[n] \Longleftrightarrow z/(z-1)$

$$u[n-6] \Longleftrightarrow \frac{1}{z^6}\frac{z}{z-1} = \frac{1}{z^5(z-1)}$$

Also, because $nu[n] \Longleftrightarrow z/(z-1)^2$

$$(n-6)u[n-6] \Longleftrightarrow \frac{1}{z^6}\frac{z}{(z-1)^2} = \frac{1}{z^5(z-1)^2}$$

Therefore

$$\begin{aligned} X[z] &= \frac{z}{(z-1)^2} - \frac{1}{z^5(z-1)^2} - \frac{6}{z^5(z-1)} \\ &= \frac{z^6-6z+5}{z^5(z-1)^2} \end{aligned}$$

EXERCISE E5.5

Using only the fact that $u[n] \Longleftrightarrow z/(z-1)$ and the right-shift property [Eq. (5.15)], find the z-transforms of the signals in Figs. 5.2 and 5.3.

ANSWERS

See Example 5.2d and Exercise E5.1a.

CONVOLUTION

The time-convolution property states that if[†]

$$x_1[n] \Longleftrightarrow X_1[z] \qquad \text{and} \qquad x_2[n] \Longleftrightarrow X_2[z],$$

then (*time convolution*)

$$x_1[n] * x_2[n] \Longleftrightarrow X_1[z]X_2[z] \tag{5.18}$$

Proof. This property applies to causal as well as noncausal sequences. We shall prove it for the more general case of noncausal sequences, where the convolution sum ranges from $-\infty$ to ∞.

We have

$$\begin{aligned}\mathcal{Z}\{x_1[n] * x_2[n]\} &= \mathcal{Z}\left[\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]\right] \\ &= \sum_{n=-\infty}^{\infty} z^{-n} \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]\end{aligned}$$

Interchanging the order of summation, we have

$$\begin{aligned}\mathcal{Z}[x_1[n] * x_2[n]] &= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{n=-\infty}^{\infty} x_2[n-m]z^{-n} \\ &= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{r=-\infty}^{\infty} x_2[r]z^{-(r+m)} \\ &= \sum_{m=-\infty}^{\infty} x_1[m]z^{-m} \sum_{r=-\infty}^{\infty} x_2[r]z^{-r} \\ &= X_1[z]X_2[z]\end{aligned}$$

[†]There is also the frequency-convolution property, which states that if

$$x_1[n]x_2[n] \Longleftrightarrow \frac{1}{2\pi j}\oint X_1[u]X_2\left[\frac{z}{u}\right]u^{-1}\,du$$

LTID System Response

It is interesting to apply the time-convolution property to the LTID input–output equation $y[n] = x[n] * h[n]$. Since from Eq. (5.14b), we know that $h[n] \Longleftrightarrow H[z]$, it follows from Eq. (5.18) that

$$Y[z] = X[z]H[z] \tag{5.19}$$

EXERCISE E5.6

Use the time-convolution property and appropriate pairs in Table 5.1 to show that $u[n] * u[n-1] = nu[n]$.

Multiplication by γ^n (Scaling in the z-Domain)

If

$$x[n]u[n] \Longleftrightarrow X[z]$$

then

$$\gamma^n x[n]u[n] \Longleftrightarrow X\left[\frac{z}{\gamma}\right] \tag{5.20}$$

Proof.

$$\mathcal{Z}\{\gamma^n x[n]u[n]\} = \sum_{n=0}^{\infty} \gamma^n x[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} x[n] \left(\frac{z}{\gamma}\right)^{-n} = X\left[\frac{z}{\gamma}\right]$$

EXERCISE E5.7

Use Eq. (5.20) to derive pairs 6 and 8 in Table 5.1 from pairs 2 and 3, respectively.

Multiplication by n

If

$$x[n]u[n] \Longleftrightarrow X[z]$$

then

$$nx[n]u[n] \Longleftrightarrow -z\frac{d}{dz}X[z] \tag{5.21}$$

Proof.

$$-z\frac{d}{dz}X[z] = -z\frac{d}{dz}\sum_{n=0}^{\infty} x[n]z^{-n}$$

$$= -z\sum_{n=0}^{\infty} -nx[n]z^{-n-1}$$

$$= \sum_{n=0}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]u[n]\}$$

EXERCISE E5.8

Use Eq. (5.21) to derive pairs 3 and 4 in Table 5.1 from pair 2. Similarly, derive pairs 8 and 9 from pair 6.

