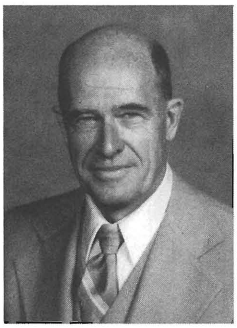
One classical technique in determining pole variations with parameters is known as the **root-locus method**, invented by W. R. Evens, which will be introduced in this chapter. It should be pointed out that in most cases *it is more important to know how to quickly sketch a root locus, which would indicate the trend of the root loci rather than the exact root-locus plot*, which can be done by using a computer if necessary. Thus, attention is not focused on the exact details of root-locus construction in this chapter. Instead, we shall concentrate on how we can use root locus as a tool for system analysis and controller design.



Walter R. Evans (1920–1999) earned his BS degree in Electrical Engineering from Washington University in 1941 and his MS Degree in Electrical Engineering from the University of California, Los Angeles, in 1951. He taught as an instructor in the Department of Electrical Engineering at Washington University from 1946 to 1948. In 1948, Mr Evans moved to Autonetics, a division of North American Aviation, now known as Rockwell International. It was during his lectures to his colleagues on analysis of servo-mechanisms in August 1948 that he finally came up with the root-locus techniques. That same year, he developed the Spirule, a tool used in conjunction with the application of the root-locus method, and The Spirule Company (formed by him) sold in the next few decades over 100,000 copies of the Spirule over 75 countries around the world. His root-locus method was published in the paper "Graphical analysis of control systems," *Transactions of the American Institute of Electrical Engineers,* vol. 67, pp. 547–551, 1948, and in the paper "Control system synthesis by root-locus method," *Transactions of the American Institute of Electrical Engineers,* vol. 69, pp. 66–69, 1950. Mr Evans worked with the technical staff of the Guidance and Control Department of the Re-Entry Systems Operation of the Ford Aeronautic Company from 1959 to 1971. He rejoined Autonetics where he worked with the technical staff of the Strategic Systems Division until his retirement in 1980. Mr Evans was awarded the prestigious Rufus Oldenburger Medal by the American Society of Mechanical Engineers in 1987 and the Richard E. Bellman Control Heritage Award of the American Automatic Control Council in 1988.

5.1 ROOT-LOCUS TECHNIQUES

Consider a standard feedback system for stabilization shown in Figure 5.1 or a unity feedback system shown in Figure 5.2.

The closed-loop poles are given by the roots of the following equation:

$$1 + P(s)C(s) = 0.$$

For simplicity of presentation, we shall denote

$$L(s) := P(s)C(s)$$

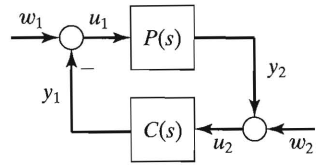


FIGURE 5.1: Feedback system for stabilization.

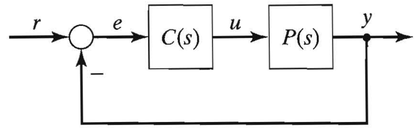


FIGURE 5.2: A unity feedback system.

and assume that $L(s)$ has the following form:

$$L(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

where $z_1, \ldots, z_m$ are the open-loop zeros; $p_1, \ldots, p_n$ are the open-loop poles; and K is a variable gain. Our objective is to study how the closed-loop poles change when K varies from 0 to ∞. We shall show later how problems involving other system parameters may be converted into problems like this.

Thus, our goal is to find all points that satisfy

$$L(s) = -1.$$

This equation can be equivalently written into two equations:

Magnitude condition:	$\lvert L(s)\rvert = 1$	(5.1)
Phase condition:	$\angle L(s) = (2k+1)180^\circ, \ k = 0, \pm 1, \ldots$	(5.2)

It is easy to see that the magnitude condition can always be satisfied by a suitable $K \geq 0$. On the other hand, the phase condition does not depend on the value of K (but depends on the sign of K):

$$\angle L(s) = \sum_{i=1}^{m} \angle(s-z_i) - \sum_{j=1}^{n} \angle(s-p_j) = (2k+1)180^\circ.$$

Thus, the key is to find all those points that satisfy the phase condition. Consider, for example, a system with open-loop transfer function

$$L(s) = \frac{K(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)(s-p_3)}, \qquad p_2 = \overline{p}_1.$$

The open-loop poles and zeros of the system are shown in Figure 5.3 where a pole is represented by "×" and a zero is represented by "∘." The phase of $L(s)$ at a

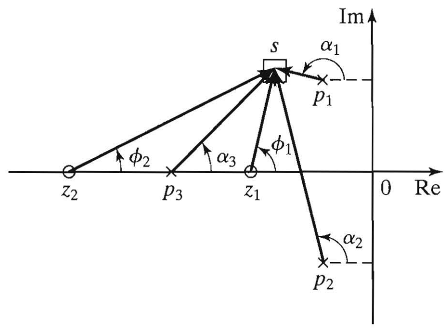


FIGURE 5.3: The phase of $L(s)$ at a point s.

point s in the complex plane is computed as

$$\begin{aligned}\angle L(s) &= \angle(s - z_1) + \angle(s - z_2) - \angle(s - p_1) - \angle(s - p_2) - \angle(s - p_3) \\ &= \phi_1 + \phi_2 - \alpha_1 - \alpha_2 - \alpha_3.\end{aligned}$$

Several basic rules can be derived from the phase condition, which will facilitate the sketching of the root locus. These rules are summarized in Table 5.1. The terminologies used in the table such as "asymptotes," "breakaway points," "angle of departure," and "angle of arrival" are illustrated in the following two examples.

EXAMPLE 5.1

In this example, asymptotes and breakaway points are illustrated. Consider an open-loop transfer function

$$L(s) = \frac{K}{s(s+4)(s+5)}.$$

The system has three poles and no zero, so the angles of the three asymptotes can be calculated as

$$\theta = \frac{(2k+1)180^\circ}{3} = 60^\circ, -60^\circ, 180^\circ$$

for $k = 0, -1$, and 1, and the intersection of the asymptotes with the real axis is given by

$$\kappa = \frac{0 - 4 - 5}{3} = -3.$$

Note that we could have set $k = 0, 1, 2$ to get $\theta = 60^\circ, 180^\circ, 300^\circ$, which are the same angles. The three asymptotes are shown in Figure 5.4.

The asymptotes clearly indicate that the system will become unstable when the gain is sufficiently large. It is quite easy to see by Rule 4 that this is always the case if the relative degree of $L(s)$ is at least 3, since in that case there will be at least one asymptote with an angle less than 90°.

$\boxed{1}$. The root locus is symmetric with respect to the real axis.

$\boxed{2}$. The root loci start from n poles p_i (when $K = 0$) and approach the n zeros (m finite zeros z_i and $n - m$ infinite zeros when $K \to \infty$).

$\boxed{3}$. The root locus includes all points on the real axis to the left of an odd number of open-loop real poles and zeros.

$\boxed{4}$. As $K \to \infty$, $n - m$ branches of the root-locus approach asymptotically $n - m$ straight lines (called **asymptotes**) with angles

$$\theta = \frac{(2k+1)180^\circ}{n-m}, \quad k = 0, \pm 1, \pm 2, \ldots$$

and the starting point of all asymptotes is on the real axis at

$$\kappa = \frac{\sum_{i=1}^{n} p_i - \sum_{j=1}^{m} z_j}{n-m} = \frac{\sum \text{poles} - \sum \text{zeros}}{n-m}.$$

5. The **breakaway points** (where the root loci meet and split away, usually on real axis) and the **breakin points** (where the root loci meet and enter the real axis) are among the roots of the equation: $\dfrac{dL(s)}{ds} = 0$. (On the real axis, only those roots that satisfy Rule 3 are breakaway or breakin points.)

6. The **departure angle** ϕ_k (from a pole, p_k) is given by

$$\phi_k = \sum_{i=1}^{m} \angle(p_k - z_i) - \sum_{j=1, j\neq k}^{n} \angle(p_k - p_j) \pm 180^\circ.$$

(In the case p_k is l repeated poles, the departure angle becomes ϕ_k/ℓ.)

The **arrival angle** ψ_k (at a zero, z_k) is given by

$$\psi_k = -\sum_{i=1, i\neq k}^{m} \angle(z_k - z_i) + \sum_{j=1}^{n} \angle(z_k - p_j) \pm 180^\circ.$$

(In the case z_k is l repeated zeros, the arrival angle becomes ψ_k/ℓ.)

TABLE 5.1: Root locus rules: $0 \leq K \leq \infty$.