TIME REVERSAL

If

$$x[n] \Longleftrightarrow X[z]$$

then†

$$x[-n] \Longleftrightarrow X[1/z] \tag{5.22}$$

Proof.

$$\mathcal{Z}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n]z^{-n}$$

Changing the sign of the dummy variable n yields

$$\mathcal{Z}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{n}$$

$$= \sum_{n=-\infty}^{\infty} x[n](1/z)^{-n}$$

$$= X[1/z]$$

The region of convergence is also inverted, that is, if ROC of $x[n]$ is $|z| > |\gamma|$, then, the ROC of $x[-n]$ is $|z| < 1/|\gamma|$.

†For complex signal $x[n]$, the time-reversal property is modified as follows:

$$x^*[-n] \Longleftrightarrow X^*[1/z^*]$$

EXERCISE E5.9

Use the time-reversal property and pair 2 in Table 5.1 to show that $u[-n] \Longleftrightarrow -1/(z-1)$ with the ROC $|z| < 1$.

TABLE 5.2 *z*-Transform Operations

Operation	$x[n]$	$X[z]$
Addition	$x_1[n] + x_2[n]$	$X_1[z] + X_2[z]$
Scalar multiplication	$ax[n]$	$aX[z]$
Right-shifting	$x[n-m]u[n-m]$	$\frac{1}{z^m}X[z]$
	$x[n-m]u[n]$	$\frac{1}{z^m}X[z] + \frac{1}{z^m}\sum_{n=1}^{m} x[-n]z^n$
	$x[n-1]u[n]$	$\frac{1}{z}X[z] + x[-1]$
	$x[n-2]u[n]$	$\frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2]$
	$x[n-3]u[n]$	$\frac{1}{z^3}X[z] + \frac{1}{z^2}x[-1] + \frac{1}{z}x[-2] + x[-3]$
Left-shifting	$x[n+m]u[n]$	$z^m X[z] - z^m \sum_{n=0}^{m-1} x[n]z^{-n}$
	$x[n+1]u[n]$	$zX[z] - zx[0]$
	$x[n+2]u[n]$	$z^2X[z] - z^2x[0] - zx[1]$
	$x[n+3]u[n]$	$z^3X[z] - z^3x[0] - z^2x[1] - zx[2]$
Multiplication by γ^n	$\gamma^n x[n]u[n]$	$X\left[\frac{z}{\gamma}\right]$
Multiplication by n	$nx[n]u[n]$	$-z\frac{d}{dz}X[z]$
Time convolution	$x_1[n] * x_2[n]$	$X_1[z]X_2[z]$
Time reversal	$x[-n]$	$X[1/z]$
Initial value	$x[0]$	$\lim_{z\to\infty} X[z]$
Final value	$\lim_{N\to\infty} x[N]$	$\lim_{z\to 1}(z-1)X[z]$ poles of $(z-1)X[z]$ inside the unit circle

INITIAL AND FINAL VALUES

For a causal $x[n]$,

$$x[0] = \lim_{z \to \infty} X[z] \tag{5.23a}$$

This result follows immediately from Eq. (5.9).

We can also show that if $(z-1)X(z)$ has no poles outside the unit circle, then†

$$\lim_{N \to \infty} x[N] = \lim_{z \to 1}(z-1)X[z] \tag{5.23b}$$

All these properties of the z-transform are listed in Table 5.2.

5.3 z-TRANSFORM SOLUTION OF LINEAR DIFFERENCE EQUATIONS

The time-shifting (left-shift or right-shift) property has set the stage for solving linear difference equations with constant coefficients. As in the case of the Laplace transform with differential equations, the z-transform converts difference equations into algebraic equations that are readily solved to find the solution in the z domain. Taking the inverse z-transform of the z-domain solution yields the desired time-domain solution. The following examples demonstrate the procedure.

EXAMPLE 5.5

Solve

$$y[n+2] - 5y[n+1] + 6y[n] = 3x[n+1] + 5x[n] \tag{5.24}$$

if the initial conditions are $y[-1] = 11/6$, $y[-2] = 37/36$, and the input $x[n] = (2)^{-n}u[n]$.

As we shall see, difference equations can be solved by using the right-shift or the left-shift property. Because the difference equation, Eq. (5.24), is in advance operator form, the

---

†This can be shown from the fact that

$$x[n] - x[n-1] \Longleftrightarrow \left\{1 - \frac{1}{z}\right\} X[z] = \frac{(z-1)X[z]}{z}$$

and

$$\frac{(z-1)X[z]}{z} = \sum_{n=-\infty}^{\infty} \{x[n] - x[n-1]\} z^{-n}$$

and

$$\lim_{z \to 1} \frac{(z-1)X[z]}{z} = \lim_{z \to 1}(z-1)X[z] = \lim_{z \to 1} \lim_{N \to \infty} \sum_{n=-\infty}^{N} \{x[n] - x[n-1]\} z^{-n} = \lim_{N \to \infty} x[N]$$

use of the left-shift property in Eqs. (5.17a) and (5.17b) may seem appropriate for its solution. Unfortunately, as seen from Eqs. (5.17a) and (5.17b), these properties require a knowledge of auxiliary conditions $y[0], y[1], \dots, y[N-1]$ rather than of the initial conditions $y[-1], y[-2], \dots, y[-n]$, which are generally given. This difficulty can be overcome by expressing the difference equation (5.24) in delay operator form (obtained by replacing n with $n-2$) and then using the right-shift property.† Equation (5.24) in delay operator form is

$$y[n] - 5y[n-1] + 6y[n-2] = 3x[n-1] + 5x[n-2] \tag{5.25}$$

We now use the right-shift property to take the z-transform of this equation. But before proceeding, we must be clear about the meaning of a term like $y[n-1]$ here. Does it mean $y[n-1]u[n-1]$ or $y[n-1]u[n]$? In any equation, we must have some time reference $n = 0$, and every term is referenced from this instant. Hence, $y[n-k]$ means $y[n-k]u[n]$. Remember also that although we are considering the situation for $n \geq 0$, $y[n]$ is present even before $n = 0$ (in the form of initial conditions). Now

$$y[n]u[n] \Longleftrightarrow Y[z]$$

$$y[n-1]u[n] \Longleftrightarrow \frac{1}{z}Y[z] + y[-1] = \frac{1}{z}Y[z] + \frac{11}{6}$$

$$y[n-2]u[n] \Longleftrightarrow \frac{1}{z^2}Y[z] + \frac{1}{z}y[-1] + y[-2] = \frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}$$

Noting that for causal input $x[n]$,

$$x[-1] = x[-2] = \cdots = x[-n] = 0$$

We obtain

$$x[n] = (2)^{-n}u[n] = (2^{-1})^n u[n] = (0.5)^n u[n] \Longleftrightarrow \frac{z}{z-0.5}$$

$$x[n-1]u[n] \Longleftrightarrow \frac{1}{z}X[z] + x[-1] = \frac{1}{z}\frac{z}{z-0.5} + 0 = \frac{1}{z-0.5}$$

$$x[n-2]u[n] \Longleftrightarrow \frac{1}{z^2}X[z] + \frac{1}{z}x[-1] + x[-2] = \frac{1}{z^2}X[z] + 0 + 0 = \frac{1}{z(z-0.5)}$$

In general

$$x[n-r]u[n] \Longleftrightarrow \frac{1}{z^r}X[z]$$

Taking the z-transform of Eq. (5.25) and substituting the foregoing results, we obtain

$$Y[z] - 5\left[\frac{1}{z}Y[z] + \frac{11}{6}\right] + 6\left[\frac{1}{z^2}Y[z] + \frac{11}{6z} + \frac{37}{36}\right] = \frac{3}{z-0.5} + \frac{5}{z(z-0.5)} \tag{5.26a}$$

†Another approach is to find $y[0], y[1], y[2], \dots, y[N-1]$ from $y[-1], y[-2], \dots, y[-n]$ iteratively, as in Section 3.5-1, and then apply the left-shift property to Eq. (5.24).