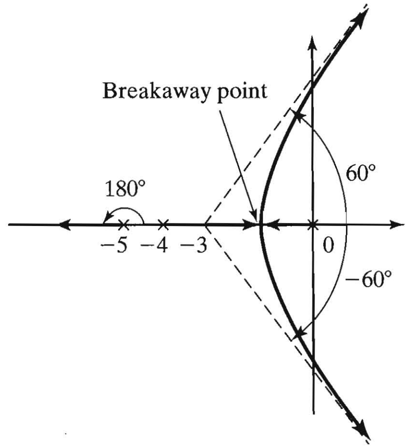


FIGURE 5.4: Breakaway point and asymptotes.

EXAMPLE 5.2

In this example, departure and arrival angles are illustrated. Consider an open-loop transfer function

$$L(s) = \frac{K(s^2+4s+8)}{(s+3)(s^2+2s+2)} = \frac{K(s-z_1)(s-z_2)}{(s-p_1)(s-p_2)(s-p_3)}$$

with zeros at $z_1 = -2+j2$ and $z_2 = -2-j2$ and poles at $p_1 = -3$, $p_2 = -1+j$, and $p_3 = -1-j$.

The departure angle ϕ at $p_2 = -1+j$ satisfies the following equation

$$\angle(p_2 - z_1) + \angle(p_2 - z_2) - \angle(p_2 - p_1) - \phi - \angle(p_2 - p_3) = -180°,$$

i.e.,

$$-45° + \tan^{-1} 3 - 90° - \phi - \tan^{-1}\frac{1}{2} = -180°$$

which gives $\phi = 90°$, while the arrival angle at the zero $z_1 = -2+j2$ can be calculated from

$$\psi + \angle(z_1 - z_2) - \angle(z_1 - p_1) - \angle(z_1 - p_2) - \angle(z_1 - p_3) = -180°,$$

i.e.,

$$\psi + 90° - \tan^{-1} 2 - 135° - (180° - \tan^{-1} 3) = -180°$$

which gives $\psi = 36.87°$.

The departure angle at $p_3 = -1-j$ and the arrival angle at $z_2 = -2-j2$ are $-\phi$ and $-\psi$, respectively. All these angles are shown in Figure 5.5.

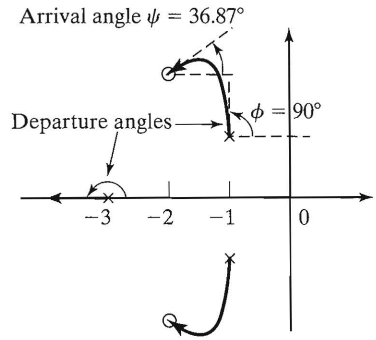


FIGURE 5.5: Departure and arrival angles.

The detailed derivations of these rules in Table 5.1 are given in the next section. We shall only look at Rule 3 here. Take s to be a point on the real axis as shown in Figure 5.6. It is easy to see that any complex pair of poles or zeros will contribute totally zero degree of phase for any test point on the real axis. Thus, we only need to look at real zeros and poles. It is also clear that any real pole or real zero on the left-hand side of the testing point s (real number) will also contribute zero degree of phase, and any pole or zero on the right-hand side will contribute 180 degree of phase. Hence, Rule 3 is verified.

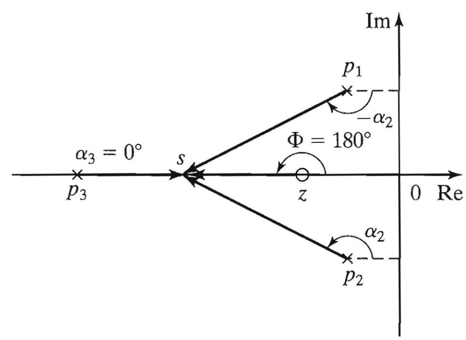


FIGURE 5.6: The phase of $L(s)$ at a point s on the real axis.

Normally, a quick sketch of the root-locus can be obtained by using only Rules 1–4. Rules 5 and 6 are rarely used nowadays since the exact root-locus can be easily generated using a computer program.

EXAMPLE 5.3

Consider a feedback system shown in Figure 5.2 with

$$L(s) = P(s)C(s) = \frac{8K(s+2)}{(s+1)(s+5)(s+10)}.$$

We would like to study how the closed-loop poles change when K varies from 0 to $+\infty$. This is done by constructing a root-locus plot of the system with respect to K. From Rule 3, we know there are root loci on the real axis in two intervals: $[-2, -1]$ and $[-10, -5]$. By Rule 2, one root locus starts from -1, a pole, and ends at -2, a zero, and other two root loci start from -10 and -5, respectively, and then approach two infinite zeros with two asymptotes. The two asymptotes start at

$$\kappa = \frac{(-1 - 5 - 10) - (-2)}{3 - 1} = -7$$

on the real axis with $90°$ and $-90°$, respectively. With the above information, a rough root locus can be sketched.