or

$$\left(1-\frac{5}{z}+\frac{6}{z^2}\right)Y[z]-\left(3-\frac{11}{z}\right)=\frac{3}{z-0.5}+\frac{5}{z(z-0.5)} \tag{5.26b}$$

from which we obtain

$$(z^2-5z+6)Y[z]=\frac{z(3z^2-9.5z+10.5)}{(z-0.5)}$$

so that

$$Y[z]=\frac{z(3z^2-9.5z+10.5)}{(z-0.5)(z^2-5z+6)} \tag{5.27}$$

and

$$\frac{Y[z]}{z}=\frac{3z^2-9.5z+10.5}{(z-0.5)(z-2)(z-3)}$$

$$=\frac{(26/15)}{z-0.5}-\frac{(7/3)}{z-2}+\frac{(18/5)}{z-3}$$

Therefore

$$Y[z]=\frac{26}{15}\left(\frac{z}{z-0.5}\right)-\frac{7}{3}\left(\frac{z}{z-2}\right)+\frac{18}{5}\left(\frac{z}{z-3}\right)$$

and

$$y[n]=\left[\tfrac{26}{15}(0.5)^n-\tfrac{7}{3}(2)^n+\tfrac{18}{5}(3)^n\right]u[n] \tag{5.28}$$

This example demonstrates the ease with which linear difference equations with constant coefficients can be solved by z-transform. This method is general; it can be used to solve a single difference equation or a set of simultaneous difference equations of any order as long as the equations are linear with constant coefficients.

Comment. Sometimes, instead of initial conditions $y[-1], y[-2], \ldots, y[-n]$, auxiliary conditions $y[0], y[1], \ldots, y[N-1]$ are given to solve a difference equation. In this case, the equation can be solved by expressing it in the advance operator form and then using the left-shift property (see later: Exercise E5.11).

EXERCISE E5.10

Solve the following equation if the initial conditions $y[-1]=2$, $y[-2]=0$, and the input $x[n]=u[n]$:

$$y[n+2]-\tfrac{5}{6}y[n+1]+\tfrac{1}{6}y[n]=5x[n+1]-x[n]$$

ANSWER

$y[n]=\left[12-15\left(\tfrac{1}{2}\right)^n+\tfrac{14}{3}\left(\tfrac{1}{3}\right)^n\right]u[n]$

EXERCISE E5.11

Solve the following equation if the auxiliary conditions are $y[0] = 1$, $y[1] = 2$, and the input $x[n] = u[n]$:

$$y[n] + 3y[n-1] + 2y[n-2] = x[n-1] + 3x[n-2]$$

ANSWER

$y[n] = \left[\frac{2}{3} + 2(-1)^n - \frac{5}{3}(-2)^n\right] u[n]$

ZERO-INPUT AND ZERO-STATE COMPONENTS

In Example 5.5 we found the total solution of the difference equation. It is relatively easy to separate the solution into zero-input and zero-state components. All we have to do is to separate the response into terms arising from the input and terms arising from initial conditions. We can separate the response in Eq. (5.26b) as follows:

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y[z] - \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial condition terms}} = \underbrace{\frac{3}{z-0.5} + \frac{5}{z(z-0.5)}}_{\text{terms arising from input}} \tag{5.29}$$

Therefore

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y[z] = \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial condition terms}} + \underbrace{\frac{(3z+5)}{z(z-0.5)}}_{\text{input terms}}$$

Multiplying both sides by z^2 yields

$$(z^2 - 5z + 6)Y[z] = \underbrace{z(3z-11)}_{\text{initial condition terms}} + \underbrace{\frac{z(3z+5)}{z-0.5}}_{\text{input terms}}$$

and

$$Y[z] = \underbrace{\frac{z(3z-11)}{z^2-5z+6}}_{\text{zero-input response}} + \underbrace{\frac{z(3z+5)}{(z-0.5)(z^2-5z+6)}}_{\text{zero-state response}} \tag{5.30}$$

We expand both terms on the right-hand side into modified partial fractions to yield

$$Y[z] = \underbrace{\left[5\left(\frac{z}{z-2}\right) - 2\left(\frac{z}{z-3}\right)\right]}_{\text{zero input}} + \underbrace{\left[\frac{26}{15}\left(\frac{z}{z-0.5}\right) - \frac{22}{3}\left(\frac{z}{z-2}\right) + \frac{28}{5}\left(\frac{z}{z-3}\right)\right]}_{\text{zero state}}$$

and

$$\begin{aligned} y[n] &= \left[\underbrace{5(2)^n - 2(3)^n}_{\text{zero input}} \underbrace{- \tfrac{22}{3}(2)^n + \tfrac{28}{5}(3)^n + \tfrac{26}{15}(0.5)^n}_{\text{zero state}}\right] u[n] \\ &= \left[-\tfrac{7}{3}(2)^n + \tfrac{18}{5}(3)^n + \tfrac{26}{15}(0.5)^n\right] u[n] \end{aligned}$$

which agrees with the result in Eq. (5.28).