`z=-2;`	vector of zeros
`p=[-1,-5,-10];`	vector of poles
`k=8;`	gain (not the variable gain K)
`L=zpk(z,p,k)`	form the transfer function of the system
`rlocus(L)`	generate a root-locus plot with an automatically chosen range of gain K

An accurate root-locus plot can be generated by using the sequence of MATLAB commands listed in the box. The root-locus plot is shown in Figure 5.7.

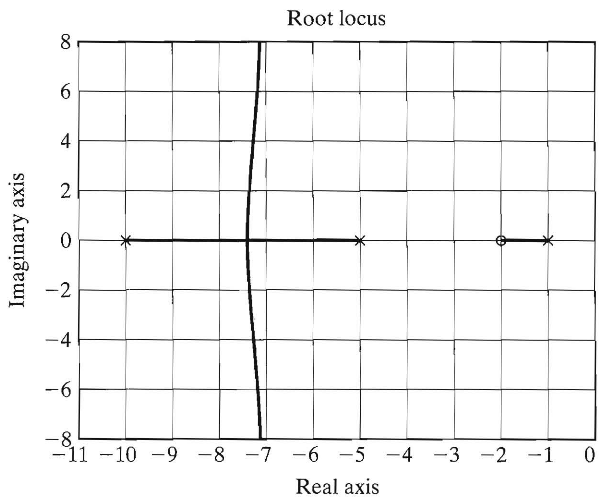


FIGURE 5.7: Root locus for Example 5.3.

Table 5.2 shows some typical root-locus plots with the given open-loop pole and zero patterns.

In many applications, the varying parameters do not necessarily appear as gains and they can, in general, appear anywhere in the transfer functions. The following example shows how to convert a nonstandard root-locus problem into a standard one.

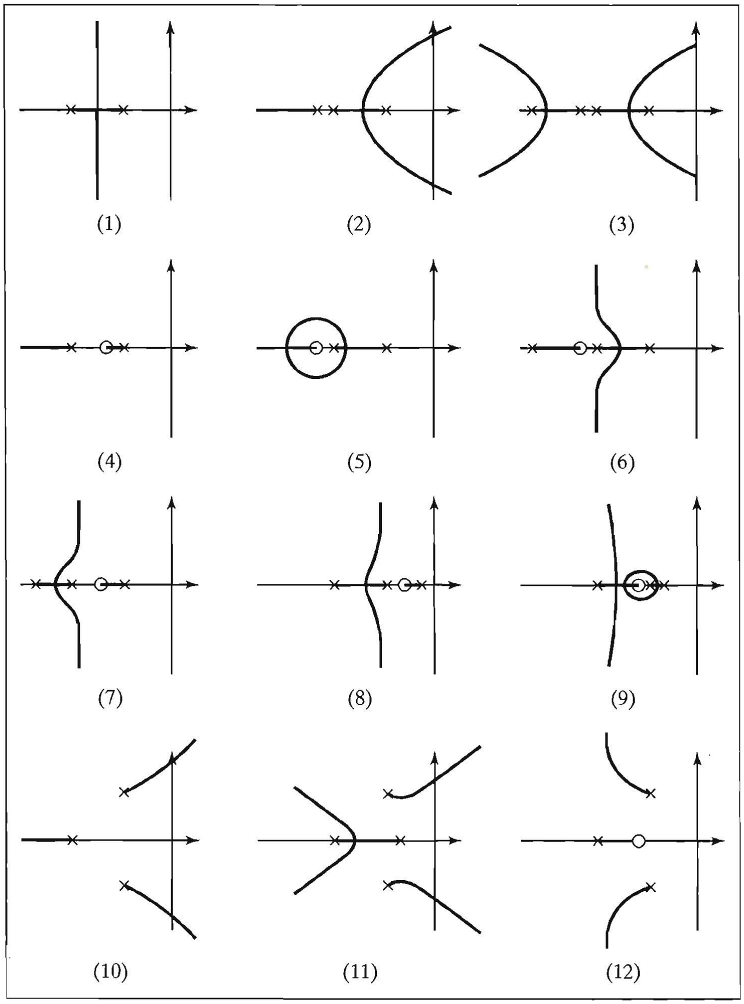


TABLE 5.2: Some typical root-locus plots.