EXERCISE E5.12

Solve

$$y[n+2] - \tfrac{5}{6}y[n+1] + \tfrac{1}{6}y[n] = 5x[n+1] - x[n]$$

if the initial conditions are $y[-1] = 2$, $y[-2] = 0$, and the input $x[n] = u[n]$. Separate the response into zero-input and zero-state components.

ANSWER

$$y[n] = \Bigg(\underbrace{\left[3\left(\tfrac{1}{2}\right)^n - \tfrac{4}{3}\left(\tfrac{1}{3}\right)^n\right]}_{\text{zero input}} + \underbrace{\left[12 - 18\left(\tfrac{1}{2}\right)^n + 6\left(\tfrac{1}{3}\right)^n\right]}_{\text{zero state}}\Bigg)u[n]$$

$$= \left[12 - 15\left(\tfrac{1}{2}\right)^n + \tfrac{14}{3}\left(\tfrac{1}{3}\right)^n\right]u[n]$$

5.3-1 Zero-State Response of LTID Systems: The Transfer Function

Consider an Nth-order LTID system specified by the difference equation

$$Q[E]y[n] = P[E]x[n] \tag{5.31a}$$

or

$$\begin{aligned}(E^N + a_1E^{N-1} + \cdots + a_{N-1}E + a_N)y[n] \\ = (b_0E^N + b_1E^{N-1} + \cdots + b_{N-1}E + b_N)x[n]\end{aligned} \tag{5.31b}$$

or

$$\begin{aligned}y[n+N] + a_1y[n+N-1] + \cdots + a_{N-1}y[n+1] + a_Ny[n] \\ = b_0x[n+N] + \cdots + b_{N-1}x[n+1] + b_Nx[n]\end{aligned} \tag{5.31c}$$

We now derive the general expression for the zero-state response: that is, the system response to input $x[n]$ when all the initial conditions $y[-1] = y[-2] = \cdots = y[-N] = 0$ (zero state). The input $x[n]$ is assumed to be causal so that $x[-1] = x[-2] = \cdots = x[-N] = 0$.

Equation (5.31c) can be expressed in the delay operator form as

$$\begin{aligned}y[n] + a_1y[n-1] + \cdots + a_Ny[n-N] \\ = b_0x[n] + b_1x[n-1] + \cdots + b_Nx[n-N]\end{aligned} \tag{5.31d}$$

Because $y[-r] = x[-r] = 0$ for $r = 1, 2, \ldots, N$

$$y[n-m]u[n] \Longleftrightarrow \frac{1}{z^m}Y[z]$$

$$x[n-m]u[n] \Longleftrightarrow \frac{1}{z^m}X[z] \qquad m = 1, 2, \ldots, N$$

Now the z-transform of Eq. (5.31d) is given by

$$\left(1+\frac{a_1}{z}+\frac{a_2}{z^2}+\cdots+\frac{a_N}{z^N}\right)Y[z]=\left(b_0+\frac{b_1}{z}+\frac{b_2}{z^2}+\cdots+\frac{b_N}{z^N}\right)X[z]$$

Multiplication of both sides by z^N yields

$$\begin{aligned}(z^N+a_1z^{N-1}+\cdots+a_{N-1}z+a_N)Y[z]\\ =(b_0z^N+b_1z^{N-1}+\cdots+b_{N-1}z+b_N)X[z]\end{aligned}$$

Therefore

$$Y[z]=\left(\frac{b_0z^N+b_1z^{N-1}+\cdots+b_{N-1}z+b_N}{z^N+a_1z^{N-1}+\cdots+a_{N-1}z+a_N}\right)X[z] \tag{5.32}$$

$$=\frac{P[z]}{Q[z]}X[z] \tag{5.33}$$

We have shown in Eq. (5.19) that $Y[z]=X[z]H[z]$. Hence, it follows that

$$H[z]=\frac{P[z]}{Q[z]}=\frac{b_0z^N+b_1z^{N-1}+\cdots+b_{N-1}z+b_N}{z^N+a_1z^{N-1}+\cdots+a_{N-1}z+a_N} \tag{5.34}$$

As in the case of LTIC systems, this result leads to an alternative definition of the LTID system transfer function as the ratio of $Y[z]$ to $X[z]$ (assuming all initial conditions zero).

$$H[z]\equiv\frac{Y[z]}{X[z]}=\frac{\mathcal{Z}[\text{zero-state response}]}{\mathcal{Z}[\text{input}]} \tag{5.35}$$

ALTERNATE INTERPRETATION OF THE z-TRANSFORM

So far we have treated the z-transform as a machine, which converts linear difference equations into algebraic equations. There is no physical understanding of how this is accomplished or what it means. We now discuss more intuitive interpretation and meaning of the z-transform.

In Chapter 3, Eq. (3.71a), we showed that LTID system response to an everlasting exponential z^n is $H[z]z^n$. If we could express every discrete-time signal as a linear combination of everlasting exponentials of the form z^n, we could readily obtain the system response to any input. For example, if

$$x[n]=\sum_{k=1}^{K}X[z_k]z_k^n \tag{5.36a}$$

the response of an LTID system to this input is given by

$$y[n]=\sum_{k=1}^{K}X[z_k]H[z_k]z_k^n \tag{5.36b}$$

Unfortunately, a very small class of signals can be expressed in the form of Eq. (5.36a). However, we can express almost all signals of practical utility as a sum of everlasting exponentials over a

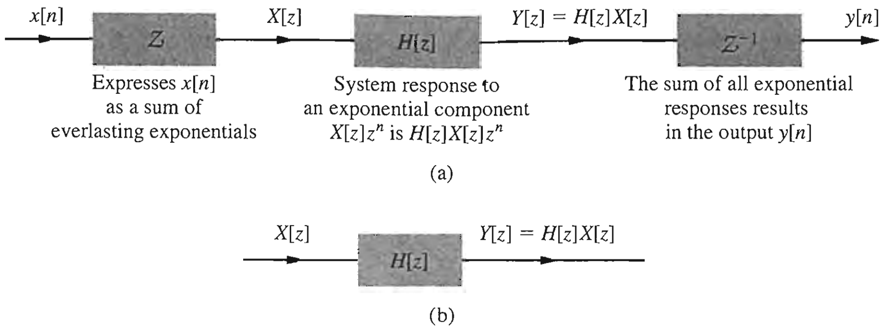

Figure 5.6 The transformed representation of an LTID system.

continuum of values of z. This is precisely what the z-transform in Eq. (5.2) does.

$$x[n] = \frac{1}{2\pi j} \oint X[z]z^{n-1}\, dz \qquad (5.37)$$

Invoking the linearity property of the z-transform, we can find the system response $y[n]$ to input $x[n]$ in the Eq. (5.37) as†

$$y[n] = \frac{1}{2\pi j} \oint X[z]H[z]z^{n-1}\, dz = \mathcal{Z}^{-1}\{X[z]H[z]\}$$

Clearly

$$Y[z] = X[z]H[z]$$

This viewpoint of finding the response of LTID system is illustrated in Fig. 5.6a. Just as in continuous-time systems, we can model discrete-time systems in the transformed manner by representing all signals by their z-transforms and all system components (or elements) by their transfer functions, as shown in Fig. 5.6b.

The result $Y[z] = H[z]X[z]$ greatly facilitates derivation of the system response to a given input. We shall demonstrate this assertion by an example.

EXAMPLE 5.6

Find the response $y[n]$ of an LTID system described by the difference equation

$$y[n+2] + y[n+1] + 0.16y[n] = x[n+1] + 0.32x[n]$$

or

$$(E^2 + E + 0.16)y[n] = (E + 0.32)x[n]$$

†In computing $y[n]$, the contour along which the integration is performed is modified to consider the ROC of $X[z]$ as well as $H[z]$. We ignore this consideration in this intuitive discussion.

for the input $x[n] = (-2)^{-n}u[n]$ and with all the initial conditions zero (system in the zero state).