EXAMPLE 5.4

Consider a feedback system with

$$P(s)C(s) = \frac{4(s+3)}{s(s+1)(s+K)}$$

where K is a variable pole position. We would like to analyze how K affects the system stability and performance. This problem is obviously not in the standard root-locus format. Nevertheless, the closed-loop poles are given by the roots of

$$1 + P(s)C(s) = 0$$

or

$$s(s+1)(s+K)+4(s+3)=0$$

which can be written as

$$(s+2)(s^2-s+6)+Ks(s+1)=0$$

or

$$1+\frac{Ks(s+1)}{(s+2)(s^2-s+6)}=0.$$

Let

$$L(s)=\frac{Ks(s+1)}{(s+2)(s^2-s+6)}.$$

`num=[1,1,0];` numerator coefficients of $L(s)$ excluding K

`den=conv([1,2], [1,-1,6]);` denominator coefficients of $L(s)$

`L=tf(num,den)` create the transfer function

`rlocus(L)` generate a root-locus plot

We can then construct the root locus for this system as usual. We can also use the above MATLAB commands for the task. The root locus of this system is shown in Figure 5.8.

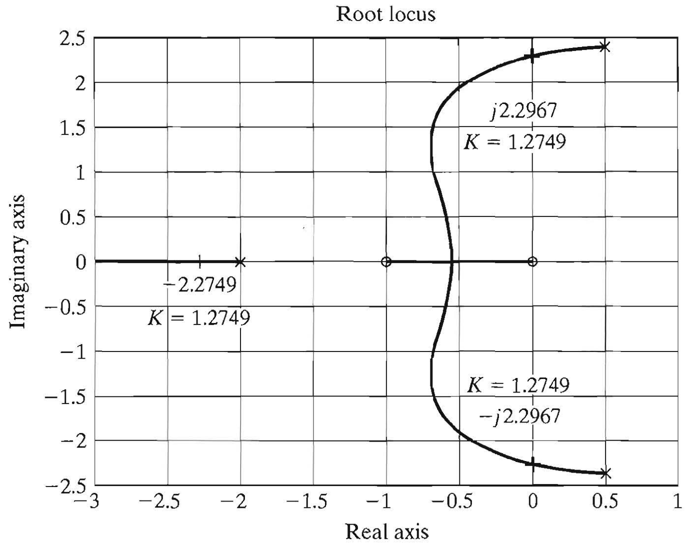


FIGURE 5.8: Root locus for Example 5.4 (The root locations for $K=1.2749$ are shown with "+").

We can also get the numerical value of any point on the root-locus plot and the corresponding gain value by using `rlocfind`:

>> `[K, poles]=rlocfind(L)` (just point and click on any desired point on the plot after entering this command)

For example, to get the critical value of K where the root locus enters the left half plane, simply enter the above command and then click on the intersection of the root locus and the imaginary axis. We get

$$K = 1.2749, \quad \text{poles} = -2.2749, \pm j2.2967.$$

(The actual numerical value may be slightly different from the above depending on how accurate the point was actually clicked.) This shows that the closed-loop system will only be stable for $K > 1.2749$.

This critical value of K can also be determined exactly by applying the Routh–Horwitz stability test in Chapter 3 to the characteristic polynomial:

$$s(s+1)(s+K) + 4(s+3) = s^3 + (1+K)s^2 + (K+4)s + 12.$$

The stability criterion then shows that the system is stable if and only if

$$(K+1)(K+4) - 12 > 0,$$

i.e.,

$$K > \frac{\sqrt{57}-5}{2} = 1.2749.$$

5.2 DERIVATIONS OF ROOT-LOCUS RULES*

Here we shall revisit the root-locus rules. Note that the closed-loop characteristic equation is

$$c(s) := (s-p_1)(s-p_2)\cdots(s-p_n) + K(s-z_1)(s-z_2)\cdots(s-z_m) = 0. \quad (5.3)$$

Rule 1. This is obvious. Since (5.3) has real coefficients, all roots of the equation appear in complex conjugate pairs.

Rule 2. When $K = 0$, (5.3) reduces to

$$(s-p_1)(s-p_2)\cdots(s-p_n) = 0$$

such that the solutions are $s = p_i, i = 1, 2, \ldots, n$. When K increases, the roots are moving continuously away from p_i.

When $K \to \infty$, we can rewrite (5.3) as

$$\frac{1}{K}(s-p_1)(s-p_2)\cdots(s-p_n) + (s-z_1)(s-z_2)\cdots(s-z_m) = 0.$$

If $|s|$ is finite, then the equation approaches

$$(s-z_1)(s-z_2)\cdots(s-z_m) = 0,$$