From the difference equation we find

$$H[z] = \frac{P[z]}{Q[z]} = \frac{z + 0.32}{z^2 + z + 0.16}$$

For the input $x[n] = (-2)^{-n}u[n] = [(-2)^{-1}]^n u(n) = (-0.5)^n u[n]$

$$X[z] = \frac{z}{z + 0.5}$$

and

$$Y[z] = X[z]H[z] = \frac{z(z + 0.32)}{(z^2 + z + 0.16)(z + 0.5)}$$

Therefore

$$\frac{Y[z]}{z} = \frac{(z + 0.32)}{(z^2 + z + 0.16)(z + 0.5)} = \frac{(z + 0.32)}{(z + 0.2)(z + 0.8)(z + 0.5)}$$

$$= \frac{2/3}{z + 0.2} - \frac{8/3}{z + 0.8} + \frac{2}{z + 0.5} \tag{5.38}$$

so that

$$Y[z] = \frac{2}{3}\left(\frac{z}{z + 0.2}\right) - \frac{8}{3}\left(\frac{z}{z + 0.8}\right) + 2\left(\frac{z}{z + 0.5}\right) \tag{5.39}$$

and

$$y[n] = \left[\tfrac{2}{3}(-0.2)^n - \tfrac{8}{3}(-0.8)^n + 2(-0.5)^n\right] u[n]$$

EXAMPLE 5.7 (The Transfer Function of a Unit Delay)

Show that the transfer function of a unit delay is $1/z$.

If the input to the unit delay is $x[n]u[n]$, then its output (Fig. 5.7) is given by

$$y[n] = x[n - 1]u[n - 1]$$

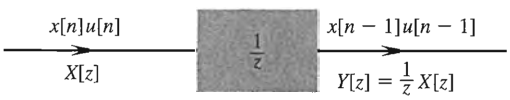

Figure 5.7 Ideal unit delay and its transfer function.

The z-transform of this equation yields [see Eq. (5.15a)]

$$Y[z] = \frac{1}{z}X[z]$$
$$= H[z]X[z]$$

It follows that the transfer function of the unit delay is

$$H[z] = \frac{1}{z} \tag{5.40}$$

EXERCISE E5.13

A discrete-time system is described by the following transfer function:

$$H[z] = \frac{z - 0.5}{(z + 0.5)(z - 1)}$$

(a) Find the system response to input $x[n] = 3^{-(n+1)}u[n]$ if all initial conditions are zero.

(b) Write the difference equation relating the output $y[n]$ to input $x[n]$ for this system.

ANSWERS

(a) $y[n] = \frac{1}{3}\left[\frac{1}{2} - 0.8(-0.5)^n + 0.3\left(\frac{1}{3}\right)^n\right]u[n]$

(b) $y[n+2] - 0.5y[n+1] - 0.5y[n] = x[n+1] - 0.5x[n]$

5.3-2 Stability

Equation (5.34) shows that the denominator of $H[z]$ is $Q[z]$, which is apparently identical to the characteristic polynomial $Q[\gamma]$ defined in Chapter 3. Does this mean that the denominator of $H[z]$ is the characteristic polynomial of the system? This may or may not be the case: if $P[z]$ and $Q[z]$ in Eq. (5.34) have any common factors, they cancel out, and the effective denominator of $H[z]$ is not necessarily equal to $Q[z]$. Recall also that the system transfer function $H[z]$, like $h[n]$, is defined in terms of measurements at the external terminals. Consequently, $H[z]$ and $h[n]$ are both external descriptions of the system. In contrast, the characteristic polynomial $Q[z]$ is an internal description. Clearly, we can determine only external stability, that is, BIBO stability, from $H[z]$. If all the poles of $H[z]$ are within the unit circle, all the terms in $h[z]$ are decaying exponentials, and as shown in Section 3.10, $h[n]$ is absolutely summable. Consequently, the system is BIBO stable. Otherwise the system is BIBO unstable.

If $P[z]$ and $Q[z]$ do not have common factors, then the denominator of $H[z]$ is identical to $Q[z]$.† The poles of $H[z]$ are the characteristic roots of the system. We can now determine

†There is no way of determining whether there were common factors in $P[z]$ and $Q[z]$ that were canceled out because in our derivation of $H[z]$, we generally get the final result after the cancellations are already effected. When we use internal description of the system to derive $Q[z]$, however, we find pure $Q[z]$ unaffected by any common factor in $P[z]$.

internal stability. The internal stability criterion in Section 3.10-1 can be restated in terms of the poles of $H[z]$, as follows.

1. An LTID system is asymptotically stable if and only if all the poles of its transfer function $H[z]$ are within the unit circle. The poles may be repeated or simple.
2. An LTID system is unstable if and only if either one or both of the following conditions exist: (i) at least one pole of $H[z]$ is outside the unit circle; (ii) there are repeated poles of $H[z]$ on the unit circle.
3. An LTID system is marginally stable if and only if there are no poles of $H[z]$ outside the unit circle, and there are some simple poles on the unit circle.

EXERCISE E5.14

Show that an *accumulator* whose impulse response is $h[n] = u[n]$ is marginally stable but BIBO unstable.

5.3-3 Inverse Systems

If $H[z]$ is the transfer function of a system $\mathcal{S}$, then $\mathcal{S}_i$, its inverse system has a transfer function $H_i[z]$ given by

$$H_i[z] = \frac{1}{H[z]}$$

This follows from the fact the inverse system $\mathcal{S}_i$ undoes the operation of $\mathcal{S}$. Hence, if $H[z]$ is placed in cascade with $H_i[z]$, the transfer function of the composite system (identity system) is unity. For example, an *accumulator*, whose transfer function is $H[z] = z/(z-1)$ and a *backward difference system* whose transfer function is $H_i[z] = (z-1)/z$ are inverse of each other. Similarly if

$$H[z] = \frac{z - 0.4}{z - 0.7}$$

its inverse system transfer function is

$$H_i[z] = \frac{z - 0.7}{z - 0.4}$$

as required by the property $H[z]H_i[z] = 1$. Hence, it follows that

$$h[n] * h_i[n] = \delta[n]$$

EXERCISE E5.15

Find the impulse response of an accumulator and the backward difference system. Show that the convolution of the two impulse responses yields $\delta[n]$.

5.4 System Realization

Because of the similarity between LTIC and LTID systems, conventions for block diagrams and rules of interconnection for LTID are identical to those for continuous-time (LTIC) systems. It is not necessary to rederive these relationships. We shall merely restate them to refresh the reader's memory.

The block diagram representation of the basic operations such as an adder a scalar multiplier, unit delay, and pick off points were shown in Fig. 3.11. In our development, the unit delay, which was represented by a box marked D in Fig. 3.11, will be represented by its transfer function $1/z$. All the signals will also be represented in terms of their z-transforms. Thus, the input and the output will be labeled $X[z]$ and $Y[z]$, respectively.

When two systems with transfer functions $H_1[z]$ and $H_2[z]$ are connected in cascade (as in Fig. 4.18b), the transfer function of the composite system is $H_1[z]H_2[z]$. If the same two systems are connected in parallel (as in Fig. 4.18c), the transfer function of the composite system is $H_1[z]H_2[z]$. For a feedback system (as in Fig. 4.18d), the transfer function is $G[z]/(1 + G[z]H[z])$.

We now consider a systematic method for realization (or simulation) of an arbitrary Nth-order LTID transfer function. Since realization is basically a synthesis problem, there is no unique way of realizing a system. A given transfer function can be realized in many different ways. We present here the two forms of *direct realization*. Each of these forms can be executed in several other ways, such as cascade and parallel. Furthermore, a system can be realized by the transposed version of any known realization of that system. This artifice doubles the number of system realizations. A transfer function $H[z]$ can be realized by using time delays along with adders and multipliers.

We shall consider a realization of a general Nth-order causal LTID system, whose transfer function is given by

$$H[z] = \frac{b_0 z^N + b_1 z^{N-1} + \cdots + b_{N-1} z + b_N}{z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N} \tag{5.41}$$

This equation is identical to the transfer function of a general Nth-order proper LTIC system given in Eq. (4.60). The only difference is that the variable z in the former is replaced by the variable s in the latter. Hence, the procedure for realizing an LTID transfer function is identical to that for the LTIC transfer function with the basic element $1/s$ (integrator) replaced by the element $1/z$ (unit delay). The reader is encouraged to follow the steps in Section 4.6 and rederive the results for the LTID transfer function in Eq. (5.41). Here we shall merely reproduce the realizations from Section 4.6 with integrators ($1/s$) replaced by unit delays ($1/z$). The direct form I (DFI) is shown in Fig. 5.8a, the canonic direct form (DFII) is shown in Fig. 5.8b and the transpose of canonic direct is shown in Fig. 5.8c. The DFII and its transpose are canonic because they require N delays, which is the minimum number needed to implement an Nth-order LTID transfer function in Eq. (5.41). In contrast, the form DFI is a noncanonic because it generally requires $2N$ delays. The DFII realization in Fig. 5.8b is also called a *canonic direct* form.